Symmetry Viewpoint and Linear Algebra

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1 General Theme: Set-Theoretic Basis, and Symmetry Viewpoint

From times immemorial, "Space" and "Number" are recognised as two categories of mathematical thought. But in the 20th century, after the compelling advocacy by Hermann Weyl, "Symmetry" is also becoming recognised as a category of mathematical thought. We have now agreed to base all mathematics on set theory, (or more generally category theory). So "number" refers to the usual development from natural numbers to real and complex numbers, or more generally, sets with two binary operations of addition, and <u>commutative</u> multiplication. Space refers to pictures of material objects which we can see with our physical eyes, or more generally the mental pictures we can build on these objects, again formulated in terms of finite or countable sets, or the sets constructed out of the set which we see as the "real line". The "real line" is a totally ordered set with completeness *à* la Dedekind, but devoid of any "addition" or "multiplication".

Notice that the set-theoretic foundation has freed mathematicians from the idea of "existence" restricted to sense-perception. In modern mathematics, it is not necessary that something exists because I "see" it, or "smell" it, or "taste" it, or "touch" it. In mathematics, something exists if starting from some "known" sets, the object can be constructed using axioms of set theory.

"Symmetry" is a much more subtle category than "Space" and "Number". Although the word was used from times immemorial in art, painting, sculpture, architecture... its mathematical formulation became available only after we agreed to adopt the set-theoretic foundation. It goes much beyond a vague feeling of symmetry of physical forms. The idea was formulated in terms of a "group" or a "semi-group", which is a set with binary operation satisfying the well-known properties, and the closely related notion of its "action" on a set.

We introduce these notions at UG/PG levels in algebra courses as certain definitions. But these are not merely definitions. They embody a much deeper way we actually think, consciously or unconsciously, in mathematics, or wherever mathematics is used. They derive their depth from the precise classification theorems, namely, enumeration of simple Lie groups, enumeration of finite simple groups, and enumeration of their representaions. These are some of the very high points of the 20th century mathematics. We are still filling many gaps which of course must remain in such a broad theme.

It would not be an exaggeration to say that many Nobel Laureates and Fields Medalists, and other distinguished mathematicians and scientists, have used “symmetry”, crucially in their works, perhaps not always consciosuly. In volume 16 of RMS Lecture Notes Series entitled “Symmetry: A Multi-Disciplinary Perspective”, edited by Prof. Passi, you will find the role of symmetry in many sciences. IISER (Mohali), where Prof. Passi is a Professor, has in fact introduced a regular undergraduate course on “symmetry”.

I should say that in this dicsussion, I have myself benefitted from some conversations with Nitin Nitsure of TIFR.

So the moral is: in any of our invetigations, we should consciously try to see if any symmetry is present. Moreover lot of classical mathematics is much better understood if we recognise the underlying symmetry.

Closely related to “Symmetry” is the notion of “Dynamics”. Roughly, the word “symmetry” is used in the context of group-actions, whereas “dynamics” is used in the context of semi-group-actions. In Physics, “Dynamics” refers to “motion”, in particular to “space“ (in the physical sense), “time”, “velocity”, “speed”, “force”, and “mass”. In Physics and Applied Mathematics, we still teach“Statics” and “Dynamics” as standard courses at the UG/PG levels in most universities. You see, in modern mathematics, the words “velocity”, “speed”, “force”, “mass” are no longer used. They are all formulated in set-theoretic terms. Thus “time” is a real number (after you choose where it starts), and “speed”, “mass” are non-negative real numbers, “velocity”, “force” are vectors sitting in more sophisticated sets associated to the original space. “Space-time” is no longer a mystery: mathematically, it is just 4-tuples of real numbers.

The set-theoretic idea of “dynamics” is already present in the very definition of a function f from a set A to a set B. A very important case is when $A = B$, and f is bijective. In case $A = B$, the domain and the range are the same, f gives rise to its iterates $f^1 = f$, $f^2 = \circ f$ $f^3 = f^2 \circ f, \cdots$. One sets $f^0 = i_A$, the identity map of A. If f is bijective, then one also obtains the inverse function f^{-1}, and its iterates. To the set A we can associate a group S_A, which is the set of all self-bijections of A, with composition as the binary operation. An invertible self map f on A gives rise to a homo-morphism from the group $\mathbb{Z}$ of integers with addition to S_A, namely $n \mapsto f^n$. In other words the <u>action</u> of $\mathbb{Z}$ on the set A. In the same way, we can consider the set M_A of all maps from A to itself. If f is in M_A, then its iteration gives rise to a homomorphism from the semigroup $\mathbb{W}$ of “whole“ numbers, i.e. natural numbers with 0 adjoined, to M_A.

If A is a topological space, and G is a topological group, we can formulate the notion of a topological group-action of G on A. If A is a differential manifold, and G is a Lie group, we can formulate the notion of a differential group-action of G on A. One of the profound contributions of Poincare at the end of the 19th century, is that a vector field on a differential manifold

A is nothing but an ODE on A, and it gives rise to a (local) action of the $\mathbb{R}$, the additive group of real numbers, on A. Thus Poincare turned the subject of ODE as subset of the study of group-actions.

I would like to interject a personal story. My thesis advisor was Shlomo Sternberg. He is above 75 now. He is an enthusiast of Poincare. My first serious assignemnt as a graduate student, was to read the original paper of Poincare, in French, in which the above ideas are expressed. Poincare of course does not exactly formulate a "theorem" in some technical way. But one derives inspiration from reading original papers, which bring together different ways of thinking.

I have already used the word "differential manifold" above. I do not wish to go into the explanation of this term here.

In my "vision"-statement for RMS, I suggested that RMS should start regional "School/UG/PG" activities. The activities are not restricted to pure or applicable mathematics, or discrete mathematics. In my opinion, mathematics is a science in its own right. It is not just a "language" of science. It is also a science of constructing theories in all sciences. The ideas of "symmetry", "manifold", and "dynamics" arise in all sciences, often expressed in different terms. In my opinion, these ideas have to arise for cognitive reasons. So we should consciosuly incorporate the "Symmetry" viewpoint, "Manifold" viewpoint, and "Dynamical" viewpoint, in our learning or teaching any of these subjects, and bring them in our School/UG/PG training.

There have been significant advances in mathematics in the last century. Popularly, the 20th century was named the century of Physics. However, most people here will agree that it was as much a century of Mathematics. However we have a gigantic task to incorporate those advances in our teaching of mathematics. At the undergraduate or even PG-level we cannot expect that the students will opt for a career in mathematics. Yet we should communicate to them a feel for the advances which have occured in mathematics. At the same time, many of these students will opt for career in other sciences, and we should consciously look for inter-disciplinary contributions, which will enrich us all.

In this lecture, I shall illustrate the use of symmetry in Linear Algebra. I hope, that even a faculty member in an undergraduate college should be able to use part of this lecture profitably in his teaching.

At the same time there is some new mathematics present here, which has developed in the last 6-7 years. I may not be able to do full justice to it. I refer to the references at the end.

2 Elementary Linear Algebra

First let us quickly review some elementary Linear Algebra from the perspectives mentioned above.

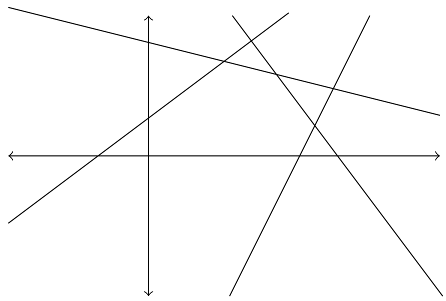
"Number"-aspect: k = a field

$$(*) \qquad \sum_{j=1}^{n} a_{ij} x_j = b_i, \ i = 1, 2, \cdots, m,$$

$$a_{ij}, b_j \in k.$$

"Space"-aspect: associated to $(*)$ a static picture of m hyperplanes in k^n.

$n = 2$



Dynamic approach: Consider

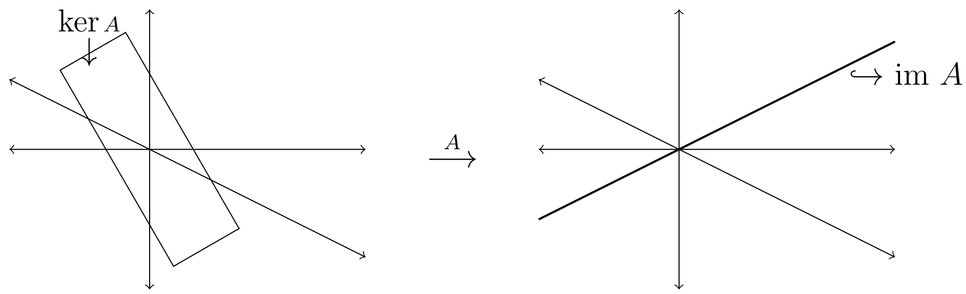
$$(**) \qquad \sum_{j=1}^{n} a_{ij} x_j = 0, \ i = 1, 2, \cdots, m,$$

$$a_{ij} \in k.$$

the associated homogeneous system,

$$A = [a_{ij}] : k^n \to k^m$$
$$V \to W$$

and the associated Linear transformation



$\dim V = \dim \ker(A) + \dim \operatorname{im}(A)$

Solution of $(*)$ exists $\Leftrightarrow \vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$ belongs to $\operatorname{im}(A)$.

If $\overrightarrow{v_o} \in V$ is one solution of $(*)$, then $\overrightarrow{v_0} + \ker(A) =$ are <u>all</u> the solutions of $(*)$ which is an affine translate of $\ker(A)$.

What is the "dynamical type" of

$$A : V \to W?$$

I.e. choose new co-ordinates in V and W so that A is readily understood?

Formulation in terms of group action:

$$\begin{aligned} X &= \mathcal{L}(V, W) \\ &= \{A : V \to W | A \text{ a linear tranformation}\}. \end{aligned}$$

$G = GL(V) \times GL(W)$ acting on X. $P \in GL(V),\ Q \in GL(W),\ A \in X$

$$(P, Q).A \stackrel{\text{def}}{=} Q.A.P^{-1}.$$

<u>Observe</u>: There are <u>finitely many</u> G-orbits in X, $\#\, G$-orbits $= 1 + \min(m, n)$. Each orbit is characterized by $r = \operatorname{rank}(A) = \dim \operatorname{im}(A)$.

This amounts to:
$\exists$ a basis $\underline{e} = \{\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_n\}$ of V and a basis $\underline{f} = \{\bar{f}_1, \bar{f}_2, \cdots, \bar{f}_m\}$ of W such that the matrix of A w.r.t $\underline{e}$, $\underline{f}$ is

$$\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$$

$r = \operatorname{rank} A$.

A much more interesting case: $V = W$,

$$X = \mathcal{L}(V) = \{A : V \to V | A \,\text{an operator}\}.$$

What are the "dynamical type" of A ?

Formulation in terms of group action :
$G = GL(V)$ acts on X by : $P \in G,\ A \in X$

$$P.A \stackrel{\text{def}}{=} P.A.P^{-1}.$$

This is the underlying subject which goes under the names "Jordan canonical form", "Rational canonical form", "Frobenius dimension formula" etc.

Note: Given $n \in \mathbb{W} = \{0, 1, 2, \cdots\}$ and $k =$ a field, $\exists!$(up to isomorphism) vector space of dimension V over k. So it is expected that the arithmetic of n, and of k should influence the "dynamical types" of $A : V \to V$.

3 Orbit and orbit-classes

$X =$ a set, $G =$ group acting on set X. For $a \in X$,

$$\begin{aligned} G(a) &= \{g \cdot a | g \in G\} \\ &= \text{ the } G\text{-orbit of a} \end{aligned}$$

$$\begin{aligned} G_a &= \{g \in G | g \cdot a = a\} \\ &= \text{ the stabizer subgroup of } a \end{aligned}$$

First Partition of X

$$X = \bigcup_{a \in X} G(a)$$

$\forall\, a, b \in X$, either $G(a) = G(b)$ or $G(a) \cap G(b) = \phi$.

Important Note: We can define the action of a semi-group in the same way. However then X may not be a disjoint union of G-orbits. We shall see a very intersting situation arising in Linear Algebra where both the group- and semigroup-actions are involved.

Note : $b \in G(a) \Leftrightarrow b = g \cdot a$ for some $g \in G$.

$\Longrightarrow G_b = gG_ag^{-1} =$ a conjugate of G_a.

$\{G\text{-orbits}\} \longrightarrow \{\text{conjugacy classes of subgroups of} G\}$.

We say $a, b \in X$ have same orbit-type if $G(a), G(b)$ are conjugate.

Notation : $a \sim_\theta b \longrightarrow a, b$ have same orbit type.

$$\begin{aligned} R(a) &\overset{\text{def}}{=} \{b \in X | a \sim_\theta b\} \\ &\bigcup_{G_a = G_b} G(b) \end{aligned}$$

Second Partition of X

$$X = \bigcup_{a \in X} R(a), \quad R(a) = \bigcup_{G_a = G_b} G(b)$$

An Important Case

$X = G$, G acts on itself by conjuagtion. Then for $a \in G$,

$$G(a) = \text{ the conjugacy class of } a$$

$$G_a = \text{ the centralizer of } a \text{ in } G$$

An Important Example :

("Dynamical types" in spherical geometry in dimension 2.)

Here $G = SO(3)$ acts on S^2 the unit sphere in $\mathbb{E}^3$.

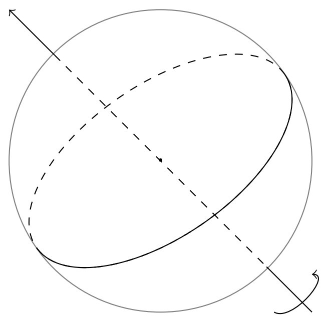
To understand the "dynamical types" we consider the orbit-types in G acting on itself by conjugation.

Conjugacy classes in $G = SO(3)$

(i) $g = e$, $G_e = G$, $C_G(e) = \{e\} \overset{notation}{=} C_0$

(ii) $g = \rho_{l,\theta}$

$= $ rotation around axis l through angle "θ", $(0 < \theta < 2\pi), \theta \neq \pi$.



Note : Let $\overrightarrow{v} \perp l$, $g \cdot \overrightarrow{v} = \overrightarrow{w}$, $(\overrightarrow{v} \neq \overrightarrow{w})$
$\pi =$ the 2-plane span $\{\overrightarrow{v}, \overrightarrow{w}\}$. Since "$\theta$" $\neq 0$ or π, $\overrightarrow{v}, \overrightarrow{w}$ are linearly independent.

We can orient π in such a way that $0 < \theta < \pi$.

Using a fixed orientation in $\mathbb{E}^3$, we can orient l also. $\rho_{l,\theta}$ leaves π invariant. Clearly the conjugacy class

$$C_G(\rho_{l,\theta}) = \{\rho_{m,\theta} | m \text{ oriented axis}\}$$
$$\overset{notation}{=} C_\theta \quad 0 < \theta < \pi$$

Now

$$\begin{aligned} G_{\rho_{l,\theta}} &= \text{the centralizer of } \rho_{l,\theta} \\ &= \{\rho_{l,\phi} | 0 \leq \phi \leq 2\pi\} \\ &\cong SO(2) \end{aligned}$$

(*iii*) $g = \rho_{l,\pi}$
$= \text{rotation around an axis } l \text{ through angle } \pi \text{ radians} = 180^o$

Let π = the 2-plane through the center $\perp l$. If we choose co-ordinates so that $l =$ the z-axis, then

$$\rho_{l,\pi} = \left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

We see that

$$\begin{aligned} G_{\rho_{l,\pi}} = &\{ \left[\begin{array}{cc|c} c & -s & 0 \\ s & c & 0 \\ \hline 0 & 0 & 1 \end{array}\right] | c = \cos\theta, s = \sin\theta\} \\ &\bigcup \{ \left[\begin{array}{cc|c} c & s & 0 \\ -s & c & 0 \\ \hline 0 & 0 & 1 \end{array}\right] | c = \cos\theta, s = \sin\theta\} \\ &\cong O(2) \end{aligned}$$

The conjugacy class

$$\begin{aligned} C_G(\rho_{l,\pi}) &= \{\rho_{m,\pi} | m \text{ un-oriented axis}\} \\ &\overset{notation}{=} C_\pi. \end{aligned}$$

First Partition of G (Conjugacy classes)

$$G = \bigcup_{0 \leq \theta \leq \pi} C_\theta$$

(Orbit classes)

(i) $R(e) = \{e\}$

(ii) For $0 < \theta < \pi,\ 0 < \phi < \pi$, $G_{\rho_{l,\theta}} = G_{\rho_{l,\phi}} (\cong SO(2))$, So

$$R(\rho_{l,\theta}) = \{\rho_{m,\phi} | m \text{ oriented axis},\ 0 < \phi < \pi\}$$

(iii) $G_{\rho_{l,\pi}}$ is conjugate to $G_{\rho_{m,\pi}}$.

$$R(\rho_{l,\pi}) = \{\rho_{m,\pi} | m \text{ un-oriented axis}\}.$$

Second Partition of G

$G = C_0 \cup R(\rho_{l,\theta_0}) \cup C_\pi$, for some fixed $\theta_0,\ 0 < \theta_0 < \pi$.

While there are infinitely many conjugacy classes in G, there are only 3 orbit classes.

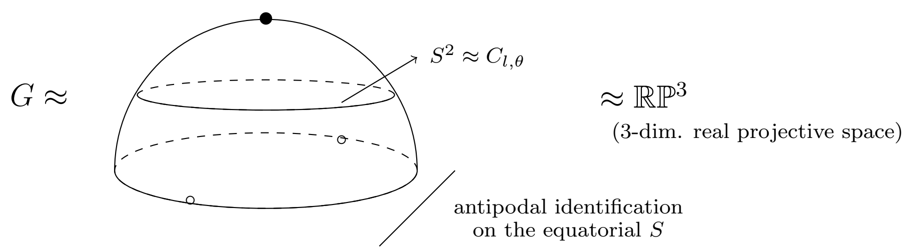
Remark : This observation is useful in understanding the topology of $SO(3)$ also. Indeed for $0 < \theta < \pi$

$$\begin{aligned} C_\theta &= \{\rho_{l,\theta} | l \text{ oriented axis}\} \\ &\cong S^2 \\ C_\pi &= \{\rho_{l,\pi} | l \text{ un-oriented axis}\} \\ &\cong \mathbb{RP}^2 (\text{two dimensional real projective space}) \\ C_0 &= \{\text{a point}\}. \end{aligned}$$

Also for fixed θ_0 such that $0 < \theta_0 < \pi$

$$\begin{aligned} R(\rho_{l,\theta_0}) &= \bigcup_{0<\theta<\pi} C_\theta \\ &= S^2 \times (0, \pi) \end{aligned}$$

So



An interesting point :

1. Each transformation $\rho_{l,\theta} \in G$ (except identity), is characterised by a spatial invariant l, and numerical invariant θ.

2. There are only finitely many orbit classes, which substantially makes precise the notion of “dynamical types” in this geometry and explains their finiteness.

We shall explain 1) and 2) by proving a general result on the structure of an orbit-class.

4 Structure of an orbit-class (α- and σ-fibrations)

Let G acts on X, $a \in X$

$$\begin{aligned} F_a &= \text{the set of fixed points of } G_a \\ &= \{b \in X | G_b \supseteq G_a\} \\ F_a' &= \{b \in X | G_b = G_a\} \\ &= \text{the "generic" elements in } F_a \\ N_a &= \text{the normalizer of } G_a \text{in } G \\ W_a &= N_a/G_a = \text{the } \underline{\text{"Weyl group"}} \text{ at } a. \end{aligned}$$

- A canonical free action of W_a on G/G_a.

  $n \in N_a$, $[n] =$ the class of n in W_a

  Definition 1. $[n] \cdot gG_a = gG_a n^{-1} = gn^{-1}G_a$

  (Note : the two "=" signs show that the action is well defined).

  Freeness :

$$\begin{aligned} &[n] \cdot gG_a = gG_a \\ &\Leftrightarrow gn^{-1}G_a = gG_a \\ &\Leftrightarrow n^{-1}G_a = G_a \\ &\Leftrightarrow n \in G_a \end{aligned}$$

- A canonical free action of W_a on $F_a' = \{b \in X | G_b = G_a\}$.

  Note : For $g \in G$,

$$\begin{aligned} F_{g\cdot a} &= \{b \in X | G_{g\cdot a} b = b\} \\ &\quad (G_{g\cdot a} b = gG_a g^{-1} b = b \Rightarrow G_a g^{-1} b = g^{-1} b) \\ &= \{b \in X | g^{-1} b \in F_a\} = g \cdot F_a. \end{aligned}$$

  So $\boxed{F_{g\cdot a} = g \cdot F_a}$

  Note :

- N_a leaves F_a invariant :

$$\begin{aligned} n \in N_a, nF_a &= F_{na} \\ &= \{b \in X | G_{n \cdot a} b = b\} \\ &(G_{n \cdot a} b = nG_a n^{-1} b = G_a) \\ &= \{b \in X | G_a b = b\} = F_a \end{aligned}$$

- N_a acts on F_a via W_a

$$n \in N_a, [n] \in W_a,\ [n]F_a = nF_a (\text{well defined}).$$

- Freeness on $F_a^{'}$: Let $b \in F_a^{'}$ and $[n]b = nb = b$. So $n \in G_b$, since $b \in F_a^{'}$. Hence $[n] = e$.

Thus W_a acts diagonally on $G/G_a \times F_a^{'}$.

Theorem 1. *The map*

$$\begin{aligned} \phi \colon G/G_a \times F_a^{'} &\to R(a) \\ (gG_a, b) &\to gb \end{aligned}$$

is well defined and induces a bijection

$$\bar{\phi} \colon \{G/G_a \times F_a^{'}\}/W_a \overset{\cong}{\to} R(a).$$

Proof. • ϕ is well defined

(i) $g \in G, b \in F_a^{'} \overset{?}{\Rightarrow} gb \in R(a)$

$$\begin{aligned} G_{g \cdot b} &\overset{?}{\sim} G_a (\sim = \text{conjugate}) \\ gG_b g^{-1} &= gG_a g^{-1}, \text{since } G_b = G_a. \end{aligned}$$

(ii) $g \in G, u \in G_a \overset{?}{\Rightarrow} g \cdot a = gua = ga$, since $u \in G_a$.

- ϕ is surjective

Let $b \in R(a)$. So $\exists g \in G$ such that

$$\begin{aligned} G_b &= gG_a g^{-1} = G_{g \cdot a} \\ \Rightarrow & b \in F_{g \cdot a}^{'} = gF_a^{'} \\ \Rightarrow & g^{-1} b \in F_a^{'} \end{aligned}$$

So $\phi(gG_a, g^{-1}b) = g \cdot g^{-1} \cdot b = b$.

- $\underline{\phi \text{ is constant on } W_a\text{- orbits}}$

  $[n] \in W_a$. Then
  $$\phi([n]gG_a, [n]b) = \phi(gn^{-1}G_a, nb) = gn^{-1}n \cdot b = gb = \phi(gG_a, b).$$
  So ϕ induces a surjective map
  $$\bar{\phi}\colon \{G/G_a \times F_a^{'}\}/W_a \twoheadrightarrow R(a).$$

- $\underline{\phi \text{ is injective}}$

  Suppose $\phi(gG_a, b) = \phi(hG_a, c)$. Let $u = h^{-1}g$. We have $ub = c$, so
  $$\begin{aligned} G_c = G_a = G_{ub} = uG_bu^{1-} &= uG_au^{-1}\ (G_a = G_b = G_c) \\ &\Rightarrow u \in N_a \\ &\Rightarrow [u] \in W_a \\ [u](g, g_a, b) = ([u]gG_a, [u]b) = (gu^{-1}G_a, ub) &= (hG_a, c) \end{aligned}$$
  So $(gG_a, b), (hG_a, c)$ are in the same W_a-orbit. q.e.d.

$\square$

<u>Two fibrations of $R(a)$</u>

The word "fibration" is used in a purely set-theoretic sense.
<u>Note</u> : $(G/G_a)/W_a = G/N_a$

$$\begin{array}{ccccc} G/G_a \times F_a^{'} & \xrightarrow{p} & \{G/G_a \times F_a^{'}\}/W_a & \xrightarrow{\bar{\phi}} & R(a) \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle p\bar{r}_1} & & \downarrow{\scriptstyle \bar{\sigma}} \\ G/G_a & \xrightarrow{p} & (G/G_a)/W_a & \xrightarrow{\cong} & G/N_a \end{array}$$

$\sigma =$ a "homogeneous space"-invariant of an orbit class.

$$\begin{array}{ccccc} G/G_a \times F_a^{'} & \xrightarrow{p} & \{G/G_a \times F_a^{'}\}/W_a & \xrightarrow{\bar{\phi}} & R(a) \\ \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle p\bar{r}_2} & & \downarrow{\scriptstyle \bar{\alpha}} \\ G/G_a & \xrightarrow{p} & (G/G_a)/W_a & \xrightarrow{=} & \text{"a fundamental domain in the the free } W_a\text{-action on } F_a^{'}\text{"} \end{array}$$

Important case: $G = X$, and G acts on itself by conjugation.
For $a \in X = G$, $G_a = Z_G(a) =$ centralizer of $a \in G$.
$F_a = \{b \in X = G | Z_G(b) \supseteq Z_G(a)\}$
$(Z_G(b) \supseteq Z_G(a) \Leftrightarrow b$ commutes with all elements in $Z_G(a) \Leftrightarrow b \in$ the center of $Z_G(a))$. So $\boxed{F_a = \text{the center of } Z_G(a)}$. Thus
$F_a^{'} =$ the "generic" elements in the center of centralizer of a. Realizing $F_a^{'}/W_a$ as the "fundamental domain" of the free W_a-action on $F_a^{'}$, we see that the map

$$\alpha\colon R(a) \to F_a^{'}/W_a$$

has values in abelian group.

The fibrations

$$\begin{array}{ccc} F_a^{'} \longrightarrow & R(a) & \\ & \Big\downarrow \sigma & \\ & G/N_a & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} G/G_a \longrightarrow & R(a) & \\ & \Big\downarrow \alpha & \\ & F_a^{'}/W_a & \end{array}$$

is a partial explaination of the "spatial" and "numerical" invariants of transformations.

Example : $G = SO(3)$ acting on itself by conjugation.

$$\rho_0 = \rho_{l,\theta_0} \ 0 < \theta_0 < \pi$$

choosing $l =$ the z-axis, we have

$$\rho_0 = \left[\begin{array}{cc|c} c_0 & -s_0 & 0 \\ s_0 & c_0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \quad c = \cos\theta, s = \sin\theta$$

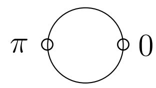
$$G_{\rho_0} = Z_G(\rho_0) = \{ \left[\begin{array}{cc|c} c & -s & 0 \\ s & c & 0 \\ \hline 0 & 0 & 1 \end{array}\right] | c = \cos\theta, s = \sin\theta, 0 \leq \theta \leq 2\pi \}.$$

$$\begin{aligned} N_{\rho_0} &= \text{the normalizer of } \ G_{\rho_0} \\ &= < G_{\rho_0}, z > \end{aligned}$$

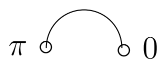
where $z = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$. So $N_{\rho_0}/G_{\rho_0} = W_{\rho_0} \cong \mathbb{Z}_2$.

Since G_{ρ_0} is abelian, we have

$$\begin{aligned} F_{\rho_0} &= G_{\rho_0} \cong SO(2) = \{e^{i\theta} | 0 \le \theta \le 2\pi\} \\ F'_{\rho_0} &= \text{the "generic" elements of } F_{\rho_0} \\ &= \{g \in G | G_g = G_{\rho_0}\} \\ &= \{e^{i\theta} | \theta \ne 0, \theta \ne \pi\} \\ &= \end{aligned}$$



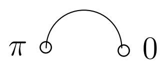
$F'_{\rho_0}/W_{\rho_0} =$

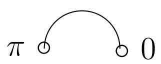


$$G/G_{\rho_0} \cong SO(3)/SO(2) \cong S^2.$$
$$\|$$
{"oriented axis"}

Thus we have the α-fibration

$$\begin{array}{ccc} S^2 & \longrightarrow & R(\rho_0) \\ & & \downarrow \alpha \end{array}$$



Since 

is a contractible space the α-fibration is trivial, and we have $R(\rho_0) \cong S^2 \times (0, \pi)$ as observed previously.

Remark : The analysis which we have carried out for $SO(3)$ can be carried out for $SO(n)$, $U(n)$, $sp(n)$, in fair detail, giving detailed information about the internal structure of these groups.

Back to linear algebra!

We apply the themes of orbits and orbit-classes, and ask for their parametrizations for $G = GL_n(k)$ acting on $X = M_n(k)$ by conjugation.

5 Dynamical notions in linear algebra

Let k be a field, $n \in \mathbb{W} = \{0, 1, 2, \cdots\}$. Let V = an n-dimensional vector space over k (unique up to isomorphism !) and $A\colon V \to V$ a linear transformation.

Note : Arithmetic of n and k must influence A!

A trivial, yet basic case : dim $V = 1$. Then

$$\mathcal{L}(V) \underset{\longleftrightarrow}{\text{1-1}} k$$
$$\{A\} \longmapsto \{\overrightarrow{v} \mapsto \alpha \overrightarrow{v} | \alpha \in k\}$$

Definition $W \leq V$ is A-invariant if $A(W) \subseteq W$. W is proper if $0 \neq W \neq V$.

So if $\dim_k V \leq 1$, then $\nexists$ a proper subspace.

Definition(Extreme case) A acts irreducibly on V, if $\nexists$ a proper A-invaraint subspace.

(So if $\dim_k V \leq 1$, then A acts irreducibly, for $\nexists$ any proper subspace.)

Definition V is decomposable w.r.t. A, if V is a direct sum of proper A-invariant subspaces.

indecomposable = not decomposable.

By pure thought : since dim $V < \infty$, V is a direct sum of A-indecomposable subspaces. Also, an A-indecomposable subspace, which is minimal w.r.t. inclusion, must be A-irreducible.

Definition A acts completely reducibly or A is dynamical semi-simple, if A is a direct sum of A-irreducible subspaces.

Natural Questions : Given $A\colon V \to V$, understand all A-invaraint subspaces.

- Understand all A-invariant decompositions of V.
- For each $n \in \mathbb{W}$, determine all $A's$ which act irreducibly, resp. indecomposibly, resp. completely reducibly. "Determine" means "enumerate" or "parametrize" in terms of "known" sets.

6 Minimal polynomials

<u>Minimal polynomial of $A\colon V \to V$</u>

Let $\mathbb{W} = \{0, 1, 2, \cdots\}$ be considered as a semi-group w.r.t. addition. The A-action naturally leads to $\mathbb{W}$-<u>action</u> on V. Namely $n \in \mathbb{W}$, $\overrightarrow{v} \in V$, then $n \cdot \overrightarrow{v} \overset{\text{def}}{=} A^n(\overrightarrow{v})$. If $\mathbb{W}$ is written multiplicatively, $\mathbb{W} = \{1, x, x^2, \cdots\}$, then the k-semigroup algebra

$$k[\mathbb{W}] \cong k[x] = \text{the algebra of polynomials in one variable.}$$

Also $\mathcal{L}(V)$ has the structure of k-algebra, and $A \in \mathcal{L}(V)$ generates a commutative k-subalgebra $= k[A] \leq \mathcal{L}(V)$. Now dim $\mathcal{L}(V) = n^2$, so dim $k[A] < \infty$.

$$\phi\colon k[x] \twoheadrightarrow k[A]$$

the canonical surjective homomorphism.

Ker $\phi \neq (1) = k[x]$ for $\phi(1) = 1$.

Ker $\phi \neq < 0 >$, for dim $k[x] = infty$, dim $k[A] < \infty$.
$k[x]$ is a PID. So Ker $\phi = < m_A(x) >$. $m_A(x)$ = the uniquely determine monic polynomial. It is called the minimal polynomial of A.

Let $W \leq V$ be A-invariant. Then We have $A_W = A|_W\colon W \to W$ and $\bar{A}_{V/W}\colon V/W \to V/W$. Clearly,

(i) $m_{A_W}(x) | m_A(x)$

(ii) $m_{\bar{A}_{V/W}}(x) | m_A(x)$

(iii) $m_A(x) | m_{A_W}(x) \cdot m_{\bar{A}_{V/W}}(x)$.

Also

<u>Proposition</u> : deg $m_A(x) \leq \dim_k(V)$

<u>Proof</u> : By induction on $\dim_k(V)$.

<u>case 1</u> A has a proper invariant subspace W. Then by (iii) and induction,

$$\begin{aligned} \deg m_A(x) &\leq \deg m_{A_W}(x) + \deg m_{\bar{A}_{V/W}}(x) \\ &\leq \dim W + \dim V/W = \dim V \end{aligned}$$

<u>case 2</u> $\nexists$ a proper A-invariant subspace.
($\dim_k(V) \leq 1$ - trivial case)

Let $\overrightarrow{v} \in V$, $\overrightarrow{v} \neq \overrightarrow{0}$. Let

$$\begin{aligned} W &= \text{span}\{A\text{-orbit of} \overrightarrow{v}\} \\ &= \text{span}\{\overrightarrow{v}, A\overrightarrow{v}, A^2\overrightarrow{v}, \cdots\} \end{aligned}$$

W is A-invariant. Let l be te least integer (≥ 1) such that $A^2\overrightarrow{v}$ can be expressed as alinear combination of $A^j\overrightarrow{v}$, $j < l$. Then it is easy to see that $\{\overrightarrow{v}, A\overrightarrow{v}, A^2\overrightarrow{v}, \cdots, A^{l-1}\overrightarrow{v}\}$ are linearly independent, and $A^m\overrightarrow{v}$, $m \geq l$, can be expressed as linear combination of $\overrightarrow{v}, A\overrightarrow{v}, A^2\overrightarrow{v}, \cdots, A^{l-1}\overrightarrow{v}$. So

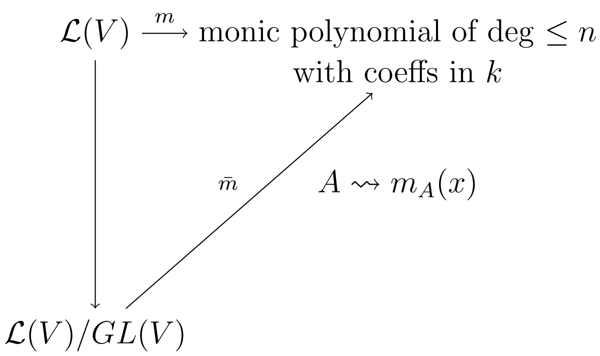
$$W = \text{span}\{\overrightarrow{v}, A\overrightarrow{v}, \cdots, A^{l-1}\overrightarrow{v}\}$$

$\dim W = l \leq n$.

Let $A^l\overrightarrow{v} + a_{l-1}A^{l-1}\overrightarrow{v} + \cdots + a_1A\overrightarrow{v} + a_0 = 0$ be a non-trivial linear combination. Consider $f(x) = \sum_{i=0}^{l} a_i x^i$. Then $f(A)\overrightarrow{v} = \overrightarrow{0}$.

Also $f(A)A\overrightarrow{v} = Af(A)\overrightarrow{v} = \overrightarrow{0} \cdots$. So $f(A)|_W = \overrightarrow{0} \Rightarrow f(A) = 0$. Hence $m_A(x)|f(x)$ has degree $\leq n$. q.e.d.

It is also clear that $m_A(x)$ depends only on the conjugacy class of A. Thus we have a map



A natural question :

{Monic polynomial of deg $\leq n$ with coeffs in k} is a "known" set. So given a monic polynomial $f(x) \in k[x]$ understand $\bar{m}^{-1}(f(x))$. In particular, Is $\bar{m}^{-1}(f(x)) \neq \phi$? Ans. Yes. Is $\bar{m}^{-1}(f(x))$ a finite set ? Ans. Yes. How many elements in $\bar{m}^{-1}(f(x))$? Can we enumerate the set ?

Remark : All these questions can be answered from the standard structure theorem based on appealing to the theorem on finitely generated modules over PIDs. This is the "Number" way of thinking. However they can also be answered from the "Symmetry-Dymanics" viewpoint. Understanding both viewpoints, and relating them, is a more pleasurable activity, than just working out the answers in one way. Incidentally, the answers to the questions posed above are not given in the standard texts - Bourbaki, Halmos, Herstein, Hoffman-Kunze, $\cdots$. Please let me know, if they are in the literature. (There are some errors even in Bourbaki.)

<u>Remark</u> :

- So far we have not appealed to the characteristic polynomial $\chi_A(x) = \det(xI - A)$. In fact from a dynamical viewpoint, linear algebra can be developed without appeal to $\chi_A(x)$, or to the Cayley-Hamilton theorem, and a part of it is applicable to the infinite-dimensional vector spaces. For example the existence of minimal polynomial does not need finite-dimensionality of the underlying vector space!
- The C-H theorem is true for an $n \times n$ matrix over any commutative ring. (One needs commutativity to make sense of the detreminant.) So it may be formulated for $A \in \mathrm{End}_R(M)$ where M is a finitely generated module over a commutative ring R. It is indeed a tautology when interpreted as a statement on $\wedge^n(M)$ ($n = \#$ { a system of generators}). The operator "$xI - A|_{x=A} = 0$"! When R = a commutative field, the "characteristic matrix" $xI - A$ carries full information about the conjugacy class of A. (The Smith normal form of $xI - A$ contains the invariant factors of A on the diagonal.)

<u>Remark</u> : The map

$$\bar{\chi}\colon \mathcal{L}(V) \to \{\text{monic poly of } \deg n \text{ in } k[x]\}$$

which associates to an operator its characteristic polynomial, is surjective. But the more subtle map

$$\bar{m}\colon \mathcal{L}(V) \to \{\text{monic poly of } \deg \leq n \text{ in } k[x]\}$$

which associates to an operator its minimal polynomial, is not surjective!

7 Basic lemma

Let $f(x) \in k[x]$, $f(x) = r(x)s(x)$, g.c.d$(r(x), s(x)) = 1$. $A \colon V \to V$ a linear transformation.

1. im $r(A)+$ im $s(A) = V$
2. ker $r(A) \cap$ ker $s(A) = 0$
3. ker $r(A) \subseteq$ im $s(A)$, ker $s(A) \subseteq$ im $r(A)$.
4. If $f(A) = 0$, then $V =$ ker $r(A) \oplus$ im $s(A)$.

   Moreover ker $r(A)$, ker $s(A)$ are invariant under $Z(A) = \{B \in \mathcal{L}(V) | BA = AB\}$.

5. Let $f(x) = m_A(x) = \prod_{i=1}^{r} P_i(x)^{d_i}$ decompose into monic irreducible factors, $i, j, i \neq j \Rightarrow P_i(x) \neq P_j(x)$. Then

   $$V = \oplus_{i=1}^{r} \ker P_i(A)^{d_i} \quad \text{direct sum}$$

   and $V_i = \ker P_i(A)^{d_i}$ are $Z(A)$-invariant. Moreover for any A-invariant subspace $W \leq V$

   $$W = \oplus_{i=1}^{r} W \cap V_i$$

   Corollary If V is A-indecomposable, then $m_A(x) = P(x)^d$, where $P(x)$ is a monic irreducible polynomial $\in k[x]$.

   Corollary If V is A-irreducible, then $\chi_A(x)$, is a monic irreducible polynomial $\in k[x]$.

   Proof Straight-forward.

8 The z-Classes of irreducible operators

K-structures and k-views

Let K = a field extension of k. $V = V_K$ a vector space over K. $A = A_K\colon V_K \to V_K$ a K-linear endomorphism. By restriction of scalars, we can consider $V = V_k$ as a k-vector space and $A = A_k$ as a k-linear endomorphism. We call A_k = a k-view of A_K.

Example $K = \mathbb{C} \supseteq k = \mathbb{R}$, $\dim_K V_K = 1$

$$\begin{aligned} A_K\colon z &\mapsto \alpha z \quad \alpha = a + ib, a, b \in \mathbb{R} \\ \| \quad & \quad \| \\ x + iy &\mapsto (a + ib) \cdot (x + iy) \\ & \qquad\qquad \| \\ & (ax - by) + i(bx + ay) \end{aligned}$$

So $A_k = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ is a k-view of A_K.

Conversely let V be a vector space over k. A K-structure on V is a family of maps

$$\sigma_K = \{\mu_\alpha | \alpha \in K,\ \mu_\alpha\colon V \to V \ k\text{-linear}\}$$

such that $\mu_{\alpha+\beta} = \mu_\alpha + \mu_\beta$, $\mu_\alpha \circ \mu_\alpha = \mu_{\alpha\cdot\beta}$, μ_1 = identity.

σ_K is said to be invariant under a k-linear operator A, if A commutes with μ_α for all $\alpha \in K$. Thus

$$\text{an } A\text{-invariant} K\text{-structure} \longleftrightarrow k\text{-algebra injection } K \to Z(A) \subseteq \mathcal{L}(V)$$

where $Z(A)$ = the centralizer of A in $\mathcal{L}(V)$.

$A\colon V \to V$, $m_A(x)$ = monic irreducible

Suppose $m_A(x) = p(x) \in k[x]$ monic irreducible. Then

$$k[A] \cong k[x]/(p(x)) = K, \text{ a field extension}$$

Clearly $k[A] \subseteq Z(A)$. So V has an A-invariant K-structure. Let $\alpha = [x]$ the class of $x \in k[x]$, modulo $p(x)$. So $K = k[\alpha] = k(\alpha)$ i.e K is a simple field extension (i.e generated by a single element α- a primitive element). Thus $A_K = \mu_\alpha\colon \vec{v} \mapsto \alpha\vec{v}$ scalar multiplication by $\alpha \in K$. In other words, $A = A_k$ is simply a k-view of μ_α! Here μ_α is a very simple operator to understand. The original $A = A_k$ is complicated k-view of μ_α!

$\{A\text{-invariant subspaces}\} \overset{\text{1-1}}{\longleftrightarrow}$ K-subspaces of $V_K = V$ considered as a vector space over K. In particular, A is irreducible $\Leftrightarrow$ $\dim_K V_K = 1$. Now let us consider the $G = GL(V)$-action on $X = \mathcal{L}(V)$. Let

$$X_{\text{irred}} = \{A \in X | A \text{ is irreducible}\}$$

We wish to understand the G-orbits and G-orbit classes in this G-action.

As observed above if A is irreducible then $K = k[A] = k[x]/(p(x))$, $p(x) = m_A(x)$ irred, and moreover we can identify (up to strong equivalence of actions)

$$(V, A) \longleftrightarrow (K = k[A] = k[x]/(p(x)), \mu_\alpha).$$

Thus to $A \in X_{\text{irred}}$, we can associate (σ_K, K, α) where

(i) σ_K is a K-structure on V.

(ii) $K =$ a field extension of k such that $(K : k) = \dim V$.

(iii) α is a primitive element of k.

Conversely given (σ_K, K, α) satisfying (i), (ii), (iii) we can simply take $A = \mu_\alpha$.

<u>G-orbits in X_{irred}</u> : Let $K =$ be a simple field extension of k with primitive element α and $(K : k) = \dim V$. Let σ_1, σ_2 be two K-structure on V. Let $A_i = \mu_\alpha$ in the structure $\sigma_i, i = 1, 2$. Then

$$(V, A_1) \approx (K = k[x]/(p(x)), \mu_\alpha) \approx (V, A_2).$$

In other words, any two K-structures on V are conjugate.

Let α_1, α_2 be two primitive elements of field extensions K_1, K_2. We say, the pairs (K_1, α_1), (K_2, α_2) are <u>equivalent</u> if $\exists$ an isomorphism $\phi : K_1 \to K_2$ such that $\phi(\alpha_1) = \alpha_2$. Clearly this happens iff α_1, α_2 are roots of the same monic irreducible polynomials $p(x) \in k[x]$, deg $p(x) = \dim V$. We denote by $[K, \alpha]$ the equivalence class of the pair (K, α). Clearly

the equivalence classes $[K, \alpha]$ of the pair (K, α) with $(K : k) = \dim V$
$\overset{\text{1-1}}{\longleftrightarrow} \{p(x) \in k[x] \mid p(x) \text{ monic irred}\}$.

Thus

$$\begin{aligned} X_{\text{irred}}/G \overset{\text{1-1}}{\longleftrightarrow} & \{[K,\alpha] \mid K = \text{be a simple field extension of } k \\ & \text{with primitive element}\, \alpha, (K:k) = \dim V\} \\ \overset{\text{1-1}}{\longleftrightarrow} & \{p(x) \in k[x] \mid p(x) \text{ monic irred}\}. \end{aligned}$$

<u>G-orbit classes in X_{irred}</u>

<u>Note</u> : $A, B \in X_{\text{irred}}$ are in the same orbit class iff G_A, G_B are conjugate. Here

$$\begin{aligned} G_A &= \{g \in G \mid gAg^{-1} = A\} = \{g \in G \mid gA = Ag\} = Z^*(A) \\ &= \text{the invertible elements in } Z(A) \end{aligned}$$

$$\| $$

$$\text{the centralizer of } A \text{ in } \mathcal{L}(V)$$

<u>Proposition</u> $Z(A) = k[A] \approx K$

<u>Proof</u> Clearly $Z(A) \supseteq k[A]$. To see the reverse inclusion let us work in the model

$$V = k[x]/(p(x)),\ A = \mu_\alpha$$

Let $B \in Z(A)$. Let $B[1] = [u(x)]$. Then

$$\begin{aligned} B[x] &= BA[1] = AB[1] = [xu(x)] \\ B^2[x] &= BA^2[1] = A^2B[1] = [x^2u(x)] \\ &\cdots \end{aligned}$$

Thus $B[1]$ determines B uniquely and $B \longleftrightarrow [u(x)]$ is a k-algebra isomorphism of $Z(A)$ with $k[x]/(p(x)) = K = k[A]$. q.e.d.

So $Z^*(A)$ is simply $= Z(A) - \{0\}$. Thus $G_A = Z^*(A)$ is conjugate to $G_B = Z^*(B)$ iff the corresponding fields K_A, K_B are isomorphic. Let $[K]$ denote the isomorphism class of a simple field extension K of k with $(K: k) =$ dim V. Thus we have proved

<u>Theorem</u> 1) $X_{\text{irred}} \longleftrightarrow \{(\sigma_K, K, \alpha)\}$.

2) $X_{\text{irred}}/G \longleftrightarrow \{[K,\alpha]\}$.

3) orbit classes $\longleftrightarrow \{[K]\}$.

(Here $(K:k) =$ dim V, $\alpha =$ a primitive element of K, $\sigma_K =$ a K-structure on V.)

Corollary : Suppose the arithmetic of k is such that there are only finitely many extensions of degree n of k. $\Rightarrow \exists$ only finitely many orbit classes in the G-action in X_{irred}.

Remark The argument given above extends from X_{irred} to $X = \mathcal{L}(V)$, and explains, and embellishes the theory of Jordan and rational canonical forms, and the Frobenius theory of centralizers of $n \times n$ matrices. See Frobenius [4], or Jacobson [8], for a traditional exposition.

Example 1) $k = \mathbb{R}$ - The only finite degree field extensions are $\mathbb{R}$ and $\mathbb{C}$. The argument above extends from $G = GL_n(\mathbb{R})$ to $G =$ a reductive Lie group. The transformation groups in classical geometries are often reductive Lie groups, or their affine extensions. So the above arguments explain the finiteness of "dynamical types" in classical geometries.

Ingredients of a "dynamical type"

i) orbit classes of G acting on itself by conjugtion

ii) whether given $g \in G$, $< \bar{g} >$ is compact or connected.

2) $k = p$-adic fields : These enjoy the property that given $n \in N$, $\exists$ only finitely many field extensions of degree n. So again "fiiteness of dynamical types" continues to field.

3) $k = \mathbb{Q}$ $G = GL_n(\mathbb{Q}) \subseteq M_n(\mathbb{Q})$. "Allmost all" $A \in M_n(\mathbb{Q})$ are irreducible! So their dynamics is essentially related to the field extensions of $\mathbb{Q}$!

9 Invariant subspaces

Example

$$\begin{aligned} V &= k[x]/(p(x)^d),\ p(x) \text{ monic irred},\ d \in \mathbb{N} \\ A &= \mu_x \\ B &= p(A) \\ V_i &= \text{ Ker } B^i \\ 0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_d &= V \end{aligned}$$

Assertion $\{V_i\}_{i=0}^d$ are precisely the A-invariant subspaces. If $d \geq 2$ then (V, A) is indecomposable, but not irreducible.

More generally

$$\begin{aligned} V = k[x]/(f(x)),\ f(x) \text{ monic irred},\ A = \mu_x \\ \{A\text{-invariant subspaces}\} \longleftrightarrow \{\text{monic factors of } \ f(x)\} \\ W \longrightarrow m_{A_W}(x) \\ \ker g(A) \longleftarrow g(x) \end{aligned}$$

Proposition i) (V, A) indecomposable $\Longrightarrow m_A(x) = p(x)^d$, $p(x)$ monic.

(ii) $m_A(x) = p(x)^d$, deg $m_A(x) =$ dim $V \Longrightarrow (V, A) \sim (k[x]/(p(x)^d), \mu_x)$, hence indecomposable.

Example Let $l, d \in \mathbb{N}, l \geq d$.

$$\begin{aligned} \pi\colon l = d_1 + d_2 + \cdots + d_r \ \text{ a partition of } l \\ d_1 \leq d_2 \leq \cdots d_r = d \end{aligned}$$

i.e π is a partition of l with largest summand d. Let $p(x) \in k[x]$ monic irred, deg $p(x) = m$. Let

$$\begin{aligned} V_\pi &= \oplus_{i=0}^r k[x]/(p(x)^{d_i}) \\ A_\pi &= \mu_x \text{ (on each summand)} \\ &\Longrightarrow \dim V = m \cdot l,\ m_{A_\pi}(x) = p(x)^d. \end{aligned}$$

If $l > d$, then V_π is decomposable.

Basic Theorem $A\colon V \to V, m_A(x) = p(x)^d$, $p(x)$ monic irreducible. deg $p(x) = m$, dim $V = n$

$\Rightarrow$ 1) $m|n$, and $n \geq ml$, $l \geq d$.

2) $\exists!$ partition π of l with largest summand d such that $(V, A) \sim (V_\pi, A_\pi)$.

Remarks The theorem follows from the usual approach of applying the general theorem on finitely generated modules over PIDs. The "dynamic" approach we have followed is "dual" to the usual approach.

Idea $B = p(A)$, $V_i =$ ker B^i

(∗) $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_d = V$.

$Z(A)$-invariant flag

- $\bar{A}_{V_i/V_{i-1}} V_i/V_{i-1} \to V_i/V_{i-1}$, $i \geq$ has $m_{\bar{A}}(x) = p(x)$.

  So V_i/V_{i-1} has K-structure, where $K = k[x]/(p(x))$.

- $(*)$ is refined to $Z(A)$-invariant flag, from which one reads π and one sets up the equivalence $(V, A) \sim (V_\pi, A_\pi)$.

The space of dynamically semisimple (completely reducible) operators

$$X_s = \{A \in \mathcal{L}(V) \mid A \text{ is dynamically semisimple}\}$$
$$G = GL(V) \text{ acts on } X_s$$

Theorem dim $V = n$

1. $X_s \overset{1\text{-}1}{\longleftrightarrow} \{(\mathcal{D}_\pi, K_i, \alpha_i, \sigma_{K_i})\}_{i=1}^r$.
2. G-orbits in $X_s \overset{1\text{-}1}{\longleftrightarrow} \{(\pi, [K_i, \alpha_i])\}_{i=1}^r$.
3. G-orbit classes $\overset{1\text{-}1}{\longleftrightarrow} \{(\pi, [K_i]\}_{i=1}^r$.

Explanation $\pi\colon n = \sum_{i=1}^{r} n_i$ a partition.

- $\mathcal{D}_\pi\colon V = \oplus_{i=1}^r V_i$, dim $V_i = n_i$.
- $K_i =$ a simple field extension of k with primitive element α_i,
- $\sigma_{K_i} =$ a K_i-structure on V_i.
- It is assumed in 1) and 2) that $[K_i, \alpha_i] \neq [K_j, \alpha_j], \ \forall\ i \neq j$.
- In 3), it is assumed that # primitive elements in $K_i \geq$ multiplicity of K_i. (This condition is automatically fulfilled if k is infinite.)

10 The z-Classes of all operators

The determination of the socalled similarity classes of operators on a finite-dimensional vector space is a major result of the classical Linear Algebra, which goes under the name "Jordan Canonical Form", in case the underlying field is algebraically closed, or "Rational Canonical Form" in the general case. This amounts to the determination of orbits of the $GL(V)$-action on $\mathcal{L}(V)$, or recognising $\mathcal{L}(V)/GL(V)$ as a set. The major result, given in algebraic terms, is: "the invariant factors", or "elementary divisors", of an operator determine its similarity class.

However the additional structure on $\mathcal{L}(V)/GL(V)$ comes from the orbit-classes. There is a further twist. Since $GL(V)$ is just the set of invertible elements of $\mathcal{L}(V)$, which is invariant under the $GL(V)$-action by conjugation, we can restrict the action to this subset. In this case, the orbit-classes are just the z-classes in $GL(V)$. For brevity, we shall call the orbit-class of the $GL(V)$-action on $\mathcal{L}(V)$ again as a z-class in $\mathcal{L}(V)$.

The Frobenius's theory of centralisers of operators, in particular the "Frobenius dimension formula", (which gives the dimension of the $Z(A)$ as a k-algebra), or the "Frobenius double-centraliser theorem", are steps in understanding these z-classes.

The determination, actually a "parametrization", of these z-classes is the subject of my paper "Dynamics of Liner and Affine Maps", which appeared in the Asian Journal of Mathematics.

Instead of stating the general result, let me just state an important ingredient. Let $Z^*(A) = Z(A) \cap GL(V)$. This is a group. Now $Z(A)$ contains $k[A]$. Set $k^*[A] = k[A] \cap GL(V)$. A significant case is when the minimal polynomial of A is power of an irreducible polynomial. Assume that this is the case. The algebras $Z(A), k[A]$ as well as the groups $Z^*(A), k^*[A]$, act on V. In the usual approach we only consider $k[A]$. Note that the orbits of a semi-group are not necessarily disjoint. If A is decomposable, (and k is infinite) then $k[A]$ and $k^*[A]$ have infinitely many orbits.

But $Z(A)$ and $Z^*(A)$ have only finitely many orbits on V. Moreover the $Z(A)$-orbits form a canonical maximal flag of subspaces of V, and the $Z^*(A)$-orbits consist of the complements of a smaller subspace in the larger subspace, in the two successive terms in this flag. Using this flag we strengthen the classical theory in a number of ways.

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