Quaternions and Rotations in $\mathbb{R}^3$

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Abstract. In this expository note, we show that (a) rotations in 3-dimensional space are completely described by an axis of rotation and an angle of rotation about that axis. It is then natural to enquire about the axis and angle of rotation of the composite of two rotations whose axes and angles of rotation are given. We then (b) use the algebra of quaternions to answer this question.

1. Rotations

We recall that Euclidean space $\mathbb{R}^n$ comes equipped with the natural *dot product* or *inner product* defined by

$$\langle v, w\rangle = \sum_{i=1}^{n} v_i w_i$$

for two vectors $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$. We will denote the ith basis vector $(0, \ldots, 1 \ldots, 0)$ with 1 in the ith spot and 0 elsewhere by e_i. The collection $\{e_i\}_{i=1}^n$ is called the *standard basis* of $\mathbb{R}^n$. If $v = \sum_{i=1}^n v_i e_i$ is a vector in $\mathbb{R}^n$, its *length* or *norm* is defined as $\|v\| = \langle v, v\rangle^{1/2} = \left(\sum_{i=1}^n v_i^2\right)^{1/2}$. A set of vectors $\{v_1, \ldots, v_r\}$ is called an *orthonormal set* if $\langle v_i, v_j\rangle = \delta_{ij}$ for all $1 \leq i, j \leq r$ (Here δ_{ij} is the *Kronecker delta* symbol defined by $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$).

If an orthonormal set is a basis, it is called an *orthonormal basis*. For example, the standard basis $\{e_i\}_{i=1}^n$ of $\mathbb{R}^n$ is an orthonormal basis.

Definition 1.1 (Orthogonal transformations and rotations in $\mathbb{R}^n$). *A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is called an orthogonal transformation if it preserves all inner products, i.e. if $\langle Tv, Tw\rangle = \langle v, w\rangle$ for all $v, w \in \mathbb{R}^n$. The set of all orthogonal transformations of $\mathbb{R}^n$ is denoted as $O(n)$. An orthogonal transformation T is called a rotation if its determinant* $\det T > 0$. *The set of all rotations of $\mathbb{R}^n$ is denoted as $SO(n)$. Thus $SO(n) \subset O(n)$.*

We note that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ determines an $n \times n$ matrix with respect to the standard basis as follows. We expand the image of the jth standard basis vector e_j under T in terms of the standard basis:

$$Te_j = \sum_{i=1}^{n} T_{ij} e_i \tag{1}$$

Thus the jth *column* of the matrix $[T_{ij}]$ is the image Te_j of the jth basis vector e_j. Clearly, with this same prescription we are free to define the matrix of a linear transformation with respect to *any fixed chosen basis* of $\mathbb{R}^n$, though in the present discussion we will mostly use the standard basis. If $v = \sum_{j=1}^n v_j e_j$, then for the linear transformation T we have from the relation (1) above:

$$Tv = T\left(\sum_j v_j e_j\right) = \sum_j v_j Te_j = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} T_{ij} v_j\right) e_i.$$

In other words, if we use the common representation of the vector v as a column vector (with column entries v_i), also denoted by v, then the column vector representing Tv is given by the matrix product $T.v$.

We next examine the condition on T_{ij} for T to be an orthogonal transformation, and for it to be a rotation.

Proposition 1.2 (Matrix characterisation of orthogonal transformations and rotations). *The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal transformation if and only if $T^tT = TT^t = I$, where T^t denotes the transpose of T defined by $T^t_{ij} = T_{ji}$. This is summarised by saying that the column vectors (resp. row vectors) of the matrix of T form an orthonormal set. For an orthogonal transformation T,* $\det T = \pm 1$. *Finally, an orthogonal transformation T is a rotation if* $\det T = 1$.

Proof. If T is orthogonal, we have $\delta_{ij} = \langle e_i, e_j\rangle = \langle Te_i, Te_j\rangle$ for all i and j. Plugging in $Te_i = \sum_k T_{ki} e_k$ and $Te_j = \sum_m T_{mj} e_m$, we find $\delta_{ij} = \langle Te_i, Te_j\rangle = \sum_{k,m} T_{ki} T_{mj} \langle e_k, e_m\rangle = \sum_{k,m} T_{ki} T_{mj} \delta_{km} = \sum_k T^t_{ik} T_{kj} = (T^tT)_{ij}$. Thus T is orthogonal implies $T^tT = I$. This shows that $T^t = T^{-1}$, and hence also that $TT^t = I$.

Conversely, if $T^tT = I$, it follows by reversing the steps above that $\langle Te_i, Te_j\rangle = \langle e_i, e_j\rangle$ for all i and j. This implies that $\langle Tv, Tw\rangle = \sum_{i,j} v_i w_j \langle Te_i, Te_j\rangle = \sum_{i,j} v_i w_j \langle e_i, e_j\rangle = \langle v, w\rangle$ for all $v, w \in \mathbb{R}^n$. This proves the first assertion. Writing out the relation $T^tT = I$ (resp. $TT^t = I$) shows that the column (resp. row) vectors of T form an orthonormal set.

Since T orthogonal implies $T^tT = I$, we have $\det(T^tT) = \det(T^t)\det T = (\det T)^2 = \det I = 1$. Hence $\det T = \pm 1$. The second and third assertions follow. □

The above matrix description of orthogonal transformations and rotations leads to the following important property of the sets $O(n)$ and $SO(n)$.

Proposition 1.3 (The group structure on $O(n)$ and $SO(n)$). *The sets $G = O(n)$ or $SO(n)$ are groups. That is, they have a product or group operation defined by composition: $T.S := T \circ S$, where $(T \circ S)v = T(Sv)$ for $v \in \mathbb{R}^n$. Furthermore, this operation satisfies the following axioms of a group:*

(i) *(Associativity) $T.(S.R) = (T.S).R$ for all $T, S, R \in G$.*
(ii) *(Existence of identity) There is an identity transformation $I \in G$ such that $T.I = I.T = T$ for all $T \in G$.*
(iii) *(Existence of inverse) For each $T \in G$, there exists an element $T^{-1} \in G$ such that $T.T^{-1} = T^{-1}.T = I$.*

Indeed, since $SO(n) \subset O(n)$ inherits the group operation from $O(n)$, we call $SO(n)$ a subgroup of $O(n)$. The group $O(n)$ is called the **orthogonal group of** $\mathbb{R}^n$ *and $SO(n)$ the* **special orthogonal group** *of* $\mathbb{R}^n$.

Proof. We first remark that the matrix of the composite transformation $T.S$, by definition, is given by:

$$(T.S)e_j = \sum_i (T.S)_{ij} e_i.$$

However the left hand side, by definition and (1), is

$$\begin{aligned} T(Se_j) &= T\left(\sum_k S_{kj} e_k\right) = \sum_k S_{kj}(Te_k) \\ &= \sum_k S_{kj}\left(\sum_i T_{ik} e_i\right) = \sum_i \left(\sum_k T_{ik} S_{kj}\right) e_i \end{aligned}$$

which implies that $(T.S)_{ij} = \sum_k T_{ik} S_{kj}$. That is, if we compose orthogonal transformations, the matrix of the composite transformation is the product of their respective matrices.

Let $T, S \in O(n)$. Since $\langle (T.S)v, (T.S)w\rangle = \langle T(Sv), T(Sw)\rangle = \langle Sv, Sw\rangle = \langle v, w\rangle$ from the fact that T and $S \in O(n)$, it follows that $T.S \in O(n)$. If further $S, T \in SO(n)$, then $\det T = \det S = 1$. But then $\det(T.S) = (\det T)(\det S) = 1$, showing that $S.T \in SO(n)$ as well. This proves the existence of a *binary operation* on $O(n)$ and $SO(n)$.

The assertion (i) follows because multiplication of matrices (or for that matter, composition of maps) is easily verified to be associative. The statement (ii) is obvious, since the identity transformation $I \in SO(n)$ and in $O(n)$ as well. The assertion (iii) follows from Proposition 1.2, where we saw that $T^t = T^{-1}$, and also that $T \in O(n)$ implies $T^t \in O(n)$, because the conditions $TT^t = I$ and $T^tT = I$ are equivalent. Further $T \in SO(n)$ implies $\det T^t = \det T = 1$, so that $T^{-1} = T^t \in SO(n)$ as well. □

2. Rotations in $\mathbb{R}^2$ and $\mathbb{R}^3$

The following propositions give a quantitative characterisation of all the matrices that define rotations in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Proposition 2.1 (Planar Rotations). *If $T \in SO(2)$, then the matrix of T is given by:*

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

which, geometrically, describes a counterclockwise rotation in $\mathbb{R}^2$ by the angle $\theta \in [0, 2\pi)$. If we denote this rotation by R_θ, then it is easily checked that $R_\theta.R_\phi = R_{\theta+\phi} = R_{\phi+\theta}$. (Here $\theta + \phi$ means we go modulo 2π, viz. integer multiples of 2π have to be subtracted to get the value of $\theta + \phi$ to lie in $[0, 2\pi)$). In particular, the group $SO(2)$ is abelian, viz $T.S = S.T$ for all $T, S \in SO(2)$. Geometrically, we can think of the group $SO(2)$ as the group S^1, defined as the circle consisting of complex numbers in $\mathbb{C}$ of modulus 1 (and group operation being multiplication of complex numbers).

Proof. We know that $T \in SO(2)$ implies firstly that $T \in O(2)$, so that by the Proposition 1.2 above, the matrix:

$$T = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

satisfies the condition $T^tT = I$, which implies that $a_{11}^2 + a_{21}^2 = 1 = a_{12}^2 + a_{22}^2$ and $a_{11}a_{12} + a_{21}a_{22} = 0$. The first relation implies, since a_{ij} are all real numbers, that $a_{11} = \cos\theta$

and $a_{21} = \sin\theta$ for some $\theta \in [0, 2\pi)$. The conditions $a_{11}a_{12} + a_{21}a_{22} = 0$ and $a_{12}^2 + a_{22}^2 = 1$ now imply that $(a_{12}, a_{22}) = \pm(-\sin\theta, \cos\theta)$. The sign $+$ is now determined by the condition that $\det T = a_{11}a_{22} - a_{12}a_{21} = 1$ since $T \in SO(2)$. This shows that T is of the form stated. The second assertion is an easy consequence of trigonometric formulas, and is left as an exercise to the reader.

For the last remark, note that if we write a vector $v = xe_1 + ye_2 \in \mathbb{R}^2$ as the complex number $z = x+iy$ (where $i = \sqrt{-1}$), then $e^{i\theta}z = (\cos\theta + i\sin\theta)(x + iy) = (x\cos\theta - y\sin\theta) + i(x\sin\theta + y\cos\theta)$. Thus the components of $e^{i\theta}z$ are exactly the same as those of the rotated vector $R_\theta v$. $\square$

Exercise 2.2. Describe the matrix of a general element $T \in O(2)$.

Now we can understand rotations in $\mathbb{R}^3$. The next proposition says that a rotation in $\mathbb{R}^3$ is only slightly more complicated than a planar rotation. Indeed, once we pin down the "axis of rotation", it is just a planar rotation in the plane perpendicular to this axis.

Proposition 2.3 (Rotations in 3-space). *Let $T \in SO(3)$. Then there is a vector $v \in \mathbb{R}^3$ such that the line $\mathbb{R}v \subset \mathbb{R}^3$ is fixed by T. This line is called the axis of rotation of T. If we restrict T to the plane H which is perpendicular to v (denoted by $H = (\mathbb{R}v)^\perp$), then this restriction $T_{|H}$ is just a planar rotation as described in the last proposition.*

In fact, if we choose a new basis of $\mathbb{R}^3$ defined by $f_3 = \frac{v}{\|v\|}$, and f_1, f_2 an orthonormal basis of H such that the vector cross product $f_1 \times f_2 = f_3$ (viz. $\{f_1, f_2, f_3\}$ gives a *right-handed* orthonormal basis of $\mathbb{R}^3$) then the matrix of T with respect to the new basis $\{f_i\}_{i=1}^3$ is given by:

$$T = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Proof. We need to use the *characteristic polynomial* of T defined by:

$$P_T(X) := \det(XI - T).$$

It is easily seen that:

$$P_T(X) = X^3 + aX^2 + bX - 1$$

where $a, b \in \mathbb{R}$, and the constant term is $-\det T$, which is -1 because $T \in SO(3)$. A root to this characteristic polynomial P_T is called an *eigenvalue of T*. We first make the

Claim 1. 1 is an eigenvalue of T.

If β is any real root of P_T, then $P_T(\beta) = \det(\beta I - T) = 0$. This implies that $(\beta I - T)$ is a *singular* linear transformation of $\mathbb{R}^3$, and hence must kill some non-zero vector $v \in \mathbb{R}^3$. This implies: $(\beta I - T)v = 0$, i.e. $\beta v = Tv$. That is, $v \in \mathbb{R}^3$ is an *eigenvector* of T corresponding to the real *eigenvalue* β. Furthermore $T \in O(n)$ implies $\beta^2 \langle v, v\rangle = \langle \beta v, \beta v\rangle = \langle Tv, Tv\rangle = \langle v, v\rangle$, and since $v \neq 0$, $\langle v, v\rangle > 0$, so that $\beta^2 = 1$. Hence *a real root β of P_T is forced to be* ± 1. Also, the real cubic polynomial P_T has either (i) two non-real complex conjugate roots $\alpha, \bar{\alpha}$ and a real root β, or (ii) all three real roots.

The negative of the constant term which equals 1 is the product of the roots. So in the first case (i) we get $1 = |\alpha|^2 \beta$. This implies that $\beta > 0$, and hence $\beta = 1$. In the second case (ii), all the three real roots are ± 1, and all cannot be (-1) since their product is 1. Hence in both cases $\beta = 1$ is a root of P_T (i.e. an eigenvalue). This proves Claim 1.

Thus the eigenvector v corresponding to the eigenvalue 1 satisfies $Tv = v$, i.e. it is *fixed* by T. Since T is linear, all scalar multiples of v, viz. all of the line $\mathbb{R}v$ in the direction v, is fixed by T. This is the required "axis of rotation" of T.

Now, set $H := (\mathbb{R}v)^\perp := \{w \in \mathbb{R}^3 : \langle w, v\rangle = 0\}$, the *orthogonal plane* to v. The relation $Tv = v$ and the fact that T is an orthogonal transformation implies $\langle Tw, v\rangle = \langle Tw, Tv\rangle = \langle w, v\rangle = 0$ for each $w \in H$. This shows that T sends vectors in H to other vectors in H. The restriction $T_{|H}$ is then an orthogonal transformation of H (it still preserves inner products). It is clear that $\det T = \det T_{|H}.1$, and hence $T_{|H}$ also has determinant 1. Now the matrix form of T follows from Proposition 2.1. $\square$

Main question of this note

Suppose we are given two rotations $A, B \in SO(3)$ in terms of their axes and angles of rotation in accordance with the Proposition 2.3 above. Can we figure out the axis of rotation and angle or rotation of the composite $A.B$ in terms of the given data?

The following sections will be devoted to answering this question. First we need some preliminaries on the algebra of quaternions.

3. The Algebra of Quaternions

Definition 3.1 (Quaternion multiplication). *We rename the standard basis of $\mathbb{R}^4$ as $1, i, j, k$. The algebra of quaternions is defined as the 4-dimensional vector space:*

$$\mathbb{H} := \{x_0.1 + x_1 i + x_2 j + x_3 k : x_i \in \mathbb{R}\}$$

with multiplication defined by linearity and the relations $i.1 = 1.i = i$; $j.1 = 1.j = j$; $k.1 = 1.k = k$; $1.1 = 1$; $ij = -ji = k$; $ki = -ik = j$; $jk = -kj = i$; $i^2 = j^2 = k^2 = -1$. From this one easily deduces that:

$$\begin{aligned}&(x_0.1 + x_1 i + x_2 j + x_3 k)(y_0.1 + y_1 i + y_2 j + y_3 k)\\ &\quad = z_0.1 + z_1.i + z_2 j + z_3.k\end{aligned}$$

where: $z_0 = x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3$, $z_1 = x_0y_1 + x_1y_0 + x_2y_3 - x_3y_2$, $z_2 = x_0y_2 + x_2y_0 + x_3y_1 - x_1y_3$ and $z_3 = x_0y_3 + x_3y_0 + x_1y_2 - x_2y_1$.

For a quaternion $x = x_0.1 + x_1 i + x_2 j + x_3 k$ we define its conjugate as $\bar{x} = x_0 - x_1 i - x_2 j - x_3 k$.

Notation 3.2. To simplify notation, we will now write the quaternion x above as $x_0 + x_1 i + x_2 j + x_3 k$, suppressing the basis vector 1 from the first term. Because of the way multiplication is defined, this causes no confusion. Also x_0 is called the *real part of* x, denoted Re x and $x_1 i + x_2 j + x_3 k$ is called the *imaginary part of* x, denoted Im x.

Proposition 3.3 (Properties of quaternion multiplication). *The multiplication of quaternions has the following properties:*

(i) *$x\bar{x} = \bar{x}x = \|x\|^2 = \sum_{i=0}^{3} x_i^2$ for all $x \in \mathbb{H}$. Furthermore, $\|xy\| = \|x\|\,\|y\|$ for all $x, y \in \mathbb{H}$.*

(ii) *If $x \neq 0$ is a quaternion, then $xx^{-1} = x^{-1}x = 1$ for $x^{-1} := \frac{\bar{x}}{\|x\|^2}$. In particular, if x is of unit length (i.e. a unit quaternion), then x^{-1} is of unit length as well.*

(iii) *The set $\mathbb{H}$ forms a non-commutative, associative algebra over $\mathbb{R}$. The fact that non-zero elements have inverses makes it what is called a* **division algebra**. *In particular, by (i) and (ii) above, the set* $\mathrm{Spin}(3) := \{v \in \mathbb{H} : \|v\| = 1\}$ *forms a non-commutative group called the* **group of unit quaternions**.

(iv) *If $x \in \mathbb{H}$ is of unit length, then left and right multiplications by x (denote them by L_x and R_x respectively) are elements of $SO(4)$.*

Proof. The statements (i), (ii) and (iii) are straighforward verifications following from the definitions in 3.1.

To see (iv), one just brutally writes down the 4×4 matrices corresponding to L_x and R_x. To write the matrix of L_x, we need to apply L_x to the basis vectors $e_1 = 1, e_2 = i$, $e_3 = j, e_4 = k$ of $\mathbb{H} = \mathbb{R}^4$ and write them down as columns as noted immediately after Definition 1.1. That is, we left multiply the basis vectors $1, i, j$ and k by x, and obtain the four columns of L_x. Thus the matrix representation of L_x is:

$$L_x = \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{pmatrix}.$$

Since $\|x\|^2 = \sum_{i=0}^{3} x_i^2 = 1$, we see that each column has norm 1. All the columns are mutually orthogonal by inspection. Hence by Proposition 1.2, L_x is an element of $O(4)$. Also, one can compute the determinant of this matrix brutally, and using the fact that $\|x\|^2 = 1$, check that $\det L_x = 1$. Hence $L_x \in SO(4)$. The matrix computations for R_x are similar. The details are left to the energetic reader. □

4. The Angle and Axis of Rotation of the Composite of Two Rotations

Notation 4.1. In this section it will be useful to denote vectors in $\mathbb{R}^4$, viz. quaternions, by small letters (such as v) as we have been doing earlier, and vectors in $\mathbb{R}^3$ by boldface letters (such as $\mathbf{v}$). We also note that we are applying the "corkscrew or right-hand rule" for rotations, viz.: The unit vector $\mathbf{v}$ points in the direction of a screw that is being rotated by the given angle θ. Since a counterclockwise rotation by θ is the same as a clockwise rotation by $2\pi - \theta$, we may change the vector $\mathbf{v}$ to $-\mathbf{v}$, and thus assume without loss of generality, that $\theta \in [0, \pi]$.

Definition 4.2 (Spin Representation). *The group of unit quaternions $G = \mathrm{Spin}(3)$ (see (iii) of Proposition 3.3 above for the definition) acts on $\mathbb{R}^3$ as follows. Let $v = v_0.1 + v_1 i + v_2 j + v_3 k$ be a quaternion of unit length in G. Let $y = y_0.1 + y_1 i + y_2 j + y_3 k \in \mathbb{H} = \mathbb{R}^4$ be any quaternion. Consider the action of G on $\mathbb{H}$ by quaternionic conjugation or adjoint action:*

$$\begin{aligned}&G \times \mathbb{H} \to \mathbb{H}\\ &(v, y) \mapsto (\mathrm{Ad}\, v)(y) := vyv^{-1} = vy\bar{v} = (L_v.R_{\bar{v}})y\end{aligned}$$

where $\bar{v} = v_0.1 - v_1 i - v_2 j - v_3 k = v^{-1}$ is the conjugate of v. Note that this action pointwise fixes the scalars $\mathbb{R}.1 \subset \mathbb{H}$. Since L_v and $R_{\bar{v}}$ are in $SO(4)$ by (iv) of Proposition 3.3, it follows that Ad $v \in SO(4)$. *Since it fixes $\mathbb{R}.1$,* Ad v *sends the orthogonal space consisting of imaginary quaternions $\mathbb{R}^3 := \mathbb{R}i + \mathbb{R}j + \mathbb{R}k \subset \mathbb{H}$ to imaginary quaternions. Thus G acts on $\mathbb{R}^3$ via $SO(3)$ elements, and we get the famous spin representation of $G =$* Spin(3) *on $SO(3)$, viz.*

$$\rho\colon G \to SO(3)$$
$$v \mapsto \rho(v)$$

where $\rho(v).(\mathbf{y}) = \mathrm{Im}\left[(\mathrm{Ad}\ v)\mathbf{y}\right] = (\mathrm{Ad}\ v)\mathbf{y}$, where $\mathbf{y} = y_1 i + y_2 j + y_3 k \in \mathbb{R}^3$ is regarded as a pure imaginary quaternion, and "Im" denotes imaginary part. This map ρ is easily checked to be a homomorphism (viz. it satisfies $\rho(vw) = \rho(v)\rho(w)$ for all v, w) since Ad $vw = (\mathrm{Ad}\ v)(\mathrm{Ad}\ w)$. *The kernel of ρ is $\rho^{-1}(Id) = \mathbb{Z}_2 = \{+1, -1\}$.*

We now come to a key lemma needed for proving the main proposition of this note.

Lemma 4.3 (Surjectivity of the homomorphism ρ). *The homomorphism ρ is surjective. If A is a rotation of angle θ about the axis $\mathbf{v} \in \mathbb{R}^3$ (with the right-hand corkscrew rule of Notation 4.1) then $A = \rho(v)$, where $v \in$* Spin(3) *is given by:*

$$v = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(v_1 i + v_2 j + v_3 k) \tag{2}$$

where $\theta \in [0, \pi]$.

Proof. If $A = \rho(v)$ for some $v = v_0.1 + v_1 i + v_2 j + v_3 k \in$ Spin(3), then let us determine the axis in $\mathbb{R}^3$ that it fixes. Clearly the unit quaternion v is fixed by conjugation Ad v, since Ad $v(v) = v.v.v^{-1} = v$. Also the real part Re $(v) = v_0.1$ is fixed by this conjugation. Thus

Claim 1. The fixed axis of $\rho(v)$ in $\mathbb{R}^3$ is the $\mathbb{R}$-span of $\mathrm{Im}\, v = v_1 i + v_2 j + v_3 k$ which we identify as $\mathbf{v} = \sum_{i=1}^{3} v_i e_i$. Note that if $\mathrm{Im}\, v = 0$, then $v = \pm 1$ and $\rho(v) = Id$. In this case there is nothing to be done. Thus we assume from now on (without loss of generality) that A is not the identity, and hence that $\theta \in (0, \pi]$.

What is the interpretation of v_0? Note that for a rotation $A \in SO(3)$, if we write it as a $\theta \in (0, \pi]$ rotation about the axis $\mathbf{v}$, and complete $\mathbf{v}$ into a orthonormal basis $\{\mathbf{a}, \mathbf{b}, \mathbf{v}\}$ which is right handed (viz. $\mathbf{a} \times \mathbf{b} = \mathbf{v}$), then by the last assertion of Proposition 2.3 above, with respect to this basis, A will have the matrix representation given by:

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Claim 2. The angle of rotation θ of the rotation A above is determined by the formula:

$$2\cos\theta = \mathrm{tr}A - 1$$

where tr A is the *trace* of A, the sum of its diagonal entries. We note here the fundamental fact that the trace is a property of the linear transformation, and not of the basis chosen for its matrix representation. Which is why we used the convenient basis $\{\mathbf{a}, \mathbf{b}, \mathbf{v}\}$ described above to assert this formula for θ.

We will now show that for $v = v_0 + v_1 i + v_2 j + v_3 k \in G$ a unit quaternion, the matrix representing Ad $v = v(-)v^{-1} = v(-)\bar{v}$ in the basis $\{1, i, j, k\}$ of $\mathbb{H}$ is given by:

$$\mathrm{Ad}\ v = \begin{pmatrix} 1 & * & * & * \\ 0 & v_0^2 + v_1^2 - v_2^2 - v_3^2 & * & * \\ 0 & * & v_0^2 + v_2^2 - v_1^2 - v_3^2 & * \\ 0 & * & * & v_0^2 + v_3^2 - v_1^2 - v_2^2 \end{pmatrix}$$

where the asterisks denote some entries that we are not of interest here. Clearly Ad $v(1) = 1$, which yields the first column above. For the second column, note that the i-component of Ad $v(i) = v.i.v^{-1}$ is just the term containing i in

$$v.i.v^{-1} = (v_0 + v_1 i + v_3 j + v_3 k).i.(v_0 - v_1 i - v_3 j - v_3 k)$$

which is easily computed to be $v_0^2 + v_1^2 - v_2^2 - v_3^2$.

Compute likewise for the j-component and k-component of Ad $v(k)$. It follows that the 4×4 matrix for Ad v has the diagonal entries shown above.

Clearly since $\mathbb{R}.1$ is fixed by Adv, and $\rho(v)$ is the restriction of Adv to $(\mathbb{R}.1)^{\perp}$, it follows that:

$$\mathrm{tr}\ \rho(v) = \mathrm{tr}\ (\mathrm{Ad}\ v) - 1 = 3v_0^2 - v_1^2 - v_2^2 - v_3^2 = 4v_0^2 - 1$$

since $\|v\|^2 = \sum_{i=0}^{3} v_i^2 = 1$. Thus, by using Claim 2 above, the angle of rotation θ of $\rho(v)$ about the axis $\mathbf{v} = v_1 e_1 + v_2 e_2 + v_3 e_3$ is given by:

$$2\cos\theta = \mathrm{tr}\rho(v) - 1 = 4v_0^2 - 2$$

which leads to $v_0^2 = \cos^2 \frac{\theta}{2}$, or

$$\cos \frac{\theta}{2} = \pm v_0. \tag{3}$$

The two signs correspond to the fact that both $\rho(v)$ and $\rho(-v)$ give exactly the same rotation of $\mathbb{R}^3 = \mathrm{Im}\,\mathbb{H}$. Note that since we are requiring that $\theta \in (0, \pi]$, $\cos \frac{\theta}{2}$ is always non-negative, and thus we may change v to $-v$ if necessary, so as to make v_0 non-negative. Hence, choosing the positive sign for v_0, it follows that

$$v = \cos \frac{\theta}{2} + \sin \frac{\theta}{2}(v_1 i + v_2 j + v_3 k)$$

where the coefficient $\sin \frac{\theta}{2}$ in the last three terms is needed to make v a *unit* quaternion, since $\mathbf{v} = \sum_{i=1}^{3} v_i e_i$ is assumed to be a unit vector in the statement. This finishes the proof of our Lemma. □

Example 4.4. To check that our computation above is correct, let us take $\mathbf{v} = e_3$, say. As an imaginary quaternion, this $\mathbf{v}$ is k. Then take a rotation A about $\mathbf{v}$ of angle θ (in the sense of the corkscrew rule). The Lemma 4.3 above says that $A = \rho(v)$ where $v = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}k$. Then $\rho(v)i = \mathrm{Ad}v(i) = v.i.\bar{v} = \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}k\right)i\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}k\right) = \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}k\right)\left(\cos\frac{\theta}{2}i + \sin\frac{\theta}{2}j\right) = \left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)i + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}j = \cos\theta.i + \sin\theta.j$. Similarly compute $\rho(v)j = \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}k\right)j\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}k\right) = -\sin\theta.i + \cos\theta.j$.

This shows that the matrix of A is:

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

as indeed, it should be, because rotating counterclockwise by θ in the xy plane moves a corkscrew in the direction of $\mathbf{v} = e_3$ in keeping of the right hand rule of Notation 4.1.

Remark 4.5. *Incidentally, the reader may well wonder whether the Lemma 4.3 above gives a formula for a "continuous section" for ρ (viz. a continuous map $\sigma : SO(3) \to \mathrm{Spin}(3)$ satisfying $\rho \circ \sigma = \mathrm{Id}_{SO(3)}$). The answer is no. The prescription above does not give a clear way for deciding what to do when $\theta = \pi$. In this case, the π rotations about both $\mathbf{v}$ and $-\mathbf{v}$ are the same, and so we have no way of prescribing a consistent choice between v and $-v$ to define σ all over $SO(3)$ so as to make it continuous.*

However, if we take the open subset U of all rotations in $SO(3)$ which rotate by an angle strictly less than π, the prescription given in the proof of Lemma 4.3 above provides a continuous map $\sigma : U \to \mathrm{Spin}(3)$ satisfying $\rho \circ \sigma = \mathrm{Id}_U$. In fact, it is interesting to note that U can be topologically identified with a 3-dimensional open ball of radius π. This topological equivalence comes from identifying the $\theta \in [0, \pi)$ rotation about the unit vector $\mathbf{v} \in \mathbb{R}^3$ with the vector $\theta\mathbf{v} \in \mathbb{R}^3$.

Now we are ready to answer the main question of this note.

Proposition 4.6 (Angle and axis of composite rotations). *Let $A, B \in SO(3)$. Let $\mathbf{v}$, (resp. $\mathbf{w}$) $\in \mathbb{R}^3$ be the axis of rotation of A (resp. B), by an amount $\theta \in [0, \pi]$ (resp. $\phi \in [0, \pi]$). Assume (for computational convenience) that both these vectors $\mathbf{v}, \mathbf{w}$ are unit vectors. Also we avoid the trivial case of A or B being the identity transformation by stipulating that θ, ϕ are both non-zero. Finally, we also stipulate that the unit vectors $\mathbf{v}$ and $\mathbf{w}$ are not linearly dependent, because in that case $\mathbf{v} = \pm\mathbf{w}$, and both rotations are in the same plane and the answer is obvious from Proposition 2.1.*

Then the axis of rotation of AB is the axis defined by the vector

$$\cos\frac{\phi}{2}\sin\frac{\theta}{2}\mathbf{v} + \sin\frac{\phi}{2}\cos\frac{\theta}{2}\mathbf{w} + \sin\frac{\theta}{2}\sin\frac{\phi}{2}(\mathbf{v} \times \mathbf{w})$$

(Note that this last defined vector may not be a unit vector). Also the angle of rotation of AB is given by ψ where

$$\cos\frac{\psi}{2} = \cos\frac{\theta}{2}\cos\frac{\phi}{2} - \sin\frac{\theta}{2}\sin\frac{\phi}{2}\langle\mathbf{v}, \mathbf{w}\rangle.$$

Proof. Note that, in view of our stipulation that θ, ϕ are both non-zero, we have $\theta, \phi \in (0, \pi]$. This implies that $\sin\frac{\theta}{2}\sin\frac{\phi}{2} \neq 0$. Because $\mathbf{v}$ and $\mathbf{w}$ are stipulated to be linearly independent, this means that $\sin\frac{\theta}{2}\sin\frac{\phi}{2}(\mathbf{v} \times \mathbf{w})$ is a non-zero vector. Since the vector $\mathbf{v} \times \mathbf{w}$ is perpendicular to the plane of the unit vectors $\mathbf{v}$ and $\mathbf{w}$, it follows that the vector claimed to be the axis of rotation of $A.B$ in the statement is also a *non-zero* vector. If A is a rotation by $\theta \in (0, \pi]$ with direction of rotation defined by $\mathbf{v} = \sum_{i=1}^{3} v_i e_i$ with $\sum_{i=1}^{3} v_i^2 = 1$, then by the Lemma 4.3 above, $A = \rho(v)$ where the unit quaternion $v \in G = \mathrm{Spin}(3)$ is given by:

$$v = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(v_1 i + v_2 j + v_3 k)$$

Likewise, if the second rotation B has direction of rotation $\mathbf{w} = \sum_{i=1}^{3} w_i e_i$ by an angle $\phi \in (0, \pi]$, then $B = \rho(w)$, where $w \in G$ is the unit quaternion given by

$$w = \cos\frac{\phi}{2} + \sin\frac{\phi}{2}(w_1 i + w_2 j + w_3 k).$$

By Claim 1 in the proof of Lemma 4.3 above, the fixed direction of $AB = \rho(v)\rho(w) = \rho(vw)$ is then given by computing the imaginary part of the quaternion $v.w$. Indeed,

$$\begin{aligned}\mathrm{Im}\,(v.w) = \Big[& w_1 \sin\frac{\phi}{2}\cos\frac{\theta}{2} + v_1 \sin\frac{\theta}{2}\cos\frac{\phi}{2} + \sin\frac{\theta}{2}\sin\frac{\phi}{2} \\ & \times (v_2 w_3 - v_3 w_2)\Big] i \\ & + \Big[w_2 \sin\frac{\phi}{2}\cos\frac{\theta}{2} + v_2 \sin\frac{\theta}{2}\cos\frac{\phi}{2} + \sin\frac{\theta}{2}\sin\frac{\phi}{2} \\ & \times (v_3 w_1 - v_1 w_3)\Big] j \\ & + \Big[w_3 \sin\frac{\phi}{2}\cos\frac{\theta}{2} + v_3 \sin\frac{\theta}{2}\cos\frac{\phi}{2} + \sin\frac{\theta}{2}\sin\frac{\phi}{2} \\ & \times (v_1 w_2 - v_2 w_1)\Big] k \\ = & \sin\frac{\theta}{2}\cos\frac{\phi}{2}\mathbf{v} + \cos\frac{\theta}{2}\sin\frac{\phi}{2}\mathbf{w} + \sin\frac{\theta}{2}\sin\frac{\phi}{2}(\mathbf{v}\times\mathbf{w})\end{aligned}$$

which proves the first part of the proposition (where, as usual, $\mathrm{Im}\,\mathbb{H}$ is being identified with $\mathbb{R}^3$ by $i \mapsto e_1$, $j \mapsto e_2$, $k \mapsto e_3$).

To see the angle of rotation ψ, we know that by Claim 2 in the proof of Lemma 4.3, $\cos\frac{\psi}{2}$ is the real part of the product quaternion vw. Thus it is

$$\begin{aligned}\cos\frac{\psi}{2} = \mathrm{Re}\,(vw) &= \cos\frac{\theta}{2}\cos\frac{\phi}{2} - \sin\frac{\theta}{2}\sin\frac{\phi}{2} \\ &\quad \times (v_1 w_1 + v_2 w_2 + v_3 w_3) \\ &= \cos\frac{\theta}{2}\cos\frac{\phi}{2} - \sin\frac{\theta}{2}\sin\frac{\phi}{2}\langle \mathbf{v}, \mathbf{w}\rangle\end{aligned}$$

which is exactly the second part of the proposition. The proposition follows. Notice how the dot product $\langle \mathbf{v}, \mathbf{w}\rangle$ enters into the formula for the angle, and the cross product $\mathbf{v} \times \mathbf{w}$ enters into the formula for the axis of rotation. □

The Differentiable Sphere Theorem

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Abstract. In this expository article, we discuss the Differentiable Sphere Theorem and outline the main ideas involved in its proof by Brendle and Schoen.

1. Introduction

One of the basic problems in Riemannian geometry is to study the influence of curvature of a Riemannian manifold on its topology. In the present article we shall confine ourselves only to certain types of positive curvature.

The famous Gauss-Bonnet theorem says that the total Gaussian curvature of a compact surface in $\mathbb{R}^3$ equals 2π times the Euler characteristic of the surface. This, together with the classification theorem for compact surfaces, implies that the only compact surface in $\mathbb{R}^3$ which has positive Gaussian curvature must be the two dimensional sphere.

In his habilitation lecture in 1854, Riemann generalized the notion of Gaussian curvature to higher dimensional manifolds. In modern language it can be described briefly as follows:

Let M be a smooth manifold of dimension n. Recall that a Riemannian metric g on M is a smooth choice of inner product on the tangent space T_pM at each point p of M. The Levi-Civita connection ∇ on M is given by

$$\begin{aligned}2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &\quad + g([X, Y], Z) - g([X, Z], Y) \\ &\quad - g([Y, Z], X)\end{aligned}$$

for vector fields X, Y, Z on M. The curvature of M is expressed by the Riemann curvature tensor R. For tangent vectors X, Y, Z, W in T_pM, choose vector fields $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}$ on M such that $\tilde{X}(p) = X$, $\tilde{Y}(p) = Y$, $\tilde{Z}(p) = Z$, $\tilde{W}(p) = W$. The curvature tensor is defined by

$$R(X, Y, Z, W) = g(\nabla_{\tilde{Y}}\nabla_{\tilde{X}}\tilde{Z} - \nabla_{\tilde{X}}\nabla_{\tilde{Y}}\tilde{Z} + \nabla_{[\tilde{X},\tilde{Y}]}\tilde{Z}, \tilde{W}).$$

It can be easily verified that the right hand side of the above equality is independent of the extension vector fields chosen.

For a two dimensional plane P in T_pM, we define the sectional curvature of P to be

$$K(P) = \frac{R(X, Y, X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2},$$

where $\{X, Y\}$ is any basis for P. Due to the symmetries of the curvature tensor, this definition is independent of the basis chosen.

For a two-dimensional Riemannian manifold there is only one two-dimensional plane to consider at a point and the resulting sectional curvature coincides with the Gaussian curvature at that point.

We also consider the Ricci and the scalar curvatures, which are defined by

$$\mathrm{Ric}(X, Y) = \sum_{i=1}^{n} R(X, e_i, Y, e_i)$$

and

$$S = \sum_{i,j=1}^{n} R(e_i, e_j, e_i, e_j),$$

respectively. Here $\{e_1, \ldots, e_n\}$ is an orthonormal basis for T_pM and $X, Y \in T_pM$. These definitions are independent of the orthonormal basis chosen.

Observe that the Ricci tensor, like the Riemannian metric, is a symmetric bilinear form at each point of the manifold whereas the scalar curvature is just a smooth function on the manifold.

We say that a manifold has positive sectional curvature if the sectional curvature of any two-dimensional plane at each point of the manifold is positive. The unit sphere $S^n \subset R^{n+1}$ equipped with the induced Riemannian metric has constant positive sectional curvature 1. Conversely, in 1926, H. Hopf showed that a compact simply connected manifold with constant positive sectional curvature 1 is isometric to the unit sphere (of same dimension). This led him to pose the following question (See [2], page 581):

If all the sectional curvatures are "sufficiently close to 1", does it follow that the manifold is homeomorphic to the sphere?

The meaning of "sufficiently close" in the above question is made precise by the notion of curvature pinching first considered by Hopf and Rauch.

Definition 1. *Let* $0 < \delta < 1$. *A Riemannian manifold* (M, g) *is said to be strictly* δ*-pinched in the global sense if its sectional curvature* K *satisfies the inequality* $\delta < K(P) \leq 1$ *for all two-dimensional planes* P *at each* $p \in M$.

In a celebrated paper [17], Rauch showed in 1951 that a compact simply connected manifold which is strictly δ-pinched must be homeomorphic to the sphere, with $\delta \approx 0.75$. Around 1960, Berger [1] and Klingenberg [11] proved the following quarter pinched Sphere theorem.

Theorem 1 (Berger [1], Klingenberg [11]). *Let* (M, g) *be an* n *dimensional compact simply connected Riemannian manifold which is strictly* $\frac{1}{4}$*-pinched in the global sense. Then* M *must be homeomorphic to* S^n.

We remark here that Theorem 1 is optimal in view of the fact that compact rank-1 symmetric spaces have nonstrict $\frac{1}{4}$-pinched curvatures.

It is natural to ask whether the conclusion of the above theorem can be strengthened from homeomorphism to diffeomorphism. This question has since been known as the Differentiable Sphere Theorem. This is a nontrivial question since the differentiable structure on S^n is not unique in general, as famously shown by Milnor [14].

Many attempts were made to prove the Differentiable Sphere Theorem under various pinching hypotheses and many partial results were obtained. Finally, in 2007, Brendle and Schoen [6] settled the Differentiable Sphere Theorem under the weaker hypothesis of pointwise quarter pinching.

Definition 2. *A Riemannian manifold* (M, g) *is said to be strictly* $\frac{1}{4}$*-pinched in the pointwise sense if there exists a positive continous function* $\delta : M \to \mathbb{R}$ *with* $\delta(p) < K(P) \leq 4\delta(p)$ *for all two-planes* P *in* T_pM.

Theorem 2 (Brendle-Schoen [6], 2007). *Let* (M, g) *be an* n *dimensional compact simply connected Riemannian manifold which is strictly* $\frac{1}{4}$*-pinched in the pointwise sense. Then* M *must be diffeomorphic to* S^n.

In fact, they proved a stronger result. See Theorem 6 below.

2. Positive Isotropic Curvature

In 1989, Micallef and Moore [12] introduced a new notion of curvature, namely positive isotropic curvature, which has turned out to be very useful.

Definition 3. *A Riemannian manifold of dimension $n \geq 4$ is said to have positive isotropic curvature if*

$$R(e_1, e_3, e_1, e_3) + R(e_1, e_4, e_1, e_4) + R(e_2, e_3, e_2, e_3) \\ + R(e_2, e_4, e_2, e_4) - 2R(e_1, e_2, e_3, e_4) > 0$$

for all points $p \in M$ and orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset T_pM$. If the nonstrict inequality holds in place of the strict inequality then we say that (M, g) has nonnegative isotropic curvature.

An interesting example of a Riemannian manifold with positive isotropic curvature is $S^n \times S^1$ equipped with it's standard product metric.

The relation of this new curvature condition with the "classical curvatures" is not quite clear. However, we note that positive isotropic curvature implies positive scalar curvature [13].

The notion of positive isotropic curvature arises naturally in the study of minimal (or harmonic) two-spheres in Riemannian manifolds, when one considers the complexified index form of energy. More precisely, for any smooth map $f : S^2 \to M$, the energy of f is given by

$$\mathcal{E}(f) = \frac{1}{2}\int_{S^2}\left(\left|\frac{\partial f}{\partial x}\right|^2 + \left|\frac{\partial f}{\partial y}\right|^2\right) dxdy,$$

where (x, y) are stereographic coordinates on S^2. We say a map $f : S^2 \to M$ is harmonic if it is a critical point of the energy functional $\mathcal{E}$. Under the assumption of positive isotropic curvature, it is possible to get a lower bound on the Morse index of energy at a nonconstant harmonic map. This enabled Micallef and Moore to obtain the following result.

Theorem 3 (Micallef-Moore [12], 1989). *Let (M, g) be an $n(n \geq 4)$ dimensional compact simply connected Riemannian manifold which has positive isotropic curvature. Then M must be homeomorphic to S^n.*

It must be mentioned here that strict $\frac{1}{4}$-pinching in the pointwise sense implies positive isotropic curvature. Therefore the above theorem is a strengthening of the classical sphere theorem of Berger and Klingenberg from strict global pinching to the weaker notion of strict pointwise pinching.

For more topological consequences of positive isotropic curvature see [13,7,8].

Recall that, at each point $p \in M$, we have a symmetric endomorphism of $\wedge^2(T_pM)$, viz, the curvature operator $\Re$ which is defined by

$$\langle \Re(X \wedge Y), (Z \wedge W)\rangle = R(X, Y, Z, W).$$

Here the inner product on $\wedge^2(T_pM)$ is the one induced from g. We say that (M, g) has positive curvature operator if the operator $\Re$ is positive or, equivalently, all the eigenvalues of $\Re$ are positive; a manifold has positive curvature operator implies that the manifold has positive isotropic curvature.

3. Ricci Flow

The main analytical tool employed in the proof of the Differentiable Sphere Theorem is Ricci flow. This is an intrinsic geometric flow introduced in a seminal paper [9] by R. S. Hamilton in 1982. The basic idea is to start with a Riemannian manifold (M, g_0) and evolve the Riemannian metric by the equation

$$\frac{\partial g(t)}{\partial t} = -2\mathrm{Ric}(g(t)), \quad g(0) = g_0.$$

Here $\mathrm{Ric}(g(t))$ denotes the Ricci tensor of the metric $g(t)$.

Hamilton showed in [9] that for any initial metric the Ricci flow admits a unique solution on some maximal time interval $[0, T), T > 0$.

Ricci flow is a nonlinear heat equation for the Riemannian metric. For example, the scalar curvature S evolves according the equation

$$\frac{\partial S}{\partial t} = \Delta S + 2|\mathrm{Ric}|^2,$$

where ΔS denotes the Laplace-Beltrami operator acting on the function S. It then follows from the maximum principle that, if the scalar curvature of the initial metric g_0 is positive, then so is the scalar curvature of the evolving metric $g(t)$. Moreover, the maximal time interval of the existence of solution must be finite in this case. Note that the last situation does occur if the initial metric has positive isotropic curvature.

As an example, if g_0 is the standard metric on S^n with constant curvature 1, then one has $g(t) = (1 - 2(n-1)t)g_0$ which is defined on the time interval $\left[0, \frac{1}{2(n-1)}\right)$.

We say that a Riemannian manifold has positive Ricci curvature if $\mathrm{Ric}(v, v) > 0$ for every non-zero vector v. Using Ricci flow, Hamilton proved the following fundamental result for three-dimensional manifolds.

Theorem 4 (Hamilton [9], 1982). *Let (M, g_0) be a compact simply connected three-manifold with positive Ricci curvature and let $g(t), t \in [0, T)$, be the maximal solution to the Ricci flow on M with initial metric g_0. Then the rescaled metrics $\frac{1}{4(T-t)} g(t)$ converge to a Riemannian metric on M of constant sectional curvature as $t \to T$. In particular, M must be diffeomorphic to S^3.*

If the assumption of simple connectedness is dropped from the statement of the previous theorem, then M will be diffeomorphic to a quotient of S^3 by a finite group of isometries of S^3 acting without fixed points, known as a spherical space form.

Hamilton developed an analogue of maximum principle for tensors in [10] which has since been widely used in the literature. Using this he was able to classify four-manifolds with positive curvature operator as spherical space forms in 1986.

The curvature tensor R evolves along Ricci flow according to the equation

$$\frac{\partial R}{\partial t} = \Delta R + Q(R),$$

where $Q(R)$ is a certain quadratic expression in the components of R. The ODE

$$\frac{dR}{dt} = Q(R)$$

is known as the curvature ODE. In view of Hamilton's maximum principle for tensors mentioned above, in many situations the study of the above PDE can be reduced to that of the curvature ODE.

By introducing new methods for studying the curvature ODE, Bohm and Wilking generalized Hamilton's work on four-manifolds with positive curvature operator to higher dimensions in 2006.

Theorem 5 (Bohm and Wilking [3], 2006). *Let (M, g_0) be a compact Riemannian manifold with positive curvature operator. Let $g(t)$ be the Ricci flow defined on the maximal time interval $[0, T)$ with initial metric g_0. Then $g(t)$ converges to a metric of constant sectional curvature 1 as $t \to T$, after rescaling. If, in addition, M is simply connected, then M must be diffeomorphic to a sphere.*

4. Proof of the Differentiable Sphere Theorem

For any finite dimensional real vector space V equipped with an inner product, one has the space of algebraic curvature tensors $\mathcal{C}_B(V)$ on V; this is the vector space consisting of all four-linear maps on V which satisfy the algebraic symmetries of the curvature tensor. If we have a Riemannian manifold (M, g) of dimension n, then by choosing a linear isometry between the tangent space T_pM at each point $p \in M$ and R^n, we get an identification of $\mathcal{C}_B(T_pM)$ with $\mathcal{C}_B(\mathbb{R}^n)$.

Hamilton gave a convergence criterion in [10] which asserts that if (M, g_0) is a compact Riemannian manifold of dimension $n \geq 3$ which has positive scalar curvature and if the curvature tensor of (M, g_0) at each point of M lies in a so called *pinching set* in $\mathcal{C}_B(\mathbb{R}^n)$, then the Ricci flow on M with g_0 as the initial metric converges to a metric of constant sectional curvature 1 on M, after rescaling.

By adapting the methods of Bohm and Wilking used in the proof of theoerem 5, Brendle and Schoen were able to construct a suitable pinching set for the case when the manifold satisfies a certain positive curvature condition (See Theorem 6 below for the precise condition). Hamilton's convergence criterion mentioned above then gives the following convergence result.

Theorem 6 (Brendle-Schoen [6], 2007). *Let (M, g_0) be a Riemannian manifold of dimension $n \geq 4$. Assume that it has the property that for every orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$ one has that $R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4) + R(e_2, e_3, e_2, e_3) + \mu^2 R(e_2, e_4, e_2, e_4) - 2\lambda\mu R(e_1, e_2, e_3, e_4)$ is strictly positive for all $\lambda, \mu \in [0, 1]$. Let $g(t), t \in [0, T)$, be the maximal solution to the Ricci flow on M with initial metric g_0. Then the rescaled metrics $\frac{1}{2(n-1)(T-t)} g(t)$ converge to a metric of constant positive sectional curvature 1 on M. In particular, if M is simply connected then it must be diffeomorphic to S^n.*

An important step in the proof of the above theorem was the proof of the fact that nonnegative isotropic curvature is preserved under Ricci flow. This was accomplished using Hamilton's maximum principle for tensors and some rather delicate algebraic calculation. This fact was also proved independently by Nguyen in [15].

The Differentiable Sphere Theorem now follows at once from Theorem 6 because strict $\frac{1}{4}$-pinching in the pointwise sense implies the curvature hypothesis of Theorem 6. See [6].

Remarks.

1. When the strict inequality in the hypothesis of Theorem 6 is replaced by the nonstrict inequality, the resulting condition is equivalent to nonnegative isotropic curvature on the product $M \times \mathbb{R}^2$. Observe, however, that the product $M \times \mathbb{R}^2$ can never have positive isotropic curvature.
2. In 2008, Brendle [4] obtained the same conclusion as in Theorem 6 under the weaker hypothesis of positive isotropic curvature on the product $M \times \mathbb{R}$. This result is a generalization of Hamilton's theorem for three-manifolds with positive Ricci curvature.

Motivated by the Differentiable Sphere Theorem one can ask the following question:

Question. Suppose (M, g) is a compact simply connected Riemannian manifold of dimension $n \geq 4$. Assume further that (M, g) has positive isotropic curvature. Is M diffeomorphic to S^n?

We conclude by remarking that, in general, Ricci flow will develop singularities for such initial metrics. If the assumption of simple connectedness is dropped, then one has other examples, most notably the Riemannian product $S^n \times S^1$.

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Happy 100th Birthday, Paul Erdös!

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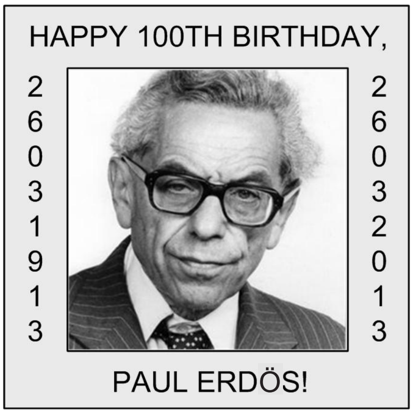

Paul Erdös (26 March 1913–20 September 1996, died at 83) was an influential Hungarian mathematician who spent a significant portion of his later life living out of a suitcase and writing papers with those of his colleagues willing to provide him room and board [1,2]. He worked on problems in combinatorics, graph theory, number theory, classical analysis, approximation theory, set theory, and probability theory. He published more papers than any other mathematician in history, working with hundreds of collaborators. He wrote over 1,500 mathematical articles in his lifetime, mostly with co-authors. Erdös is also known for his "legendarily eccentric" personality.

Erdös strongly believed in and practiced mathematics as a social activity, having over 500 collaborators during his life. Due to his prolific output, his friends created the *Erdös* number as a humorous tribute to his outstanding work and productivity [2]. Erdös number describes the "collaborative distance" between a person and mathematician Paul Erdös, as a measure by authorship of mathematical papers.

Tuesday, 26 March 2013 marks Erdös's centennial birthday. As I was looking at numbers related to Erdös's birthday, I noticed some interesting numerical coincidences and connections. I decided to report my findings in this article, as a centennial brainteaser birthday gift for Erdös.

The following are my numerical findings related to Erdös's birthday:

1. Erdös's birth date, 26 March, can be expressed in day-month date format as 26-3, or simply 263. Coincidentally, the 263rd day of 2013 and any other non-leap year is 20th of September, the day Erdös died in 1996.
2. Erdös's 100th full birthday in day-month-year date format can be written as 26-3-2013 or simply, 2632013. This number can be expressed in terms of its prime factors as follows:

$$2632013 = 19 \times 83 \times \underbrace{1669}_{\text{263rd prime}} = \underbrace{1577}_{19\times 83} \times \underbrace{1669}_{\text{263rd prime}}$$

   Notice that Erdös's 100th birthday expressed as 2632013 is divisible by the 263rd prime number 1669, where 263 represent Erdös's birth date, 26 March! Also, if 1669 is split in the middle as 16 and 69, these two numbers add up to 85, where the 85th day of 2013 (and any other non-leap year) is amazingly 26 March! In addition, prime 83 represents Erdös's death age and 19 is equal to half of the reverse of 83. (Also, as an aside, if number 1577 is split in the middle as 15 and 77, 15 plus 77 equals the difference of 1669 and 1577.)
3. Erdös's 100th full birthday can also be written as 26-03-2013, or simply, 26032013. The prime factors of this date number are given as follows:

$$26032013 = 157 \times 7 \times \underbrace{23687}_{\text{2636th prime}}$$

First, the prime factors 157 and 7 put side-by-side yield number 1577, one of the divisors of 2632013. Second, 23687 correspond to the 2636th prime number [3], where the leftmost three digits of 2636 is again 263, representing 26 March! Wow! Third, the rightmost digit 6 of 2636 corresponds to the sum of the digits of 2013. Fourth, 2636 differ from 2603 (representing 26 March) by 33, where 33 equals 20 plus 13, where 20 and 13 constitute the left- and right-halves of 2013. Fifth, the digits of 23687 add up to 26, the day number of Erdös's birthday. Sixth, if 2636 is split in the middle as 26 and 36, the sum of these two numbers yield 62, where the reverse of 62 is 26, again, the day number of Erdös's birthday.

4. Erdös's birth year 1913 is the 293rd prime number, where 293 is the 62nd prime number, where again the reverse of 62 is 26, the day number of Erdös's birthday.
5. Erdös's 100th full birthday in day-month-year date format is 3-26-2013, or simply, 3262013. Interestingly enough, the prime factors of the reverse of 3262013 are given as follows:

$$\overleftarrow{3102623} = 29 \times 83 \quad \underbrace{1289}_{\text{209th prime}}$$

Here, numbers 83 and 209 can be interpreted as Erdös's death age and death date (20 September).

6. 1289 is also one of the prime factors of the reverse of Erdös's full birthday expressed in day-month-year date format as 26031913 since

$$\overleftarrow{31913062} = 2 \times \underbrace{1289}_{\text{209th prime}} \times 12379$$

Thanks for transforming mathematics into a universal social activity through your modesty and humbleness Paul Erdös, and have a happy 100th birthday!

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Proximity of $\cos \mu x$ to $(\sin x)/x$ on $(0, \pi/2]$

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Abstract. We note here what seems to us the simplest proof of the result: the inequalities $\cos \mu_1 x > \frac{\sin x}{x} > \cos \mu_2 x$, $0 \le \mu_1 < \mu_2 \le 1$, $x \in \left(0, \frac{\pi}{2}\right]$ hold if and only if $\mu_1 < \frac{2}{\pi} \cos^{-1} \frac{2}{\pi} \in \left(\frac{1}{2}, \frac{1}{\sqrt{3}}\right)$, $\mu_2 \ge \frac{1}{\sqrt{3}}$.

Motivated by the inequalities

$$\cos \frac{x}{2} > \frac{\sin x}{x} > \cos \frac{x}{\sqrt{3}} \Longleftrightarrow \frac{1}{2} < \frac{1}{x} \cos^{-1}\left(\frac{\sin x}{x}\right) < \frac{1}{\sqrt{3}}, \quad x \in (0, \pi/2]$$

which Prof. K. S. K. Iyengar (1899–1944) noted in 1943 when the author was his student in the final B. Sc. (Hons.) class, both he and, independently, his student proved [1] that the inequalities

$$\cos \mu_1 x > \frac{\sin x}{x} > \cos \mu_2 x,$$

$$0 \le \mu_1 < \mu_2 \le 1, \quad x \in (0, \pi/2]$$

are valid if and only if

$$\mu_1 < \frac{2}{\pi} \cos^{-1}\left(\frac{2}{\pi}\right) \in \left(\frac{1}{2}, \frac{1}{\sqrt{3}}\right), \quad \mu_2 \ge \frac{1}{\sqrt{3}}.$$

The teacher's proof uses familiar estimates of $\cos x$ and $\sin x$ by partial sums of their power series, and the student's determines the sign of

$$\frac{\sin x}{\cos \mu x} - x, \quad 0 < \mu < 1, \quad x \in \left(0, \frac{\pi}{2}\right].$$

Perhaps the simplest proof of the above result was found by the author in 1961. It consists in defining the function

$$x \mapsto y = \cos^{-1}\left(\frac{\sin x}{x}\right), \quad 0 < x < \pi,$$

writing $\downarrow$ for 'decreasing' ($\uparrow$ for 'increasing') and proving that

$$x \mapsto \frac{y}{x} \text{ is } \downarrow, \quad \lim_{x\to 0}\frac{y}{x} = \frac{1}{\sqrt{3}}. \qquad (*)$$

This is the end result of three successive applications of the extended L' Hospital's rule (EHR) which states: if $o < \omega \leq +\infty$, $I = (0, \omega)$, $\{f, g\} \subset C^1(I)$, $f(0+) = 0 = g(0+)$ and g is $\uparrow$, then

$$\frac{f'}{g'} \text{ is } \downarrow \Longrightarrow \frac{f}{g} \text{ is } \downarrow,$$

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)} = l$$

$$\Longrightarrow \lim_{x\to 0}\frac{f(x)}{g(x)} = l \quad (l \in \mathbb{R}).$$

The extended part of this rule is easily proved: for $0 < \alpha < \beta < \omega$,

$$\begin{vmatrix} f(\alpha) & f(\beta) \\ g(\alpha) & g(\beta) \end{vmatrix} = \begin{vmatrix} f(\alpha) & f(\beta) - f(\alpha) \\ g(\alpha) & g(\beta) - g(\alpha) \end{vmatrix}$$

$$= \int_0^\alpha d\xi \int_\alpha^\beta \begin{vmatrix} f'(\xi) & f'(\eta) \\ g'(\xi) & g'(\eta) \end{vmatrix} d\eta > 0.$$

We prepare the ground for two applications of EHR:

$$0 < y < \frac{\pi}{2}, \quad \cos y = \frac{\sin x}{x},$$

$$\frac{dy}{dx}\tan y = \chi(x) = \frac{1}{x} - \cot x,$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{\phi(x)}{\psi(x)}, \quad \phi(x) = (\chi(x))^2,$$

$$\psi(x) = \left(\frac{x}{\sin x}\right)^2 - 1,$$

$$0 < \chi(x) < \frac{1-\cos x}{\sin x} = \frac{\sin x}{1 + \cos x},$$

$$\phi(0+) = 0 = \psi(0+), \quad \psi \text{ is } \uparrow,$$

$$\phi'(x) = 2\chi(x)\left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right),$$

$$\psi'(x) = \left(\frac{x}{\sin x}\right)^2 2\chi(x),$$

$$\frac{\phi'(x)}{\psi'(x)} = \frac{x^2 - \sin^2 x}{x^4} = (1 + w(x))\frac{u(x)}{v(x)},$$

$$w(x) = \frac{\sin x}{x}, \quad u(x) = x - \sin x, \quad v(x) = x^3,$$

$$u(0+) = 0 = v(0+), \quad v \text{ is } \uparrow,$$

$$u'(x) = 1 - \cos x = 2\sin^2\left(\frac{x}{2}\right),$$

$$v'(x) = 3x^2 = 12\left(\frac{x}{2}\right)^2,$$

$$\frac{u'(x)}{v'(x)} = \frac{1}{6}\left(w\left(\frac{x}{2}\right)\right)^2,$$

$$\frac{u'}{v'} \text{ is } \downarrow, \quad \lim_{x\to o}\frac{u'(x)}{v'(x)} = \frac{1}{6}.$$

By EHR with $\omega = \pi$, $f = u$ and $g = v$,

$$\frac{u}{v} \text{ is } \downarrow, \quad \lim_{x\to 0}\frac{u(x)}{v(x)} = \frac{1}{6},$$

so that

$$\frac{\phi'}{\psi'} \text{ is } \downarrow, \quad \lim_{x\to 0}\frac{\phi'(x)}{\psi'(x)} = \frac{1}{3}.$$

By EHR with $\omega = \pi$, $f = \phi$ and $g = \psi$,

$$\frac{\phi}{\psi} \text{ is } \downarrow, \quad \lim_{x\to 0}\frac{\phi(x)}{\psi(x)} = \frac{1}{3}.$$

Because $\frac{dy}{dx} > 0$, this yields

$$x \rightarrowtail \frac{dy}{dx} \text{ is } \downarrow, \quad \lim_{x\to 0}\frac{dy}{dx} = \frac{1}{\sqrt{3}}. \qquad (**)$$

Now $(**) \Longrightarrow (*)$ by EHR with $\omega = \pi$, $f(x) = y$ and $g(x) = x$ as these f and g satisfy the conditions $f(0+) = 0 = g(0+)$ and g is $\uparrow$.

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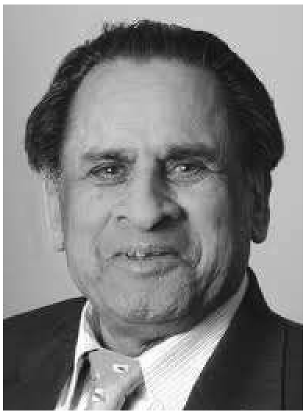
Remembering Shreeram S. Abhyankar

Sudhir R. Ghorpade



Shreeram S. Abhyankar
(July 22, 1930–November 2, 2012)

Shreeram Shankar Abhyankar, an extraordinary mathematician and educator, passed away on November 2, 2012 at the age of 82. He remained active in research as well as teaching and was constantly engaged in "doing mathematics" almost till his last breath. This, and the accompanying articles, are meant as a tribute to Prof. Abhyankar and an attempt to provide some glimpses of the person and his work.

1. The Beginnings

Shreeram Abhyankar was born on July 22, 1930 at Ujjain in central India. His father, Shankar Keshav Abhyankar, was a professor of mathematics based in Gwalior and he worked in Ujjain during 1928–1932, while his mother Uma Abhyankar (nee Tamhankar) was originally from Burhanpur. Shreeram was a prodigious child with an exceptional aptitude for mathematics. After completing his schooling in Gwalior, he came to Bombay and studied at what was then called the Royal Institute of Science. Around this time, he began visiting the newly founded Tata Institute of Fundamental Research (TIFR) and attending some lecture courses there, most notably the one by Marshall Stone. Even though he was intensely interested in mathematics, some doubts still lingered in his mind and at one time, young Shreeram felt that he should major in physics. What brought about the change is a story that I have always found awe-inspiring and goes as follows. When asked by D. D. Kosambi, one of the two professors (besides F. W. Levi) at the TIFR then, the reasons for majoring in physics in spite of being more interested in mathematics, Shreeram Abhyankar mentioned "I don't know what I should do if one day I can no longer do mathematics". To this, Kosambi replied, "Then you should kill yourself." At that moment, all the doubts in Abhyankar's mind melted away and, as we now know, he has never had a reason to kill himself! The head of the Department of Mathematics at the Royal Institute of Science at that time was Pesi Masani, a 1946 Harvard PhD with Garrett Birkhoff and a close associate of Norbert Wiener. With some encouragement from him, Abhyankar could enroll in the PhD programme at Harvard University on the eastern coast of USA. Thus, after obtaining his BSc in Mathematics from Bombay University in 1951, Shreeram Abhyankar embarked on his first overseas voyage to the United States.

2. Inception and Evolution of a Research Career

Owing to an illness on the boat while travelling to the USA, Abhyankar was detained in England for about two months and reached Harvard later than scheduled. As Sathaye has mentioned (see Appendix 1), Abhyankar met, on his first day in the US his future PhD supervisor Oscar Zariski, who was already a legendary figure in algebraic geometry. Shreeram Abhyankar received the MA and PhD degrees from Harvard University in 1952 and 1955 respectively. His thesis work was a major breakthrough where he succeeded in settling the problem of resolution of singularities of algebraic surfaces in prime characteristic. Invitations to several places followed and in the subsequent years, he held regular and visiting positions at some of the leading universities worldwide including Columbia, Cornell, Johns Hopkins, Harvard, Princeton and Yale in USA, Erlangen and Münster in Germany, Leiden in Holland, Angers, Nice, Paris, Saint-Cloud and Strasbourg in France, and Kyoto in Japan. In 1963 he moved to Purdue University, West Lafayette, Indiana, USA, and since 1967 he was the Marshall Distinguished Professor of Mathematics at Purdue. Moreover, since 1987–1988, he was also made a

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Professor in the Departments of Industrial Engineering and Computer Science at Purdue.

Throughout his life, Shreeram Abhyankar retained close ties to the country of his birth and was deeply concerned about the development of mathematics in India. He made numerous visits to academic institutions in India including TIFR, IIT Bombay, MatScience, and in fact, spent several years away from Purdue while he worked as a Professor (and for some years as the Head of Department of Mathematics) at the University of Pune. Moreover, he founded a research institution, named Bhaskaracharya Pratishthana, at Pune in 1976. His inspiring seminars and eminence in mathematics often attracted many students and several of them went on to do PhD with him.

3. Research Contributions

Shreeram Abhyankar has made numerous important contributions to many areas of mathematics, especially algebraic geometry, commutative algebra, theory of functions of several complex variables, invariant theory, and combinatorics. He has authored close to 200 research papers published in some of the leading international journals. He is also the author of about a dozen books and research monographs. Like the great David Hilbert, Abhyankar's work can be roughly divided in fairly distinct phases during which he focused mainly on one broad topic and made substantive contributions. (See also the write-up by Balwant Singh in Appendix 2.) A brief outline is given below.

Phase I: Resolution of Singularities. This phase began, as noted earlier, with the path-breaking PhD thesis work of Shreeram Abhyankar, and was predominant mainly during 1954–1969. The topic always remained close to his heart and he would return to it from time to time, most notably, in the early 1980's when he worked on canonical desingularisation. See the accompanying articles of Balwant Singh and Dale Cutkosky (Appendix 2 and Appendix 3) for more on this topic. We also refer to the brief survey by Mulay in [7] and a riveting account, intermingled with personalised history, by Abhyankar himself in his Bulletin article [6].

Phase II: Affine Geometry. The study of affine algebraic curves and surfaces together with their embeddings and automorphisms formed the focal theme of Abhyankar's researches beginning around 1970 and lasting for about a decade. Through his explicit and algorithmic methods, rooted in high school algebra, Abhyankar achieved remarkable successes that greatly advanced the research in affine algebraic geometry. Some brilliant collaborations with his PhD students, especially T. T. Moh and Avinash Sathaye, ensued during this period. Abhyankar also popularised a 1939 conjecture of Keller, now known as the Jacobian problem, that still remains open and has enticed countless mathematicians over the years. The reminiscences by Avinash Sathaye, Balwant Singh and Rajendra Gurjar (see Appendices 1, 2 and 4) shed more light on this topic and the contributions of Abhyankar in it. Further, one may refer to the relevant parts of the Ford and Chauvenet award winning article [1] and the Kyoto paper [2] of Abhyankar for more on this topic.

Phase III: Young Tableaux and Determinantal Varieties. Around 1982, motivated by some questions concerning singularities of Schubert varieties in flag manifolds, Abhyankar became interested in determinantal varieties and was fascinated by the straightening law of Doubilet–Rota–Stein. He forayed into the combinatorics of Young tableaux and by a remarkable *tour de force*, managed to discover intricate formulas for enumerating various classes of tableaux and the interrelationships among them. These could be applied to obtain Hilbert functions of determinantal varieties and to derive some geometric properties. It was during this period that I met him and became enamoured of his mathematics. Abhyankar's work on this topic is mainly found in his research monograph [3] of about 500 pages that, incidentally, has no Greek symbols! For a gentler introduction, one may refer to my survey article in [7]. Although Abhyankar's interest in combinatorics had reached a peak in the mid-80's to the extent that he was once willing to denounce being an algebraic geometer, by the late-80's, he was persuaded by engineers and computer scientists to return to algebraic geometry and to teach them his algorithmic approach to it. This resulted in profitable interactions and was partly responsible for his book *Algebraic Geometry for Scientists and Engineers* [4] that went on to become an AMS bestseller. And then there were the letters from J.-P. Serre.

Phase IV: Galois Theory and Group Theory. In a series of letters in 1988, Serre posed specific questions to Abhyankar related to his 1957 paper on covering of algebraic curves.

This led to a revival of interest by Abhyankar in his own conjectures concerning Galois groups of unramified coverings of the affine line in characteristic $p \neq 0$. By this time, group theory had progressed a great deal and Abhyankar, already in his sixties, became a student again and began a massive study of the significant advances in group theory and applied them effectively to questions in galois theory. During 1990–2005, Abhyankar wrote numerous papers giving “nice equations for nice groups”. Meanwhile, Abhyankar’s conjectures from his 1957 paper were settled using rather abstract methods, by M. Raynaud and D. Harbater (for which they received the Cole Prize from the American Mathematical Society). For an introduction to Abhyankar’s work during this phase, it may be best to refer to his two *Bulletin* articles [5] and [6].

Phase V: Jacobian Problem and Dicritical Divisors. Soon after the publication in 2006 of his book *Lectures on Algebra I* that has more than 700 pages, Abhyankar returned to the Jacobian problem, which was mentioned earlier. He began by publishing his “thoughts” in a four part paper of more than 300 pages in the *Journal of Algebra* in 2008. Around this time, he encountered the topological notion of dicritical divisors. He quickly understood its significance, but was not satisfied until he could understand them algebraically in his own way. In a series of remarkable papers, he algebracised the theory of dicritical divisors thereby making it valid for nonzero characteristic as well as mixed characteristic, and studied its connections with the Jacobian problem. As he liked to put it, using dicritical divisors, one sees that intrinsically hidden inside the belly of a bivariate polynomial there live a finite number of univariate polynomials. Sathaye had once described this work of Abhyankar’s as one of his best thus far. One is further impressed when one considers that it was done when Abhyankar was around 80 years of age!

4. Honours and Accolades

Quite naturally, Shreeram Abhyankar received numerous honours and awards during his lifetime. This includes McCoy Prize from Purdue, Lester Ford Prize and the Chauvenet Prize from the Mathematical Association of America, a Medal of Honour from the University of Valliadolid, Spain, as well as the University of Brasilia, Brazil, and the honorary title of *Vidnyan Sanstha Ratna* from the Institute of Science, Mumbai. He received a honorary doctorate from the University of Angers, France in 1998. On this occasion, Fields medalist Heisuke Hironaka wrote: “*Your long and powerful works deserve far more than the honorary doctorate you are receiving. Even so, I am happy to hear the good news. Your originality has been a gold mine for many other algebraic geometers, including myself. Now the mined gold is receiving rays of sunlight, facets after facets.*”

Abhyankar was elected as a Fellow of the Indian National Science Academy in 1987 and the Indian Academy of Sciences in 1988. Most recently, he was among the inaugural batch of Fellows of the American Mathematical Society (AMS) announced by the AMS on November 1, 2012. He has guided about 30 PhD students and has inspired many more at different stages of their education and research career. International conferences in algebra and algebraic geometry in honour of Shreeram Abhyankar were held at Purdue in 1990, 2000, 2010 and 2012 around the month of July and also at Pune, India, in December 2010.

5. Epilogue

Prof. Abhyankar is survived by his wife Yvonne, a remarkable person herself and who had been a constant companion and a source of strength to him since their marriage in 1958, son Hari (a 1999 PhD in Operations Management from MIT), daughter Kashi (a 2001 PhD in Mathematics from Berkeley), and four granddaughters Maya, Kira, Kaia and Ela. Shreeram Abhyankar’s influence on many areas of mathematics, especially algebraic geometry, through his outstanding research and on numerous students, colleagues, and admirers the world over through his inspiring lectures and also his books and articles, shall remain for years to come. Those of us who have had the pleasure and privilege of knowing him and learning something about mathematics and life from him will always cherish the fond memories of our association.

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Appendix 1

Shreeram Shankar Abhyankar

Appreciation of my Guru

I met Abhyankar when I was a third year college student, while he was visiting Pune in Summer. He had made an open invitation to anyone interested in mathematics and our mathematics Professor took me to meet him in response. My introductory meeting lasted about three hours, to be repeated over several days! It is hard to summarise what we talked about. He would certainly answer any questions about mathematics that I asked, sometimes diverting them to more interesting topics. Often, he would talk about Marathi and Sanskrit literature, philosophy, memories of learning in childhood and so on. Very rarely did he give a formal lecture, in these private chats. He was always patient in explaining the same thing over and over again.

I continued to visit him whenever he would return to Pune and finally in 1969, I moved to Purdue University after he suggested that would make it easier to continue my studies. I finished my doctorate in 1973 and have been deriving my inspiration to do mathematics by observing his work and listening to him.

Abhyankar had a unique perspective of mathematics. He often rebeled against "fancy mathematics", demanding that all theorems should have detailed concrete proofs. He also believed that papers should spell out all the necessary details and he practiced this rigorously. As a result, several of his papers are difficult to read, because you have to keep your concentration on every little detail that he has laid down. He would often say that the proofs should be so logical that a computer should be able to verify them!

He also had a sense of poetry and rhythm in his papers. He would create multiple subsections with matching words and equal number of subitems, so that the paper had a natural symmetry. Sometimes, he would spend enormous amount of time to create such intricate structures. He then would proceed to prove the main theorem by pulling together his several lemmas into a proof of the type: "The result follows by items $a, b, c, \ldots, z$".

He was also a master of proof by mathematical induction. A good induction proof needs a clear understanding of the main points and an intellectual capacity to analyse the changes as you move through different cases.

During his long mathematical career of more than 57 years, he went through several distinct periods of concentration on specific topics. When he was concentrating on a specific topic, he would be totally immersed in it, reading what he could find, asking everybody about it, and even consciously seeking the experts in the field. At the end, he would become a master of the subject himself.

I was his student when he was concentrated on affine geometry. He had just finished his monumental work in resolution of singularities and had come to the conclusion that he needed something appealing to young new students. The subject of resolution, while at his heart, required years of preparation and did not connect that well with freshmen or high school students. By his personal experience, he knew that love of mathematics is best developed early—he was younger than 10 when he discovered its beauty.

So he thought of interesting problems about the simplest mathematical structure, the polynomials. This is something one learns in middle school and usually stays in one's mind as a boring skill! He was determined to change that.

He invented the question which has now become famous by the title "Abhyankar–Moh Epimorphism Theorem". In a modern textbook, it would be stated in fancy language as follows:

Suppose that $f \in k[X, Y]$—a polynomial in two variables such that f is biregular to a line, then is f a generator of the polynomial ring?

While precise, this statement needs lots of explanation. Abhyankar reformulated it so that even a middle schooler can understand and think about it:

Suppose $p(t) = t^n + p_1 t^{n-1} + \cdots + p_n$ and $q(t) = t^m + q_1 t^{m-1} + \cdots + q_m$ are polynomials so that t can be written as a polynomial in $p(t)$ and $q(t)$. Is it true that n divides m or m divides n?

Students are familiar with linear change of variables. Building on the concept, Abhyankar described a polynomial $f(X, Y)$ to be a "variable" if there is a polynomial $g(X, Y)$ such that every polynomial in X, Y can be written as a polynomial in X, Y. In standard notation, this means $k[X, Y] = k[f, g]$.

Then the central question raised by Abhyankar was, how can you tell if a given $f(X, Y)$ is a variable? The Epimorphism Theorem gives a sufficient condition that there are polynomials p, q as described above, so that $f(p(t), q(t)) = 0$. (This is actually true only when you are working in characteristic zero, but that means it is true in the usual real or complex numbers.)

The corresponding three (or higher) dimensional question is still unresolved, but has been a key feature of numerous research papers since.

Another intriguing question is how to tell if a given pair of polynomials f, g form a pair of variables, i.e. $k[X, Y] = k[f, g]$. An answer in the form of the famous "Automorphism Theorem" is that we should be able to transform the given f, g into X, Y by a sequence of standard transformations where we hold one of them fixed and add a polynomial expression in it to the other. For example, replace f, g by $f, g + 2f - f^3$.

While this is a valid criterion, it is not that satisfactory, since we do not know the result until all the steps are carried out. So it is worth seeking other useful criteria.

Abhyankar popularised another striking question known as the Jacobian Problem which asks:

Suppose that the Jacobian of f, g written as $J(f, g) = f_X g_Y - f_Y g_X$ is a nonzero constant. Then is it true (in characteristic zero, or simply in complex numbers) that f, g is a pair of variables?

A calculus student knows and can understand this condition. This simple sounding problem has a long history. There are several published incorrect proofs and new ones are being produced with a predictable regularity. Abhyankar himself was instrumental in pointing out the flaws in many of these "proofs" and has some of the best results obtained so far (at least in the two variable case). The problem naturally extends to any number of variables and is an active area of research.

Abhyankar has also led in and inspired a lot of research in the problems of Galois Theory (especially over function fields in positive characteristic). Abhyankar used to fondly recall how his pathbreaking papers on fundamental groups (from 1950s) were born out of a flash of intuition during intense concentration. He felt that he had experienced the yogic experience of Samadhi at that time and he had practically decided to become a Yogi instead! Fortunately for the mathematical world, some accidental events in his life at the time brought him back to the material world. Yet, till the end, he always considered his mathematics as applied Yoga!

Abhyankar was convinced that mathematics is a panacea. He used to tell how, as a young child he was weak and suffered from many ailments. This continued until he discovered mathematics. Once he started reading mathematics, he did not get sick again. Even when sick, he could push aside the pain and get well by immersing in mathematics. Perhaps, his sudden death while sitting at his desk, working on mathematics, is a testimonial to his theory of mathematics over matter!

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Appendix 2

A Note on Abhyankar and His Work

Abhyankar's style of work, according to my observation, was to work thoroughly in one area of algebraic geometry for a few years, making substantial and deep contributions to it, and then move on to a different area or return to one of his earlier favourites. Some specific areas encompassed in his vast research work are resolution of singularities, tame coverings and algebraic fundamental groups, affine geometry, enumerative combinatorics of Young tableaux and Galois groups and equations. I will talk about only two of these, where I have some firsthand knowledge, namely resolution of singularities and affine geometry.

In the 1940's Zariski had obtained a rigorous proof of resolution of singularities of surfaces and threefolds in characteristic zero. The case of positive characteristic for surfaces was done by Abhyankar in his PhD thesis in 1956. An extension of the result to threefolds required, as a first step, the resolution of singularities of an embedded surface. This Abhyankar investigated over several years, developing in the process highly intricate and powerful algorithms in positive characteristic. As the material grew in size, he found it necessary to write up and update his various results in a book *Resolutions of Singularities of Embedded Algebraic Surfaces*, which appeared in 1966 and which culminated in a proof of resolution of singularities of threefolds in characteristics other than 2, 3, 5. He once told me that he had about 300 pages of handwritten notes which take care also of characteristics 2, 3, 5, but that he did not have the energy to translate these into a readable exposition. Apart from positive characteristic, Abhyankar also solved the equally difficult problem for the arithmetic case, i.e. for surfaces over the ring of integers. For a long time the only significant contributions in positive characteristic or in the arithmetic case were those due to Abhyankar.

In affine geometry, the themes of Abhyankar's work were embeddings and automorphisms. Two well-known terms here are the Epimorphism Theorem and the Jacobian Conjecture. In fact, Abhyankar's focus was on the Jacobian Conjecture, and Epimorphism Theorem was just the outcome of his first attempt at solving the Jacobian Conjecture. This was in the early 1970's. Soon after, Abhyankar moved to other areas but then returned to the Jacobian Conjecture in about the year 2002. Then this remained the area of his work in the last decade of his life. Abhyankar was very fond of *algebraicising* results from other areas, notably analysis and topology, if he thought they were relevant to his current interest. He often succeeded in doing this, and this was also one of his strengths. It is in this spirit that he developed the algebraic theory of dicritical divisors in the last few years, keeping in view their possible application to a solution of the Jacobian Conjecture.

I must single out two of his publications for special mention: His paper *On the valuations centered in a local domain* (1956) and his Princeton monograph *Ramification Theoretic Methods in Algebraic Geometry* (1959). These are repertories which continue even now to yield new techniques and insights.

I discovered early on in my association with Abhyankar that while you can learn mathematics from a book you can almost never get from it the insight which listening to his lectures provided. I learned much mathematics from his lectures and my private discussions with him.

My relationship with Abhyankar went beyond mathematics. Our families have had a close relationship for over four decades. In my young days I took pride in my knowledge of Hindu mythology. This pride evaporated quickly after I heard Abhyankar expound on the theme for the first time. He must have noticed my interest because soon mythology became as much a part of our conversations as mathematics, if not more. Mostly this was a one way street with me learning much mythology and mathematics from him. During a talk by Abhyankar at Purdue on "Relationships in Mahabharata" the audience was awestruck by his unmistakable and firm grasp on the complicated web of inter-relationships among numerous characters in the epic. I must say that over the years I developed a distinct feeling that some of his actions were guided by what he believed his mythological hero did or would do in a similar situation.

Balwant Singh obtained PhD in Mathematics from the University of Bombay in 1970 under the supervision of R. Sridharan/C. S. Seshadri at the Tata Institute of Fundamental Research, Mumbai, India, and he worked there till his retirement in 2000. Thereafter he worked for several years at the Indian Institute of Technology Bombay, and since 2009, he is at the MU-DAE Centre for Excellence in Basic Sciences at Mumbai. He works in the areas of algebraic geometry and commutative algebra.

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Appendix 3

An Homage to Prof. Abhyankar

It was with great sorrow that I heard of the passing of Prof. Abhyankar. He was a greatman. I admired and liked him, both as a mathematician and as a man. I first met him in West Lafayette in 1988, when my wife, Hema Srinivasan, and I were visiting Purdue University for a year. We frequently visited his home, and have continued to have a close contact with him since then. Always he was surrounded by his students and co-workers. His wife, Yvonne, somehow managed all of this, making everyone welcome. At that time, their son Hari was an undergraduate at Purdue and their daughter Kashi was still in High School.

Prof. Abhyankar was a charismatic man, and an excellent speaker, who could mesmerise an audience. He had a special way of talking with people about mathematics. He would insist that people explain what they were doing in elementary terms. Often when you began talking with him you realised how poor your understanding was, but at the end of the conversation you had a much deeper knowledge. Throughout his mathematical life, he took polynomials and power series, and related concepts such as determinants and discriminants as the focus of his interest, only considering the most fundamental and important problems. He liked to see things in the simplest way possible, without affectation. I have always admired his strength, being willing to stand alone if necessary, following what he believed in.

I have spent many hours closely studying Prof. Abhyankar's wonderful papers on resolution, valuations and ramification. His papers are written with remarkable care. It takes tremendous effort to read them. He created his own language to explain his mathematics. Everything is stated in pure algebra, with no recourse to possibly misleading geometric intuition. Often a key point in the argument is a clever algebraic manipulation. Considering the incredible complexity of his proofs, it is amazing how carefully written and precise they are.

I am very glad that I was able to attend his 82nd Birthday conference this summer at Purdue University. The conference was a tribute to his rich life, celebrated by his many students, collaborators colleagues.

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Appendix 4

Prof. Shreeram Shankar Abhyankar: Some Reminiscences

I first met Prof. Shreeram Abhyankar sometime in 1970 when I was still an undergraduate in Pune. Avinash Sathaye, who was then a PhD student of Abhyankar, took me to Abhyankar's house. For the next few years I used to visit him when he came to Pune. During these visits he had recommended some books in mathematics for me to read. Among them I remember the following classics: Kamke's *Set Theory*, Knopp's two volume *Theory of Functions*, Zariski-Samuel's two volume *Commutative Algebra*, Ford's *Automorphic Functions*, Siegel's three-volume *Complex Function Theory, Goursat's Complex Analysis*, ...

During my student years, and later as a more mature researcher, I managed to read all these books. The knowledge acquired from these books has helped me greatly over the years and made me a more confident researcher. I wish I had read more of the books suggested by Abhyankar!

Prof. Abhyankar helped me to go to Purdue University for PhD I had the honour of associating with him in one research project. In connection with the famous Jacobian Problem, I found a formula due to Lagrange for expressing the inverse of a power series in one variable which has a non-vanishing determinant at the origin. In Goursat's book there was a partial generalisation of this to two variables. By some heuristic argument I found a complete generalisation to arbitrary number of variables using the argument in Goursat which used complex integration. When I showed this to Abhyankar he first observed that I was expressing the formula in a somewhat complicated way. He first simplified the expression and then gave a purely algebraic proof of it! Several mathematicians, including Bass–Connell–Wright, have tried to use this formula for solving the Jacobian Problem. From this proof, and from my study of some of Abhyankar's work in later years, I consider him as one of the best commutative algebraists in the world in the last 60 years.

Before settling down in Purdue University in 1963 he had worked in major universities like Columbia, Cornell, Princeton, Johns Hopkins, Harvard, ...

Prof. Abhyankar has many outstanding results to his credit. There is still no improvement, or simplification, of his work about resolution of singularities of surfaces in 1956, and threefolds in 1965 in characteristic $p > 0$. Some of the other works for which he will always be remembered are: Uniqueness of embeddings of an affine line in the affine plane in characteristic 0, Galois groups of coverings of the affine line in characteristic $p > 0$, multiply-transitive Galois covers, Jacobian Problem in two dimension. In my opinion he went much deeper than anyone else in trying to solve the Jacobian Problem.

Prof. Abhyankar authored many books: *Resolution of Singularities of Embedded Algebraic Surfaces, Ramification Theoretic Methods in Algebraic Geometry, Local Analytic Geometry, Algebraic Space Curves, Algebraic Geometry for Scientists and Engineers, Expansion Techniques in Algebraic Geometry, Enumerative Combinatorics of Young Tableaux, Weighted Expansions for Canonical Desingularization*, a recent textbook on Algebra, ... These will be valuable for years to come.

The total body of his work (research papers, books, a very large number of conference lectures around the world, . . .) is truly staggering. It shows his tremendous hard work, technical abilities and total passion for mathematics. This in itself is greatly inspiring, but he will always be remembered for his indomitable spirit, a strong patriotic feeling towards his Indian roots, his confidence in his knowledge and a just pride in knowing the value, and place, of his research work in algebraic geometry. It can be said that his interests lay more in classical mathematics. Perhaps I am wrong, but I think he did not even once use tensor product in his research!

One of his motto was never to use a result whose proof he had not read. He broke this rule the first time when he used the classification of finite simple groups.

I met him in August during a conference on Topology of Algebraic Varieties in Montreal and we had interesting discussions about the latest paper he was writing with Artal Bartolo, hoping to make Zariski's work on complete ideals in two-dimensional regular local rings more understandable. Even at 82, he was in great spirit. In the conference he gave a talk on some nice work he had done a few years ago using classical Knot Theoretic work of Zariski related to singularities of plane algebraic curves. This work of Abhyankar is also related to his work on some version of Hilbert's 13th Problem. He remained active and thought about mathematics almost till his last breath.

Prof. Abhyankar guided more than twenty five students for PhD. More than ten of them are Indian; in fact Prof. Abhyankar took many Indian students to Purdue and guided them for PhD. He made a large number of visits to India and inspired many young mathematicians. I count myself lucky to be one of these people. In recent years, during his visits he used to lecture to many students from Pune/Mumbai on diverse topics in algebraic geometry, sometimes for hours. My son Sudarshan was one of these students; this, forty years after I had first started learning from him!

Although he lived in US for more than sixty years he contributed to Indian mathematics in many ways. His founding of the Bhaskaracharya Pratishthana in Pune is only one of these important contributions.

Prof. Abhyankar received many prestigious awards and fellowships in his long and distinguished career. Some of the prestigious awards like, Sloan Fellowship, Ford Prize, Chauvenet Award, ... are the proof of his outstanding works. He was Marshall Distinguished Professor of Mathematics in Purdue University from 1967. For the past twenty-five years he was also a professor in the departments of Industrial Engineering and Computer Science in Purdue University, in recognition of his interdisciplinary work.

Prof. Abhyankar will always remain alive in our memories.

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Problems and Solutions

Edited by Amritanshu Prasad

The section of the Newsletter contains problems contributed by the mathematical community. These problems range from mathematical brain-teasers through instructive exercises to challenging mathematical problems. Contributions are welcome from everyone, students, teachers, scientists and other maths enthusiasts. We are open to problems of all types. We only ask that they be reasonably original, and not more advanced than the MSc level. A cash prize of Rs. 500 will be given to those whose contributions are selected for publication. Please send solutions along with the problems. The solutions will be published in the next issue. Please send your contribution to `problems@imsc.res.in` with the word "**problems**" somewhere in the subject line. Please also send solutions to these problems (with the word "**solutions**" somewhere in the subject line). Selected solutions will be featured in the next issue of this Newsletter.

1. **Tom Moore, Bridgewater State University.** The earliest mention of a fourth order magic square occurred in India in the work of the great philosopher Nagarjuna, around 100 AD. Construction of magic squares was first seriously taken up by Narayana around 1356 AD. In contrast the famous third order magic square, known as the lo shu, was mentioned by the Chinese Chang Tzu in 300 BC. Here is the lo shu in which the eight lines consisting of the rows, columns and diagonals each sum to 15:

8	1	6
3	5	7
4	9	2

   To honor this magic square tradition, we offer the following puzzle. Rearrange the numbers in the lo shu in any way so as to produce five lines summing to 15 and three lines summing to 18.
2. **Tom Moore, Bridgewater State University.** Prove that infinitely many squares of natural numbers may be expressed in the form $2a^3+3b^2$ for natural numbers a and b.
3. **Tom Moore, Bridgewater State University.** It is well known and startling that the polynomial $2x^2+29$ produces primes for the consecutive integers $x = 0, 1, 2, \ldots, 28$. Prove, in contrast, that the polynomial $2x^3+29$ can produce primes for at most two consecutive nonnegative integers x.
4. **Mahan Mj., RKM Vivekananda University.** Let $ABCD$ be a qaudrilateral with AB, DC parallel.
   Let E, F be the mid-points of the diagonals AC, BD. Prove that EF is parallel to AB, DC and has length equal to half the difference between the lengths of AB, DC.
5. **K. N. Raghavan, IMSc.** For each positive integer n, let $f_n(x) = x(x-1)\cdots(x-k+1)/k!$. Also, let $f_0(x) \equiv 1$. Show that for every non-negative integer n, every polynomial $f(x)$ of degree n with real coefficients such that $f(x)$ is an integer whenever x is an integer, can be written in the form
$$\alpha_0 f_0 + \alpha_1 f_1 + \cdots + \alpha_n f_n$$
   for some positive integer n and integers $\alpha_0, \alpha_1, \ldots, \alpha_n$.
6. **Amritanshu Prasad, IMSc.** Let $f(x)$ be a rational function in x with rational coefficients, i.e.,
$$f(x) = a(x)/b(x), \text{ with } a(x), b(x) \in \mathbf{Q}[x], b(x) \not\equiv 0.$$
   If $f(x)$ is an integer for infinitely many integers x, show that f is a polynomial with rational coefficients.

Solutions to Problems from the December Issue

1. **B. Sury, ISI Bangalore.** For each positive integer n, the value $\phi(n)$ of the Euler totient function is the number of integers $1 < k < n$ that are coprime to n. Show that every factorial is a value of the Euler totient function: for every positive integer m, there exists a positive integer n such that $\phi(n) = m!$.

   Solution. The Euler totient fucntion enjoys the property that if s and t are coprime, then $\phi(st) = \phi(s)\phi(t)$. Moreover, $\phi(p^n) = p^{n-1}(p-1)$[1]. So, if the factorization of s into a product of prime powers is $s = p_1^{n_1}, \ldots, p_k^{n_k}$, with $n_1, \ldots, n_k$ positive, then $\phi(s) = p_1^{n_1-1}, \ldots, p_k^{n_k-1}$ $(p_1-1)\cdots(p_k-1)$, so that
$$\frac{s}{\phi(s)} = \frac{1}{(p_1-1)\cdots(p_k-1)}. \qquad (*)$$

[1]See for example, Theorem 2.19 in *An Introduction to the Theory of Numbers (fifth edition)* by Niven, Zuckerman, and Montgomery.

Thus if s and t have the same prime factors, then $\phi(s)/s = \phi(t)/t$. If we seek solutions to $\phi(n) = m!$ where $m!$ and n have the same prime factors, the identity $\phi(m!)/m! = \phi(n)/n$ gives us $n = m!^2/\phi(m!)$. The problem is solved if we show that n is an integer with the same prime factors as $m!$. But if a prime p divides $m!$, then $p \leq m$, whence $p-1 < m$. So $m!/\phi(m!)$ is an integer by $(*)$. It follows that n is an integer with the prime factors as $m!$.

2. **B. Sury, ISI Bangalore.** Consider the $n \times n$ matrix A whose (i,j)th entry is $a_{ij} = \gcd(i,j)$. Determine $\det A$.

   Solution. Consider the matrix B where $b_{ij} = \sqrt{\phi(j)}$ if $j|i$ and 0 otherwise. Here $\phi(r)$ denotes the number of irreducible positive fractions ≤ 1 with denominator r (equivalently, the number of positive integers $d \leq r$ such that d is coprime to r; the Euler totient function). Note that $\sum_{d|n} \phi(d) = n$ by counting all the n fractions i/n with $1 \leq i \leq n$. Now,

   $$(BB^t)_{ij} = \sum_k b_{ik} b_{jk} = \sum_{k|\gcd(i,j)} \phi(k) = \gcd(i,j) = a_{ij}$$

   Note that B is a lower triangular matrix. Thus, $\det(A) = \det(BB^t) = \det(B)^2 = b_{11}^2 b_{22}^2 \cdots b_{nn}^2 = \phi(1)\phi(2)\cdots\phi(n)$.

3. **Amritanshu Prasad, IMSc.** Show that

   $$\frac{q^{3n+3} + q^{3n-1} - q^{2n+2} - q^{2n+1} - q^{2n} - q^{2n-1} + 2q^n}{(q^2-1)(q-1)}$$

   is a polynomial in q with non-negative integer coefficients for all positive integers n.

   Solution. Firstly observe that $f(q) = (q^m - 1)/(q-1)$ is a polynomial in q with non-negative integer entries, since it is the sum of the geometric series $1 + q + \cdots + q^{m-1}$. Similarly, $g(q) = (q^{2m-1})(q^2-1)$ is a polynomial with non-negative integer coefficients. For all $n \geq 1$, at least one of n and $n-1$ is even. Therefore the expression

   $$f_n(q) = \frac{(q^n-1)(q^{n-1}-1)}{(q^2-1)(q-1)}$$

   being a product of a polynomial of type $f(q)$ and a polynomial of type $g(q)$, is also a polynomial with non-negative integer coefficients. Now, the expression in question is easily seen to be written as

   $$q^n(f_{n+2}(q) + f_n(q))$$

   whence it must have non-negative integer entries.

   Note. The polynomial $f_n(q)$ is a special case of a Gaussian binomial coefficient:

   $$\binom{n}{k}_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}.$$

   Solution received. Another solution to this problem using induction was received from Hari Kishan of D. N. College, Meerut.

4. **Priyamvad Srivastav, IMSc.** Let k be a positive integer. Suppose that $\{a_n\}_{n\geq 1}$ is a sequence of positive integers satisfying $a_m^k + a_n^k | m^k + n^k$ whenever $\gcd(m,n) = 1$. Show that $a_n = n$ for all n.

   Solution. Since $(1,1) = 1$, $2a_1^k | 2$, whence $a_1 = 1$.

   Claim. $a_n \equiv n \mod 2$ for every n.

   If n is even, then $n^k + 1$ is odd. Since $a_n^k + 1 | n^k + 1$, $a_n^k + 1$ is odd, so a_n is even. Similarly if n is odd, then using $(n,2) = 1$, we get $a_n^k + 2^k | n^k + 2^k$, so $a_n^k + 2^k$ is odd, which is only possible if a_n is odd.

   Proof by induction. Let $n \geq 3$ be a positive integer and assume that for $r < n$, $a_r = r$.

   Suppose that $a_n = m$. Since $a_n^k + 1 | n^k + 1$, we know that $m \leq n$. We consider two cases:

   Case I. If $(n,m) = 1$, then $m < n$. Since $m < n$, it follows by induction hypothesis that $a_m = m$. Moreover, $(n,m) = 1$ implies that $a_m^k + a_n^k | m^k + n^k$, so $2m^k | m^k + n^k$. Therefore $m|n$. This forces $m = 1$ and n to be odd. Let

   $$T = \{l \in \mathbf{N} : (l,j) = 1 \text{ for } 1 \leq k \leq n\}.$$

   If $l \in T$, then $a_k^l + 1$ divides $l^k + n^k$ as well $l^k + 1$. Hence $a_k^l + 1 | n^k - 1$. Let $a_l = u$ ($u \leq n-1$). We have $(l,u) = 1$ and $a_u = u$ (by induction hypothesis), so $a_u^k + a_l^k | u^k + l^k$, so $u|l$, whence $u = 1$. We have shown that $a_l = 1$ for all $l \in T$.

   Let q be a prime divisor of $2^k + 1$ and $\{p_1, \ldots, p_t\}$ be the set of distinct odd prime divisors of $n!$. That $r = qp_1, \ldots, p_t + 2$. Then r is comprime to each p_i and is odd. So $r \in T$ and hence, $a_r = 1$. Since $(r,2) = 1$, we have $a_r^k + 2^k | r^k + 2^k$ from which one can conclude that $2^k + 1 | r^k - 1$. Therefore $q | r^k - 1$. Also, since $r \equiv 2 \mod q$, $r^k - 1 \equiv 2^k - 1 \mod q$, so $q | 2^k - 1$. But q already divides $q^k + 1$, so $q = 2$, a contradiction.

   Case II. Suppose $(n,m) = r > 1$. Since $(n, n-1) = 1$, $m^k + (n-1)^k | n^k + (n-1)^k$, from which it follows that $m^k + (n-1)^k | n^k - m^k$. As r is coprime to $m^k + (n-1)^k$, $m^k + (n-1)^k | (n/r)^k - (m/r)^k$. But $m^k + (n-1)^k > (n-1)^k > (n/r)^k > (n/r)^k - (m/r)^k$, forcing $n = m$.

5. **S. Viswanath, IMSc.** Let $P_n(x)$ denote the nth Legendre polynomial given by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2-1)^n].$$

Evaluate the determinant

$$\begin{vmatrix} P_0(x_1) & P_1(x_1) & \cdots & P_{n-1}(x_1) \\ P_0(x_2) & P_1(x_2) & \cdots & P_{n-1}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ P_0(x_n) & P_1(x_n) & \cdots & P_{n-1}(x_n) \end{vmatrix}$$

Solution. We note that, for each non-negative integer i, $P_j(x)$ is a polynomial in x with leading coefficient $\frac{1}{2^j}\binom{2j}{j}$. In particular, $P_0 \equiv 1$. The linear span of the first j Legendre polynomials is the space of all polynomials of degree at most $j-1$. Therefore, by subtracting such a linear combination of the first j columns from the $j+1$st column, we can reduce the $j+1$st column to $\frac{1}{2^{j+1}}\binom{2(j+1)}{j+1} x_i^j$. Pulling out these constants leaves us with,

$$\left(\prod_{j=0}^{n} \frac{1}{2^j}\binom{2j}{j}\right) \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots & \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix},$$

a constant times the celebrated Vandermonde determinant. Therefore the determinant in question is

$$\prod_{j=0}^{n} \frac{1}{2^j}\binom{2j}{j} \prod_{1 \le i < j \le n} (x_j - x_i).$$

6. **S. Kesavan, IMSc.** Let $u_1, u_2, \ldots, u_N$ be a set of n linearly independent unit vectors in R^N. The $N \times N$ matrix $G = (g_{ij})$, where $g_{ij} = u_i \cdot u_j$ (the usual dot product of vectors) is called the Gram matrix associated to $u_1, \ldots, u_N$. Show that λ_1, the smallest eigenvalue of G satisfies

$$\lambda_1 > \frac{\det G}{e}.$$

Solution. The Gram matrix G is symmetric positive definite, so its eigenvalues are all positive. Let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$ be the eigenvalues of G. Then

$$\frac{\det G}{\lambda_1} = \prod_{i=2}^{N} \lambda_i \le \left(\frac{\Sigma_{i=2}^n \lambda_i}{N-1}\right)^{\frac{1}{N-1}}$$

by the AM-GM inequality. But since $\langle u_i, u_i \rangle = 1$ for all $1 \le i \le N$,

$$\sum_{i=2}^{n} \lambda_i = \mathrm{tr}(G) - \lambda_1 = N - \lambda_1 < N.$$

Thus

$$\frac{\det(G)}{\lambda_1} < \left(\frac{N}{N-1}\right)^{\frac{1}{N-1}} = \left(1 + \frac{1}{N-1}\right)^{\frac{1}{N-1}} < e$$

which proves the result.

7. **S. Kesavan, IMSc.** Let $\Omega \subset \mathbf{R}^N$ be a bounded and connected open set with continuous boundary $\partial\Omega$. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be such that $u > 0$ in Ω and $u = 0$ in $\partial\Omega$. If there exists a constant λ such that, for all $x \in \Omega$ and all $1 \le i, j \le N$,

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \lambda \delta_{ij},$$

show that Ω is a ball.

Solution. Since $\bar{\Omega}$ is compact and u is continuous, there exists a point $x_0 \in \Omega$ such that

$$u(x_0) = \max_{x \in \bar{\Omega}} u(x) = M > 0.$$

Let $r > 0$ be such that the open ball $B(x_0, r)$ with centre x_0 and radius r, is the largest ball centred at x_0 contained in Ω. Then there exists a point $x_1 \in \partial\Omega$ such that $|x_1 - x_0| = r$ and all points such that $x \in \mathbf{R}^N$ such that $|x - x_0| < r$ will lie in $\bar{\Omega}$.

By Taylor's theorem, we have, for any $x \in B(x_0, r)$,

$$u(x) = u(x_0) + \nabla u(x_0) \cdot (x - x_0) + \frac{1}{2}(x - x_0)^T D^2 u(\xi)(x - x_0),$$

where ξ is a point in the line segment joining x_0 to x, and $D^2 u(\xi)$ denotes the matrix of second derivatives of u at ξ and $(x - x_0)^T$ denotes the tranpose of the column vector $x - x_0$.

Since u attains its maximum at the interior point x_0, we have $\nabla u(x_0) = 0$ and using the hypothesis on second derivatives of u, we deduce that

$$u(x) = M + \frac{\lambda}{2}|x - x_0|^2. \tag{1}$$

By continuity, this is also valid for all x such that $|x - x_0| = r$. In particular, setting $x = x_1$, we get

$$0 = M + \frac{\lambda}{2}|x_1 - x_0|^2. \qquad (2)$$

But the right hand side of (1) depends only on $|x - x_0|$, and so (2) is valid for all points such that $|x - x_0| = r$. Since all such points lie in the closure of Ω and $u > 0$ in Ω, it follows that the sphere centred at x_0 with radius r must coincide with $\partial\Omega$. This proves that Ω is a ball.

Note. we also see that $\lambda < 0$ and

$$r = \sqrt{-\frac{2M}{\lambda}}.$$

International Conference and Workshop on Fractals and Wavelets

9–16 November, 2013

at

Rajagiri School of Engineering & Technology, Cochin, Kerala, India

Workshop: 9 to 12 November 2013.

Workshop is meant to give the fundamentals of Fractals and Wavelets to the participants.

Conference: 13 to 16 November 2013.

Conference will include Plenary talks, Invited talks and Contributory sessions.

Target Audience: Research scholars and Post graduate students in Mathematics, Science and Engineering, University/College teachers, Engineers, Mathematicians/Experts in Fractals/Wavelets.

Registration Deadline: 30th June 2013.

Technical Support: American Mathematical Society, Kerala Mathematical Association.

Proceedings will be published by Springer in Proceedings in Mathematics series.

Conference website: `http://www.icfwrajagiri.in`

Conference E-mail: `icfwrajagiri@gmail.com`

Location: Rajagiri School of Engineering & Technology, RajagiriValley, P.O., Cochin 682039, Kerala, India.

Contact:

VinodKumar. P. B.
Organising Convener
E-mail: `vinod_kumar@rajagiritech.ac.in`

Extracts From the Newsletter From the International Mathematical Union

2013 has been declared the International Year of Statistics ("Statistics 2013") by the following institutions: the American Statistical Association, the Institute of Mathematical Statistics, the International Biometric Society, the International Statistical Institute, the Bernoulli Society and the Royal Statistical Society. As we speak, the International Year of Statistics is a worldwide event supported by more than 1,700 organizations.

The aims of this event are as follows:

- To increase public awareness of the power and impact of Statistics on all aspects of society.
- To nurture Statistics as a profession, especially among young people.
- To promote creativity and development in the sciences of probability and statistics.

Why has 2013 been chosen? There are at least two main reasons: 300 years ago, in 1713, Jakob Bernoulli's work, "Ars Conjectandi", was published posthumously in Basel, eight years after his death. This work is considered the foundation of the combinatorial basis of the Theory of Probability. Moreover, 250 years ago, in 1763, Thomas Bayes's work, "An Essay towards solving a problem in the Doctrine of Chances", was also published posthumously, two years after the death of the author, and regarded as the fundamental basis of Bayesian Statistics.

Statistics has undergone a spectacular growth and is increasingly being applied to other sciences, technologies, medicine, biological sciences and industrial processes, etc., which makes it indispensable for our society. However, depending on countries, it could be taught very little in schools, even though knowledge of this subject is important for any citizen and vital in many university courses. To underline its role in education, ICMI and the International Association for Statistical Education (IASE) have organized the production of an

ICMI Study: "Challenges for Teaching and Teacher Education Study", published by Springer in 2011 (The 18th ICMI Study Series: New ICMI Study Series, Vol. 14) and edited by Carmen Batanero, Gail Burrill, and Chris Reading.

For its part, the Scientific Program of the International Congress of Mathematicians always includes a section (Section 12) devoted to Probability and Statistics, with between 10 and 13 talks by invited speakers.

Fully aware of the importance of Statistics, the IMU is supporting the International Year of Statistics and is planning some additional activities to be held at the ICM in Seoul in 2014 (round tables, presentation of IYS results) to underline "Statistics 2013".

The celebration of "Statistics 2013" will undoubtedly be a wonderful opportunity to extend relations between the IMU and the main statistical associations around the world.

Manuel de Leon
Member-at-large of the IMU Executive Committee

"ATCM+TIME 2013: From 12/07/2013 to 12/11/2013"

"ATCM+TIME 2013" a joint session of 18th Asian Technology Conference in Mathematics and 6th Technology & Innovations in Mathematics Education.

Location: Department of Mathematics, Indian Institute of Technology, Powai, Mumbai 400076, India.

URL: `atcm.mathandtech.org` and
`http://www.math.iitb.ac.in/TIME2013`

Description: The ATCM conferences are international conference addressing technology-based issues in all Mathematical Sciences. The 17th ATCM December 16–20, 2012 was held at SSR University, Bangkok, Thailand. About 400 participants coming from over 30 countries around the world participated in the conference.

The TIME conferences are national (Indian) conferences held every two years. TIME conferences serve a dual role: as a forum in which mathematics educators and teachers will come together to discuss and to probe major issues associated with the integration of technology in mathematics teaching and learning, and as a place where they can share their perspectives, personal experiences, and innovative teaching practices.

Contact:

Prof. Inder K. Rana, Convener
Department of Mathematics
I. I. T. Bombay Powai,
Mumbai 400076, India
E-mail: `ikr@math.iitb.ac.in`

Details of Workshop/Conferences in India

For details regarding Advanced Training in Mathematics Schools
Visit: `http://www.atmschools.org/`

Date: June 5–11, 2013
Name: National Workshop on Optimization Techniques and Their Applications (NWOTA 2013)
Location: Motilal Nehru National Institute of Technology Allahabad
Visit: `http://mnnit.ac.in/nwota2013/`

Date: July 1–3, 2013
Name: International Conference on Mathematical Modeling and Numerical Simulation
Location: Babasaheb Bhimrao Ambedkar University, Lucknow
Visit: `http://www.icmmans.org/`

Date: May 16–17, 2013
Name: Symposium on Recent Trends and Emerging Applications of Mathematical Science – 2013
Location: NIT Rourkela
Visit: `http://srteams2013.nitrkl.ac.in/`

Date: November 9–16, 2013
Name: Internationa Conference and Workshop on Fractals and Wavelets (ICFW)
Location: RAJAGIRI SCHOOL OF ENGINEERING & TECHNOLOGY, Cochin, Kerala
Visit: `http://www.icfwrajagiri.in/`

Details of Workshop/Conferences in Abroad

Date: July 1–4, 2013
Name: 2nd IMA Conference on Dense Granular Flows
Location: Isaac Newton Institute of Mathematical Sciences, Cambridge, United Kingdom
Visit: `http://www.ima.org.uk/conferences/conferences_calendar/dense_granular_flows.cfm`

Date: July 1–5, 2013
Name: Conference on Geometrical Analysis
Location: Centre de Recerca Matemática, Bellaterra, Barcelona
Visit: `http://www.crm.cat/2013/CGeometricalAnalysis/`

Date: July 1–5, 2013
Name: 28th Journées Arithmétiques JA 2013
Location: University Joseph Fourier Grenoble I, Grenoble, France
Visit: `http://www-fourier.ujf-grenoble.fr/ja2013/index-en.html`

Date: July 1–5, 2013
Name: 28th Journées Arithmétiques JA 2013
Location: University Joseph Fourier Grenoble I, Grenoble, France
Visit: `http://www-fourier.ujf-grenoble.fr/ja2013/index-en.html`

Date: July 1–5, 2013
Name: International conference on Sampling Theory and Applications 2013
Location: Jacobs University, Bremen, Germany
Visit: `http://www.jacobs-university.de/sampta`

Date: July 1–5, 2013
Name: The 6th Pacific RIM Conference on Mathematics 2013
Location: Sapporo Convention Center, Sapporo City, Japan
Visit: `http://www.math.sci.hokudai.ac.jp/sympo/130701/`

Date: July 1–12, 2013
Name: Advanced School and Workshop on Matrix Geometries and Applications
Location: The Abdus Salam International Centre for Theoretical Physics (ICTP), Trieste, Italy
Visit: `http://agenda.ictp.it/smr.php?2470`

Date: July 1–5, 2013
Name: Preconditioning of Iterative Methods-Theory and Applications 2013 (PIM 2013)
Location: Faculty of Civil Engineering, Czech Technical University in Prague, Czech Republic
Visit: `http://pim13.fsv.cvut.cz`

Date: July 1–5, 2013
Name: Oxford Conference on Challenges in Applied Mathematics (OCCAM)
Location: St Anne's College, Oxford, United Kingdom
Visit: `http://www.maths.ox.ac.uk/groups/occam/events/oxford-conference-challenges-applied-mathematics`

Date: July 1–5, 2013
Name: 7th International Summer School on Geometry, Mechanics and Control (ICMAT School)
Location: La Cristalera, Miraflores de la Sierra, Madrid, Spain
Visit: `http://gmcnetwork.org/drupal/?q=activity-detaill/867`

Date: July 1–12, 2013
Name: New Geometric Techniques in Number Theory
Location: Mathematical Sciences Research Institute, Berkeley, California
Visit: `http://www.msri.org/web/msri/scientific/workshops/summer-graduate-workshops/show/-/event/Wm9460`

Date: July 2–5, 2013
Name: The 4th International Conference on Matrix Analysis and Applications
Location: Konya, Turkey
Visit: `http://icmaa2013.selcuk.edu.tr/`

Date: July 3–5, 2013
Name: The 2013 International Conference of Applied and Engineering Mathematics
Location: Imperial College London, London, United Kingdom
Visit: `http://www.iaeng.org/WCE2013/ICAEM2013.html`

Date: July 3–6, 2013
Name: International Conference on Anatolian Communications in Nonlinear Analysis (ANCNA 2013)
Location: Abant Izzet Baysal University, Bolu, Turkey
Visit: `http://ancna.net/`

Date: July 7–13, 2013
Name: Seventh Czech-Slovak International Symposium on Graph Theory, Combinatorics, Algorithms and Applications
Location: Kosice, Slovakia
Visit: `http://csgt13.upjs.sk`

Date: July 8–10, 2013
Name: SIAM Conference on Control and Its Applications (CT13)
Location: Town and Country Resort and Convention Center, San Diego, California
Visit: `http://www.siam.org/meetings/ct13/`

Date: July 8–12, 2013
Name: AIM Workshop: Generalizations of chip-firing and the critical group
Location: American Institute of Mathematics, Palo Alto, California
Visit: `http://www.aimath.org/ARCC/workshops/chipfiring.html`

Date: July 8–12, 2013
Name: Mathematics of Planet Earth Australia 2013
Location: RMIT, Melbourne, VIC, Australia
Visit: `http://mathsofplanetearth.org.au/events/2013/`

Date: July 8–12, 2013
Name: Mathematics of Planet Earth 2013 – Pan-Canadian Thematic Program – Climate Change and the Ecology of Vector-borne Diseases
Location: Fields Institute, CDM, Toronto, Canada
Visit: `http://www.crm.umontreal.ca/act/theme/theme_2013_1_en/vector_borne13_e.php`

Date: July 8–12, 2013
Name: 2013 SIAM Annual Meeting (AN13)

Location: Town and Country Resort & Convention Center, San Diego, California
Visit: `http://www.siam.org/meetings/an13/`

Date: July 8–12, 2013
Name: Topics in Numerical Analysis for Differential Equations
Location: Instituto de Ciencias Matemáticas-ICMAT, campus de Cantoblanco, Madrid, Spain
Visit: `http://www.icmat.es/congresos/tnade2013/index.html`

Date: July 8–26, 2013
Name: RTG Summer School on Microlocal Analysis and Inverse Problems
Location: University of Washington, Seattle
Visit: `http://www.math.washington.edu/ipde/summer/index2013.html`

Date: July 9–10, 2013
Name: Random Perturbations and Statistical Properties of Dynamical Systems
Location: Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany
Visit: `http://www.mis.mpg.de/calendar/conferences/2013/randyn13/`

Date: July 9–12, 2013
Name: Moduli-Operads-Deformations
Location: Buskerud University College, Kongsberg, Norway
Visit: `http://www.agmp.eu/mod13`

Date: July 9–13, 2013
Name: Satellite Summer School to the 7th International Conference on Levy Processes: Theory and Applications
Location: Mathematical Research and Conference Center of the Institute of Mathematics of the Polish Academy of Sciences, Bedlewo, Poland
Visit: `http://bcc.impan.pl/13LevySchool/`

Date: July 11–12, 2013
Name: Variational methods and partial differential equations workshop on the occasion of Michel Willem's 60th birthday
Location: Université Catholique de Louvain, Louvain-la-Neuve, Belgium
Visit: `http://sites.uclouvain.be/pde2013/`

Date: July 14–19, 2013
Name: The Sixth International Congress of Chinese Mathematicians (ICCM)
Location: Opening ceremony on July 14 in the Big Hall of the Grand Hotel, Taipei, Taiwan. Lectures and invited talks from July 15–19 on the campus of National Taiwan University, Taipei, Taiwan
Visit: `http://iccm.tims.ntu.edu.tw/`

Date: July 15–19, 2013
Name: Finite Dimensional Integrable Systems 2013
Location: CIRM, Marseille, France
Visit: `http://www.cpt.univ-mrs.fr/fdis13/`

Date: July 15–19, 2013
Name: General Algebra and its Applications GAIA2013
Location: La Trobe University, City Campus, Melbourne, Victoria, Australia
Visit: `http://gaia.ltumathstats.com`

Date: July 15–19, 2013
Name: ICERM IdeaLab 2013: Weeklong Program for Postdoctoral Researchers
Location: ICERM, Providence, Rhode Island
Visit: `http://icerm.brown.edu/idealab_2013`

Date: July 15–19, 2013
Name: 7th International Conference on Lévy Processes: Theory and Applications
Location: University of Wroclaw and Wroclaw University of Technology, Wroclaw, Poland
Visit: `http://bcc.impan.pl/13Levy/`

Date: July 15–19, 2013
Name: Symmetries of Discrete Systems and Processes
Location: Decin, Czech Republic, North Bohemia
Visit: `http://spmdd.fjfidecin.cz/conference-details`

Date: July 15–19, 2013
Name: Women in Shape (WiSh): Modeling Boundaries of Objects in 2- and 3-Dimensions (in cooperation with AWM)
Location: Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, California
Visit: `http://www.ipam.ucla.edu/programs/awm2013/`

Date: July 15–19, 2013
Name: Workshop on Interactions between Dynamical Systems and Partial Differential Equations (JISD2013)
Location: School of Mathematics and Statistics, Universitat Politècnica de Catalunya, Barcelona, Spain
Visit: `http://www.ma1.upc.edu/recerca/seminaris-recerca/jisd2013/jisd2013`

Date: July 15–20, 2013
Name: Stochasticity in Biology: Is Nature Inherently Random?
Location: Montreal, Canada
Visit: `http://www.crm.umontreal.ca/act/theme/theme_2013_2_en/biodiversity_environment13_e.php`

Date: July 15–27, 2013
Name: XVI Summer Diffiety School
Location: Pomorski Park Naukowo-Technologiczny, Gdynia, Poland
Visit: `http://sites.google.com/site/levicivitainstitute/xvi-summer-diffiety-school`

Date: July 15–August 2, 2013
Name: School and Workshop on Geometric Measure Theory and Optimal Transport
Location: International Centre for Theoretical Physics (ICTP), Trieste, Italy
Visit: `http://agenda.ictp.it/smr.php?2459/`

Date: July 20–25, 2013
Name: European Meeting of Statisticians
Location: Eotvos Lorand University, Budapest, Hungary
Visit: `http://www.ems2013.eu`

Date: July 21–26, 2013
Name: The 2013 Progress on Difference Equations
Location: Bialystok University of Technology, Bialystok, Poland
Visit: `http://katmat.pb.bialystok.pl/pode13/`

Date: July 21–27, 2013
Name: Applied Topology – Bedlewo 2013
Location: Bedlewo, near Poznan, Poland
Visit: `http://bcc.impan.pl/13AppTop/`

Date: July 22–25, 2013
Name: MIP2013: Mixed Integer Programming Workshop

Location: University of Wisconsin-Madison, Madison, Wisconsin

Date: July 22–25, 2013
Name: Moscow International Conference "Israel Gelfand Centenary"
Location: Russian Academy of Science, Leninski Ave., Moscow, Russia
Visit: http://gelfand100.iitp.ru/

Date: July 22–26, 2013
Name: Mathematics of Planet Earth 2013 – Pan-Canadian Thematic Program – Biodiversity in a Changing World
Location: CRM, Cambam, Montréal, Canada
Visit: http://www.crm.umontreal.ca/act/theme/theme_2013_1_en/changing_world13_e.php

Date: July 22–26, 2013
Name: Planetary Motions, Satellite Dynamics, and Spaceship Orbits
Location: Centre de recherches mathématiques, Montréal, Canada
Visit: http://www.crm.umontreal.ca/2013/Satellites13/index_e.php

Date: July 22–26, 2013
Name: Positivity VII
Location: Leiden University, Leiden, The Netherlands
Visit: http://websites.math.leidenuniv.nl/positivity2013/

Date: July 22–26, 2013
Name: Samuel Eilenberg Centenary Conference
Location: Warsaw, Poland
Visit: http://eilenberg100.ptm.org.pl

Date: July 22–26, 2013
Name: VII International Meeting on Lorentzian Geometry
Location: University of Sao Paulo, Sao Paulo, Brazil
Visit: http://www.ime.usp.br/~gelosp2013/

Date: July 22–August 9, 2013
Name: Complex Geometry
Location: Institute for Mathematical Sciences, National University of Singapore, Singapore
Visit: http://www2.ims.nus.edu.sg/Programs/013complex/index.php

Date: July 23–26, 2013
Name: 4th Canadian Conference on Nonlinear Solid Mechanics (CanCNSM2013)
Location: McGill University, Montreal, Quebec, Canada
Visit: http://cancnsm2013.mcgill.ca/index.html

Date: July 29–August 2, 2013
Name: 36th Conference on Stochastic Processes and their Applications
Location: Boulder, Colorado
Visit: http://math.colorado.edu/spa2013/
E-mail: brian.rider@colorado.edu

Date: July 29–August 9, 2013
Name: Introduction to the Mathematics of Seismic Imaging
Location: Mathematical Sciences Research Institute, Berkeley, California
Visit: http://www.msri.org/web/msri/scientific/workshops/summer-graduate-workshops/show/-/event/Wm9427

Date: July 31–August 10, 2013
Name: Workshop and Conference on the Topology and Invariants of Smooth 4-Manifolds (First Announcement)
Location: University of Minnesota, Twin Cities, Minnesota
Visit: `http://www.math.umn.edu/~akhmedov/UMN2013`

Date: August 1–3, 2013
Name: 15th IMS New Researchers Conference
Location: Centre de recherches mathématiques, Montréal, Canada
Visit: `http://www.math.mcgill.ca/nrc2013/index.html`

Date: August 3–11, 2013
Name: Groups St. Andrews 2013
Location: University of St. Andrews, St. Andrews, Fife, Scotland, UK
Visit: `http://www.groupsstandrews.org/2013/index.shtml`

Date: August 5–9, 2013
Name: 1st Mathematical Congress of the Americas
Location: Guanajuato, Mexico
Visit: `http://www.mca2013.org`

Date: August 5–9, 2013
Name: AIM Workshop: Sections of convex bodies
Location: American Institute of Mathematics, Palo Alto, California
Visit: `http://aimath.org/ARCC/workshops/sectionsconvex.html`

Date: August 5–9, 2013
Name: Quantum/Affine Schubert Calculus
Location: University of Oregon, Eugene, Oregon
Visit: `http://pages.uoregon.edu/njp/qsc.html`

Date: August 5–9, 2013
Name: XXII Rolf Nevanlinna Colloquium
Location: Helsinki, Finland
E-mail: `kirsi.peltonen@tkk.fi`

Date: August 5–16, 2013
Name: Hypergeometric Functions and Representation Theory
Location: Cimpa-Unesco-Mesr-Mineco-Mongolia Research School, National University of Mongolia, Ulaanbaatar, Mongolia
Visit: `http://www.cimpa-icpam.org/spip.php?article484`

Date: August 5–16, 2013
Name: International Cimpa School: "New Trends in Applied Harmonic Analysis Sparse Representations, Compressed Sensing and Multifractal Analysis"
Location: Mar del Plata, Argentina
Visit: `http://nuhag.eu/cimpa13`

Date: August 11–12, 2013
Name: A quasiconformal life: Celebration of the legacy and work of F. W. Gehring
Location: University of Helsinki, Helsinki, Finland
Visit: `https://wiki.helsinki.fi/display/Gehring/Home`

Date: August 11–17, 2013
Name: 3rd Mile High Conference on Nonassociative Mathematics
Location: University of Denver, Denver, Colorado
Visit: `http://www.math.du.edu/milehigh`

Date: August 12–15, 2013
Name: International Conference on Algebra in Honour of Patrick Smith and John Clark's 70th Birthdays
Location: Balikesir, Turkey
Visit: `http://ica.balikesir.edu.tr/`

Date: August 12–15, 2013
Name: 12th International Workshop on Dynamical Systems and Applications
Location: Atilim University, Ankara, Turkey
Visit: `http://iwdsa2013.atilim.edu.tr/`

Date: August 12–17, 2013
Name: 18th International Summer School on Global Analysis and Applications
Location: Juraj Pales Institute, Bottova 15, Levoca, Slovakia
Visit: `http://www.lepageri.eu/ga2013/`

Date: August 12–October 11, 2013
Name: Mathematical Horizons for Quantum Physics 2
Location: Institute for Mathematical Sciences, National University of Singapore, Singapore
Visit: `http://www2.ims.nus.edu.sg/Programs/013mhqp/index.php`

Date: August 12–16, 2013
Name: AIM Workshop: Computable stability theory
Location: American Institute of Mathematics, Palo Alto, California
Visit: `http://www.aimath.org/ARCC/workshops/computestab.html`

Date: August 12–16, 2013
Name: Random Trees
Location: Centre de Recherches Mathématiques, Montréal, Canada
Visit: `http://www.crm.math.ca/Biodiversity2013/`

Date: August 12–16, 2013
Name: 2nd Strathmore International Mathematics Conference (SIMC-2013)
Location: Strathmore University, Nairobi, Kenya
Visit: `http://www.strathmore.edu/maths2/`

Date: August 18–24, 2013
Name: International Conference "Differential Equations. Function Spaces. Approximation Theory" dedicated to the 105th anniversary of the birthday of S. L. Sobolev
Location: Sobolev Institute of Mathematics, Novosibirsk, Russia
Visit: `http://www.math.nsc.ru/conference/sobolev/105/english/`

Date: August 19–23, 2013
Name: Differential Geometry and its Applications
Location: Masaryk University, Old campus, Brno, Czech Republic
Visit: `http://www.math.muni.cz/DGA2013/`

Date: August 19–23, 2013
Name: Fifth Montreal Problem Solving Workshop, A CRM-Mprime Event

Location: Centre de Recherches Mathématiques, Montréal, Canada
Visit: http://www.crm.umontreal.ca/probindustriels2013/index_e.php

Date: August 19–23, 2013
Name: GAP XI Pittsburgh – Geometry and Physics "Seminaire Itinerant"
Location: University of Pittsburgh, Pittsburgh, Pennsylvania
Visit: http://www.geometryandphysics.org/

Date: August 19–September 13, 2013
Name: Infectious Disease Dynamics
Location: Isaac Newton Institute for Mathematical Sciences, Cambridge, United Kingdom
Visit: http://www.newton.ac.uk/programmes/IDD/

Date: August 19–December 20, 2013
Name: Mathematical General Relativity
Location: Mathematical Sciences Research Institute, Berkeley, California
Visit: http://www.msri.org/web/msri/scientific/programs/show/-/event/Pm8946

Date: August 19–December 20, 2013
Name: Optimal Transport: Geometry and Dynamics
Location: Mathematical Sciences Research Institute, Berkeley, California
Visit: http://www.msri.org/web/msri/scientific/programs/show/-/event/Pm8952

Date: August 22–23, 2013
Name: Connections for Women on Optimal Transport: Geometry and Dynamics
Location: Mathematical Sciences Research Institute, Berkeley, California
Visit: http://www.msri.org/web/msri/scientific/workshops/all-workshops/show/-/event/Wm9225

Date: August 23–25, 2013
Name: 14th International Pure Mathematics Conference 2013
Location: Islamabad, Pakistan
Visit: http://www.pmc.org.pk

Date: August 24–31, 2013
Name: Second International Conference "Mathematics in Armenia: Advances and Perspectives" dedicated to the 70th anniversary of foundation of Armenian National Academy of Sciences
Location: YSU Guesthouse, Tsaghkadzor, Armenia
Visit: http://mathconf.sci.am

Date: August 25–27, 2013
Name: The 7th Global Conference on Power Control and Optimization (PCO'2013)
Location: Prague, Czech Republic
Visit: http://www.pcoglobal.com/Prague.htm
E-mail: pcoglobal@gmail.com

Date: August 26–29, 2013
Name: 2nd International Eurasian Conference on Mathematical Sciences and Applications
Location: Sarajevo, Bosnia and Herzegovina
Visit: http://www.iecmsa.org

Date: August 26–30, 2013
Name: AIM Workshop: Rigorous computation for infinite dimensional nonlinear dynamics

Location: American Institute of Mathematics, Palo Alto, California
Visit: `http://www.aimath.org/ARCC/workshops/computenonlinear.html`

Date: August 26–30, 2013
Name: Geometric Function Theory and Applications 2013
Location: Isik University, Campus of Sile, Istanbul, Turkey
Visit: `http://gfta.isikun.edu.tr`

Date: August 26–30, 2013
Name: International Conference AMMCS-2013 (Applied Mathematics, Modeling and Computational Science)
Location: Waterloo, Ontario, Canada
Visit: `http://www.ammcs2013.wlu.ca`

Date: August 26–30, 2013
Name: Introductory Workshop on Optimal Transport: Geometry and Dynamics
Location: Mathematical Sciences Research Institute, Berkeley, California
Visit: `http://www.msri.org/web/msri/scientific/workshops/all-workshops/show/-/event/Wm9226`

Date: August 26–September 7, 2013
Name: DGS 2013 – International Conference and Advanced School Planet Earth, Dynamics, Games and Science
Location: Calouste Gulbenkian Foundation (FCL) and Escola Superior de Economia e Gestão, Universidade Técnica de Lisboa (ISEG-UTL), Lisbon, Portugal
Visit: `http://sqig.math.ist.utl.pt/cim/mpe2013/DGS`

Date: August 27–30, 2013
Name: The 44th Annual Iranian Mathematics Conference
Location: Ferdowsi University of Mashhad, Mashhad, Iran
Visit: `http://imc44.um.ac.ir/index.php?&newlang=eng`

Date: August 27–31, 2013
Name: The 9th William Rowan Hamilton Geometry and Topology Workshop, on Geometry and Groups after Thurston
Location: Hamilton Mathematics Institute, Trinity College, Dublin, Ireland
Visit: `http://www.hamilton.tcd.ie/events/gt/gt2013.htm`

Date: August 27–December 20, 2013
Name: Mathematical Challenges in Quantum Information
Location: Isaac Newton Institute for Mathematical Sciences, Cambridge, United Kingdom
Visit: `http://www.newton.ac.uk/programmes/MQI/`

Date: August 28–September 2, 2013
Name: Gelfand Centennial Conference: A View of 21st Century Mathematics
Location: Massachusetts Institute of Technology, Cambridge, Massachusetts
Visit: `http://math.mit.edu/conferences/Gelfand/index.php`

Date: August 30–31, 2013
Name: 8th International Symposium on Trustworthy Global Computing (TGC 2013)
Location: Fac. de Ciencias Económicas – UBA Av Córdoba Avenida Córdoba 2122 C1120AAQ, Buenos Aires, Argentina
Visit: `http://sysma.lab.imtlucca.it/tgc2013/`

Date: September 1–5, 2013
Name: Motivic Galois Groups
Visit: `http://renyi.mta.hu/~szamuely/mgg.html`

Date: September 1–6, 2013
Name: Kangro-100, Methods of Analysis and Algebra, International Conference dedicated to the Centennial of Prof. Gunnar Kangro
Location: University of Tartu, Tartu, Estonia
Visit: `http://kangro100.ut.ee`

Date: September 1, 2013–August 31, 2014
Name: Call for Research Programmes 2013–2014
Location: Centre de Recerca Matemàtica, Bellaterra, Barcelona, Spain
Visit: `http://www.crm.cat/`

Date: September 1–December 20, 2013
Name: Research Program on Automorphisms of Free Groups: Algorithms, Geometry and Dynamics
Location: Centre de Recerca Matemàtica, Bellaterra, Barcelona, Spain
Visit: `http://www.crm.cat/2012/RPAutomorphisms`

Date: September 2–5, 2013
Name: XXII International Fall Workshop on Geometry and Physics
Location: University of Évora, Évora, Portugal
Visit: `http://www.ifwgp2013.uevora.pt/`

Date: September 2–6, 2013
Name: School and Workshop on Conformal Blocks, Vector Bundles on Curves and Moduli of Curves
Location: Mathematics Department, G. Castelnuovo Sapienza, Universita di Roma, Rome, Italy
Visit: `http://conformalmoduli.sciencesconf.org/`

Date: September 3–4, 2013
Name: Connections for Women: Mathematical General Relativity
Location: Mathematical Sciences Research Institute, Berkeley, California
Visit: `http://www.msri.org/web/msri/scientific/workshops/programmatic-workshops/show/-/event/Wm9551`

Date: September 3–6, 2013
Name: CAI 2013: 5th Conference on Algebraic Informatics
Location: ERISCS and IML, Aix-Marseille University, IGESA, Porquerolles Island, France
Visit: `http://iml.univ-mrs.fr/ati/conferences/CAI2013/`

Date: September 4–6, 2013
Name: Loday Memorial Conference
Location: Institut de Recherche Mathématique Avancée (IRMA), Strasbourg, France
Visit: `http://www-irma.u-strasbg.fr/article1351.html`

Date: September 8–14, 2013
Name: Combinatorial Methods in Topology and Algebra
Location: Il Palazzone, Cortona, Italy
Visit: `http://www.cometa2013.org/`

Date: September 9–11, 2013
Name: Aachen Conference on Computational Engineering Science (AC.CES)
Location: AC.CES takes place at RWTH Aachen University (SuperC building), Germany
Visit: `http://www.ac-ces.rwth-aachen.de/`

Date: September 9–11, 2013
Name: S.Co. 2013 – Complex Data Modeling and Computationally Intensive Statistical Methods for Estimation and Prediction

Location: Politecnico di Milano, Milano, Italy
Visit: http://mox.polimi.it/sco2013/

Date: September 9–11, 2013
Name: Seventh International Workshop Meshfree Methods for Partial Differential Equations
Location: Universität Bonn, Germany
Visit: http://wissrech.ins.uni-bonn.de/meshfree

Date: September 9–13, 2013
Name: AIM Workshop: Definability and decidability problems in number theory
Location: American Institute of Mathematics, Palo Alto, California
Visit: http://aimath.org/ARCC/workshops/definabilityinnt.html

Date: September 9–13, 2013
Name: European Conference on Combinatorics, Graph Theory and Applications – Eurocomb 2013
Location: National Research Council of Italy (CNR), Pisa, Italy
Visit: http://www.eurocomb2013.it/

Date: September 9–13, 2013
Name: Introductory Workshop: Mathematical General Relativity
Location: Mathematical Sciences Research Institute, Berkeley, California
Visit: http://www.msri.org/web/msri/scientific/workshops/programmatic-workshops/show/-/event/Wm9552

Date: September 9–December 6, 2013
Name: ICERM Semester Program on "Low-Dimensional Topology, Geometry, and Dynamics"
Location: Institute for Computational and Experimental Research in Mathematics (ICERM), Providence, Rhode Island
Visit: http://icerm.brown.edu/sp-f13

Date: September 11–13, 2013
Name: BioDynamics 2013
Location: Bristol, United Kingdom
Visit: http://www.bio-dynamics2013.org

Date: September 11–13, 2013
Name: 14th IMA Conference on Mathematics of Surfaces
Location: University of Birmingham, United Kingdom
Visit: http://www.ima.org.uk/conferences/conferences_calendar/14th_mathematics_of_surfaces.cfm

Date: September 11–14, 2013
Name: The Sixth International Workshop on Differential Equations and Applications
Location: Izmir University of Economics, Izmir, Turkey
Visit: http://dm.ieu.edu.tr/wdea2013

Date: September 15–20, 2013
Name: ICERM Workshop: Exotic Geometric Structures
Location: ICERM, Providence, Rhode Island
Visit: http://icerm.brown.edu/sp-f13-w1

Date: September 16–20, 2013
Name: Mathematics for an Evolving Biodiversity
Location: Centre de Recherches Mathématiques, Montréal, Canada
Visit: http://www.crm.math.ca/Biodiversity2013/

Date: September 16–20, 2013
Name: MatTriad'2013 – Conference on Matrix Aanalysis and its Applications
Location: Herceg-Novi, Montenegro
Visit: `http://mattriad2013.pmf.uns.ac.rs`

Date: September 16–October 11, 2013
Name: Mathematics and Physics of the Holographic Principle
Location: Isaac Newton Institute for Mathematical Sciences, Cambridge, United Kingdom
Visit: `http://www.newton.ac.uk/programmes/HOL/`

Date: September 21–23, 2013
Name: The First International Conference on New Horizons in Basic and Applied Science Session "Aspects of Mathematics and its Applications"
Location: Hurghada, Egypt
Visit: `http://www.nhbas2013.com`

Date: September 21–27, 2013
Name: "Wavelets and Related Multiscale Methods" within ICNAAM2013: 11th International Conference of Numerical Analysis & Applied Mathematics
Location: Rodos Palace Hotel, Rhodes, Greece
Visit: `http://www.icnaam.org/sessions_minisymposia.htm`

Date: September 22–26, 2013
Name: 18th Annual cum 3rd International Conference of Gwalior Academy of Mathematical Sciences (GAMS)
Location: Department of Mathematics Maulana Azad National Institute of Technology, Bhopal, India 462051
Visit: `http://www.gamsinfo.com` and `http://www.manit.ac.in`

Date: September 23–27, 2013
Name: Mathematics of Sequence Evolution: Biological Models and Applications
Location: Centre de Recherches Mathématiques, Montréal, Canada
Visit: `http://www.crm.math.ca/Biodiversity2013/`

Date: September 28–29, 2013
Name: Mathematics of Planet Earth 2013 – Pan-Canadian Thematic Program – Mathematical Modeling of Indigenous Population Health
Location: BIRS, Banff, Canada
Visit: `http://www.crm.umontreal.ca/act/theme/theme_2013_1_en/indigenous_population13_e.php`

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`www.ramanujanmathsociety.org`