Zermelo Navigation Problem and the Finsler-Poincaré Disc

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1. Introduction

In 1931, Zermelo [13] considered the following problem: Suppose a ship is sailing on open calm water and a mild wind blows along. How must the ship be steered in order to reach a given destination in the shortest time? Zermelo found an answer to his question assuming that the open sea was $\mathbb{R}^2$ with distances measured by the Euclidean metric. Z. Shen [12] generalized this problem to the setting where the sea is an arbitrary differentiable manifold M equipped with a Riemannian metric h and found that, when the wind is time-independent, the paths of shortest time are the geodesics of a certain Randers metric, which is an interesting non-Riemannian Finsler metric. Subsequently, Bao, Robles and Shen [3] showed that every strongly convex Randers metric arises as a solution to Zermelo's navigation problem on some Riemannian landscape (M, h) under the influence of wind, an appropriate vector field W on M with $|W| < 1$. More recently, Huang and Mo [6] studied the geodesics of Finsler metrics via Zermelo navigation. A geometric description of the geodesics for any Finsler metric and any homothetic field in terms of navigation representation was worked out, generalizing a result previously only known in the case of Randers metrics.

In this article, we first introduce the notion of a Finsler metric with some examples and then briefly sketch a description of the Navigation problem on a Riemannian manifold. Then we consider a Riemannian metric on the open disc $\mathbb{D}^2$ of radius 2 with a vector field V such that the resultant metric becomes a Finsler-Randers metric $F = \alpha + \beta$, where α is the usual Poincaré metric on $\mathbb{D}^2$ and β is a 1-form. This metric is called Finsler-Poincaré metric. In fact, this metric is not symmetric and we will finally see that the distance from centre of the disc to its boundary is infinite, whereas the distance from boundary of the disc to its centre is $\log 2$. Though, on the face of it, the non-symmetry of this distance feels a bit unnatural since one has mostly come across symmetric distance functions, it indeed seems more natural in real life as pointed out by Gromov in [5] with the following analogy: *The effort of climbing up to the top of a mountain, in real life as well as in mathematics, is not at all the same as descending back to the starting point.*

2. Finsler Metric

Although the origin of Finsler geometry lies in the famous lecture of Riemann in his Habilitationsvortrag '*On the Hypotheses, which lie at the Foundations of Geometry*' in 1854, where the possibility of a more general metric than a quadratic metric, in particular a metric which is fourth root of a quartic differential form, was discussed, the general case was revived by Paul Finsler in 1918, when he submitted his thesis under the direction of Carathéodory. Subsequently, the theory was further developed by L. Berwald, E. Cartan, H. Rund, M. Matsumoto, S.-S. Chern and several others. Finsler geometry is indeed an extension of Riemannian geometry. Instead of an inner product, one prescribes the so called Minkowski norms on every tangent space. We now define a Minkowski norm:

Minkowski norm: A non negative function F defined on a finite dimensional vector space V is said to be a Minkowski norm, if the following conditions are satisfied:

- $F(y) = 0$ if and only if $y = 0$;
- $F(\lambda y) = \lambda F(y)$ for any $y \in V$ and $\lambda > 0$;
- F is C^∞ on $V - \{0\}$ such that for any $y \in V$, the bilinear symmetric functional g_y on V, defined as $g_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}[F^2(y + su + tv)]_{s=t=0}$, is an inner product; i.e., for a fixed basis $\{b_i\}$ of V and $y = y^i b_i \neq 0$ the matrix $(g_{ij}(y))$ defined by $g_{ij}(y) := g_y(b_i, b_j) = \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j}[F^2](y)$ should be positive definite.

The inner product g_y is called fundamental form in the y direction. The pair (V, F) is called a Minkowski space. A Minkowski norm is said to be reversible if $F(-y) = F(y)$ for all $y \in V$.

Example 2.1. Let $V = \mathbb{R}^n$ and for $y = (y^i) \in \mathbb{R}^n$, $|y| := \sqrt{\sum_{i=1}^n (y^i)^2}$ be standard Euclidean norm, then $F(y) = |y|$ is a Minkowski norm and $(\mathbb{R}^n, F)$ is a Minkowski space.

Example 2.2. Let $V = \mathbb{R}^2$ and for $y = (y^1, y^2) \in \mathbb{R}^2$, let $F(y) = ((y^1)^4 + (y^2)^4)^{\frac{1}{4}}$, called quartic norm. This F is not a Minkowski norm on $\mathbb{R}^2$, because $\det(g_{ij}) = \frac{3(y^1)^2(y^2)^2}{(y^1)^4+(y^2)^4}$ vanishes on co-ordinate axes and hence positive definiteness of matrix (g_{ij}) is violated at some nonzero y.

Example 2.3. Let $V = \mathbb{R}^2$ and for $y = (y^1, y^2) \in \mathbb{R}^2$, let

$$F(y) = \sqrt{\sqrt{(y^1)^4 + (y^2)^4} + \lambda[(y^1)^2 + (y^2)^2]};$$

this may be viewed as a perturbation of the quartic norm. In this case we have

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \lambda + \frac{(y^1)^2((y^1)^4+3(y^2)^4)}{((y^1)^4+(y^2)^4)^{\frac{3}{2}}} & \frac{-2(y^1)^3(y^2)^3)}{((y^1)^4+(y^2)^4)^{\frac{3}{2}}} \\ \frac{-2(y^1)^3(y^2)^3)}{((y^1)^4+(y^2)^4)^{\frac{3}{2}}} & \lambda + \frac{(y^2)^2(3(y^1)^4+(y^2)^4)}{((y^1)^4+(y^2)^4)^{\frac{3}{2}}} \end{pmatrix} \tag{2.1}$$

Hence $\det(g_{ij}) = \lambda^2 + \lambda\frac{((y^1)^2+(y^2)^2)^3}{((y^1)^4+(y^2)^4)^{\frac{3}{2}}} + \frac{3(y^1)^2(y^2)^2}{(y^1)^4+(y^2)^4}$, and $\mathrm{trace}(g_{ij}) = 2\lambda + \frac{((y^1)^2+(y^2)^2)^3}{((y^1)^4+(y^2)^4)^{\frac{3}{2}}}$. If $\lambda > 0$, both determinant and trace of the matrix (g_{ij}) are positive away from the origin and hence both its eigen values are positive. Therefore the matrix (g_{ij}) is positive definite. Hence F is a Minkowski norm for positive λ. And, as seen in Example 2.2, it is not a Minkowski norm on $\mathbb{R}^2$ if $\lambda = 0$.

Finsler Metric: Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$ and by $TM := \cup_{x\in M} T_xM$ the tangent bundle of M. Each element of TM is of the form (x, y), where $x \in M$ and $y \in T_xM$. A Finsler metric on M is a function $F : TM \to [0, \infty)$ with the following properties:

(a) $F(x, y)$ is C^∞ on slit tangent bundle $TM_0 := TM - \{0\}$.
(b) The restriction of F, $F_x(y) := F(x, y)$ on each tangent space T_xM is a Minkowski norm.

A Finsler metric $F = F(x, y)$ on a manifold is said to be reversible if $F(x, -y) = F(x, y)$ for all $y \in T_xM$. Normally, one does not impose this reversibility condition on a Finsler metric, because it excludes some interesting examples such as a Randers metric. A Finsler metric F on M is said to be Riemannian, if the restriction of F, $F_x(y) := F(x, y)$ on T_xM is a Euclidean norm for any $x \in M$; that is, $F_x(y) = \sqrt{\langle y, y\rangle_x}$, for any $y \in T_xM$, where $\langle\, , \rangle_x$ is an inner product on T_xM. One usually denotes a Riemannian metric by a family of inner products $g_x = \langle y, y\rangle_x$ on tangent spaces T_xM.

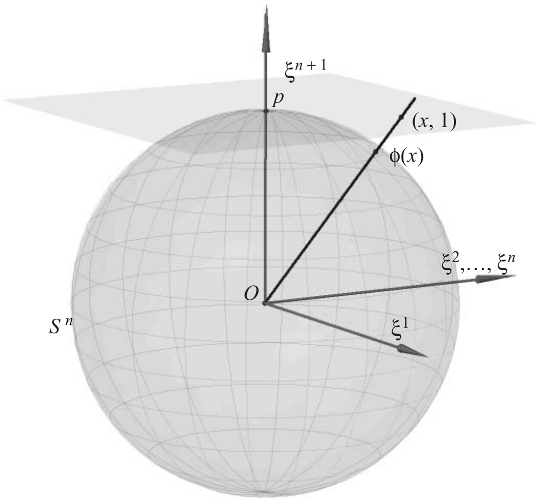

Figure 1.

The Riemannian metrics, which are reversible Finsler metrics, are important examples of Finsler metrics.

Example 2.4 (Euclidean Metric). Let $|y| := \sqrt{\sum_{i=1}^n (y^i)^2}$, for $y = (y^i) \in \mathbb{R}^n$ be the standard Euclidean norm on $\mathbb{R}^n$. Considering the identification $T_x\mathbb{R}^n \cong \mathbb{R}^n$, define $\alpha_0 = \alpha_0(x, y)$ by $\alpha_0 := |y|$, for $y \in \mathbb{R}^n$, then α_0 is a Finsler metric called the *standard Euclidean metric*.

Example 2.5 (Spherical Metric). Let S^n be the standard unit sphere in $\mathbb{R}^{n+1}$ defined as

$$S^n := \left\{ \xi = (\xi^i) \in \mathbb{R}^{n+1} : |\xi| = \sqrt{\sum_{i=1}^{n+1} (\xi^i)^2} = 1 \right\}.$$

Every tangent vector $\eta \in T_\xi S^n$ can be identified with a vector in $\mathbb{R}^{n+1}$ in a natural way. The induced metric g on S^n is defined by $g(\eta) = \|\eta\|_\xi$, for $\eta \in T_\xi S^n \subset \mathbb{R}^{n+1}$, where $\|\cdot\|_\xi$ denotes the induced Euclidean norm on $T_\xi S^n$. Let $\phi : \mathbb{R}^n \to S^n \subset \mathbb{R}^{n+1}$ be defined by (Figure 1)

$$\phi(x) := \left(\frac{x}{\sqrt{1 + |x|^2}}, \frac{1}{\sqrt{1 + |x|^2}} \right). \tag{2.2}$$

Then ϕ pulls back g on the upper hemisphere to a Riemannian metric α_{+1} on $\mathbb{R}^n$, which is given by

$$\alpha_{+1}(y) = \sqrt{\phi^*(g)(y, y)} = \sqrt{g(d\phi(y), d\phi(y))} = \|d\phi(y)\|_x,$$

for $y \in T_x\mathbb{R}^n \cong \mathbb{R}^n$. After simplification, we have

$$\alpha_{+1} := \alpha_{+1}(y) = \frac{\sqrt{|y|^2 + (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 + |x|^2}, \tag{2.3}$$

where notation $\langle\, , \rangle$ denotes usual Euclidean inner product on $\mathbb{R}^n$.

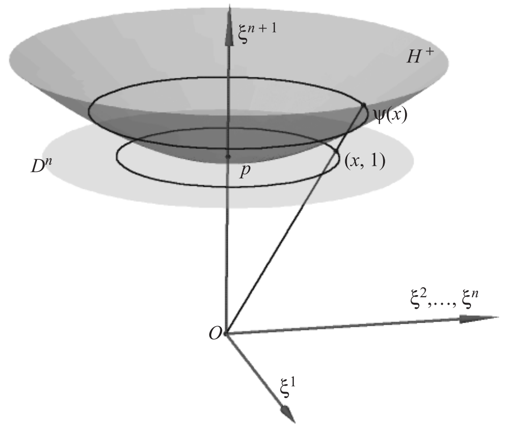

Figure 2.

Example 2.6 (Klein Hyperbolic metric). Let H_+^n be upper portion of hyperboloid of two sheets given by

$$H^n := \{\xi = (\xi^i) \in \mathbb{R}^{n+1} : -(\xi^1)^2 - (\xi^2)^2 - \cdots - (\xi^n)^2 + (\xi^{n+1})^2 = 1\}.$$

The induced metric g_{-1} on H_+^n is defined by $g_{-1}(\eta) = \|\eta\|_\xi^L$, for $\eta \in T_\xi H^n \subset \mathbb{R}^{n+1}$, where $\|\cdot\|_\xi^L$ denotes the induced Lorentzian norm on $T_\xi H^n$, defined by $\|\eta\|_\xi^L = \sqrt{(\eta^1)^2 + (\eta^2)^2 + \cdots + (\eta^n)^2 - (\eta^{n+1})^2}$, for any $\eta = (\eta^i) \in \mathbb{R}^{n+1}$. Let $\mathbb{D}^n$ denote the unit disc in $\mathbb{R}^n$ and $\psi : \mathbb{D}^n \to H_+^n \subset \mathbb{R}^{n+1}$ be the function defined by (Figure 2)

$$\psi(x) := \left(\frac{x}{\sqrt{1-|x|^2}}, \frac{1}{\sqrt{1-|x|^2}}\right). \tag{2.4}$$

Then ψ pulls back metric g_{-1} on the upper hyperboloid to a Riemannian metric, called the Klein metric, on $\mathbb{D}^n$, which is given by

$$\alpha_{-1}(y) = \sqrt{\psi^*(g_{-1})(y,y)} = \sqrt{g_{-1}(d\psi(y), d\psi(y))} = \|d\psi(y)\|_x,$$

for $y \in T_x D^n \cong R^n$. After simplification we have

$$\alpha_{-1} := \alpha_{-1}(y) = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1-|x|^2}. \tag{2.5}$$

It is remarkable that the Lorentzian norm $\|\cdot\|^L$ is not positive definite on $\mathbb{R}^{n+1}$, whereas the Klein metric α_{-1} is positive definite on the ball $\mathbb{D}^n$.

The three metrics in Examples (2.4), (2.5) and (2.6) can be incorporated in a single formula:

$$\alpha_\mu = \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1+\mu|x|^2}, \quad y \in T_x\mathbb{D}^n(r_\mu) \cong \mathbb{R}^n, \tag{2.6}$$

where $r_\mu := \frac{1}{\sqrt{-\mu}}$ if $\mu < 0$ and $r_\mu := +\infty$ if $\mu \geq 0$.

Remark. It can be easily seen that the Riemannian metric given in equation (2.6) is of constant sectional curvature μ. Moreover Cartan's local classification theorem tells us that any Riemannian metric of constant sectional curvature μ is locally isometric to α_μ.

Example 2.7 (Poincaré Hyperbolic metric). As in the previous example, we can construct Poincaré hyperbolic metric through hyperbolic stereographic projection of the upper portion of the hyperboloid

$$H_R^n := \{\xi = (\xi^i) \in \mathbb{R}^{n+1} : -(\xi^1)^2 - (\xi^2)^2 - \cdots - (\xi^n)^2 + (\xi^{n+1})^2 = R^2\}$$

onto the ball $D_{2R}^n := \{x \in \mathbb{R}^n : |x| < 2R\}$, as follows

$$\alpha = 4R^2\frac{|y|}{4R^2 - |x|^2}. \tag{2.7}$$

In particular, for dimension $n = 2$ and $R = 1$, if we consider polar coordinate system (r, θ), the said metric reduces to $\alpha = \frac{1}{1-\frac{r^2}{4}}\sqrt{dr^2 + r^2 d\theta^2}$ onto the ball D_2^2.

Example 2.8 (Funk Metric). Let

$$\Theta := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle}{1-|x|^2}, \; y \in T_x\mathbb{D}^n \cong \mathbb{R}^n. \tag{2.8}$$

Then $\Theta = \Theta(x, y)$ is a Finsler metric on $\mathbb{D}^n$, called the Funk metric on $\mathbb{D}^n$.

Example 2.9 (Berwald Metric). Let

$$B := \frac{(\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle)^2}{(1-|x|^2)^2\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}, \tag{2.9}$$

where $y \in T_x\mathbb{D}^n \cong \mathbb{R}^n$. Then $B = B(x, y)$ is a Finsler metric on $\mathbb{D}^n$, first constructed by L. Berwald.

Remark. The Finsler metrics given in examples (2.8) and (2.9) are examples of non-Riemannian Finsler metrics. These metrics have special geometrical properties. They are locally projectively flat with constant flag curvature (analogue of sectional curvature in Riemannian Geometry).

Example 2.10 (Randers Metric). This metric was first introduced by a physicist named G. Randers [11], who was concerned with the unified field theory of gravitation and electromagnetism. His justification was: 'Perhaps the most characteristic property of the physical world is the unidirection of time-like intervals. Since there is no reason why this asymmetry should disappear in the mathematical description, it is of interest to consider the possibility of a metric with asymmetrical property'. Later on, Ingarden [7] also used this metric in the context of electron microscopes and called it the Randers metric, for the first time, taking the name of G. Randers. Let $\alpha = \sqrt{a_{ij} y^i y^j}$ be a Riemannian metric and $\beta = b_i y^i$ be a 1-form on a manifold M^n. Then the metric $F(x, y) = \alpha(x, y) + \beta(x, y)$ is a Finsler metric, called Randers metric provided $\|\beta_x\| = \sqrt{a^{ij} b_i b_j} < 1$ (see [1], [4]).

A geometric interpretation of Funk and Randers metrics are given in next section as solutions to the *Zermelo-Navigation problem*.

3. Navigation Problem

Let $h = \sqrt{h_{ij}(x) y^i y^j}$ be a Riemannian metric and let $V = V^i(x)\left(\frac{\partial}{\partial x^i}\right)$ be a vector field on a manifold M with $h(x, -V_x) = \|V\|_x := \sqrt{h_{ij}(x) V^i(x) V^j(x)} < 1$, $x \in M$. Consider $|y|$ as measuring the time to travel from the basepoint of the vector y to its tip. The unit tangent sphere in each $T_x M$ consists of all those tangent vectors u such that $|u| = 1$. Now introduce a vector field W ($|W| < 1$), the spatial velocity vector of our mild wind on the Riemannian space (M, h). In the absence of vector field W, a journey from the base to the tip of any u would take unit time. In presence of the wind (vector field), within the same unit of time, one traverses not u but the resultant $v = u + W$. As an example, suppose $|W| = \frac{1}{4}$. If u points along W ($u = 4W$), then $v = \frac{5}{4}u$. Alternatively, if u points opposite to W ($u = -4W$), then $v = \frac{3}{4}u$. In these two scenarios, $|v|$ equals $\frac{5}{4}$ and $\frac{3}{4}$ instead of 1. So, in presence of wind, our Riemannian metric h no longer gives the travel time along vectors. This needs the introduction of a function F on the tangent bundle TM, to keep track of the travel time needed to traverse tangent vectors y under windy conditions. For all those resultants $v = u + W$ mentioned above, we have $F(v) = 1$.

Given any Finsler manifold (M, F), the indicatrix in each tangent space is given by $S_x(F) := \{y \in T_x M : F(x, y) = 1\}$. The indicatrices of h and the Randers metric F with navigation data (h, W) are related by a rigid translation, $S_x(F) = S_x(h) + W_x$. We will see that F is a Randers metric and the paths of shortest time are the geodesics of this F. Now we consider those $u \in T_x M$ with $|u| = 1$; equivalently, $h(u, u) = 1$. Into this, we substitute $u = v - W$ and $h(v, W) = |v||W| \cos\theta$. We have $|v|^2 - 2|W| \cos\theta |v| - \lambda = 0$, where $\lambda := 1 - |W|^2$. Since $|W| < 1$, the resultant v is never zero, hence $|v| > 0$. This leads to $|v| = |W| \cos\theta + \sqrt{|W|^2 \cos^2\theta + \lambda}$. If we put $p = |W| \cos\theta$ and $q = \sqrt{|W|^2 \cos^2\theta + \lambda}$, we have $|v| = p + q$. Since $F(v) = 1$, we have $F(v) = 1 = |v| \frac{1}{q+p} = |v| \frac{q-p}{q^2-p^2} = \frac{\sqrt{[h(W,v)]^2 + |v|^2 \lambda}}{\lambda} - \frac{h(W,v)}{\lambda}$. It remains to deduce $F(y)$ for an arbitrary $y \in TM$. Note that every nonzero y is expressible as a positive multiple c of some v with $F(v) = 1$. For $c > 0$, traversing $y = cv$ under the windy conditions should take c unit of time. Since F is positively homogeneous of degree one: $F(y) = cF(v)$, we have

$$F(y) = \frac{\sqrt{[h(W, y)]^2 + |y|^2 \lambda}}{\lambda} - \frac{h(W, y)}{\lambda}. \tag{3.1}$$

Here, $F(y)$ abbreviates $F(x, y)$, the basepoint x has been suppressed temporarily. Hence F is a Randers metric; i.e., $F(x, y) = \sqrt{a_{ij}(x) y^i y^j} + b_i(x) y^i$, where a is a Riemannian metric and b is a 1-form. Explicitly, $a_{ij} = \frac{h_{ij}}{\lambda} + \frac{W_i W_j}{\lambda^2}$, $b_i = \frac{-W_i}{\lambda}$. Here $W_i = h_{ij} W^j$ and $\lambda = 1 - W^i W_i$. In particular, there is a canonical Randers metric associated to each Zermelo navigation problem with data (h, W). Incidentally, the inverse of a is given by $a^{ij} = \lambda(h^{ij} - W^i W^j)$, and that of b by $b^i = a^{ij} b_j = -\lambda W^i$.

Under the influence of W, the most efficient navigational paths are no longer the geodesics of the Riemannian metric h, instead, they are the geodesics of the Finsler metric F. To see this, let $x(t)$, for $t \in [0, \tau]$, be a curve in M from a point p to a point q. Return to our imaginary ship sailing on M, with velocity vector u but not necessarily with constant speed. If the ship is to travel along the curve $x(t)$ while the wind blows, the captain must continuously adjust the ship's direction $\frac{u}{|u|}$ so that the resultant $u + W$ is tangent to $x(t)$. The travel time along any infinitesimal segment $\dot{x} dt$ of the curve is $F(x, \dot{x} dt) = F(x, \dot{x}) dt$, because as explained above it is the positively homogeneous F (not h) that keeps track of travel times. The captain's task is to select a path $x(t)$ from p to q that minimises the total travel time $\int_0^\tau F(x, \dot{x}) dt$. This quantity is independent of orientation-preserving parametrizations due to the positive homogeneity of F and the change-of-variables theorem. Such an efficient path is precisely a geodesic of the Finsler metric F, which is said to have solved Zermelo's problem of navigation under the external influence of W.

Example 3.1. Let $h = |y|$ be the standard Euclidean norm on $\mathbb{R}^n$, $M = \mathbb{D}^n$ be the unit ball in $\mathbb{R}^n$. In this case the Finsler metric is nothing but Funk metric as defined in Example 2.8.

Inverse Problem: Indeed, let us be given an arbitrary Randers metric F with Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and a differential 1-form $\beta = b_i(x)y^i$ satisfying $\|b\|^2 = a^{ij}b_ib_j < 1$. Set $b^i = a^{ij}b_j$ and $\epsilon = 1 - \|b\|^2$. Construct h and W as follows

$$h_{ij} = \epsilon(a_{ij} - b_ib_j), \quad W^i = -\frac{b^i}{\epsilon}. \tag{3.2}$$

It can be checked that perturbing h by the stipulated W gives back the Randers metric we started with.

4. Finsler-Poincaré disc

In this section we construct a Riemannian metric on a Euclidean open disc, $\mathbb{D}^2(2) := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 4\}$, centered at origin and radius 2 in $\mathbb{R}^2$, such that under the presence of a force field V, the Finsler metric is a Rander's metric $F = \alpha + \beta$, where α is the usual Poincaré metric on open disc $\mathbb{D}^2(2)$ (as defined in Example 2.7) and β is a 1-form globally defined on the disc. We call this the Finsler-Poincaré metric. This metric is implicitely given in [9], and is extensively discussed in [1]. This metric is interesting because its geodesic trajectories agree with those of the Riemannian Poincaré model α. However, we shall soon see that the travel time from the boundary to the center is finite ($= \log 2$), while the return trip takes infinite time along the geodesics.

Introduce polar coordinates (r, θ) such that $x = r\cos\theta$, $y = r\sin\theta$. Then the Poincaré metric (Example 2.7) in polar coordinates is written as $\alpha := \frac{1}{1-\frac{r^2}{4}}\sqrt{dr^2 + r^2d\theta^2}$. Consider a 1-form $\beta := d\big(\log\big[\frac{4+r^2}{4-r^2}\big]\big) = \frac{r}{\left(1+\frac{r^2}{4}\right)\left(1-\frac{r^2}{4}\right)}dr$, which is globally defined on $\mathbb{D}^2$. Note that α is the Riemannian Poincaré metric of constant sectional curvature -1 and the 1-form β is exact and hence closed. Now we use a technical result for a Randers metric [1]:

Lemma 4.1. *If β is a closed 1-form, then the Finslerian geodesics have the same trajectories as the geodesics of the underlying Riemannian metric α.*

Since the geodesics of α are Euclidean straight lines passing through origin and Euclidean circular arcs that intersect the boundary of the Poincaré disc at Euclidean right angles. The Zermelo navigation data is

$$h = \frac{\left(1-\frac{r^2}{4}\right)^2 dr^2 + r^2\left(1+\frac{r^2}{4}\right)^2 d\theta^2}{\left(1+\frac{r^2}{4}\right)^4},$$

$$W = \frac{-r\left(1+\frac{r^2}{4}\right)}{\left(1-\frac{r^2}{4}\right)}\partial_r. \tag{4.1}$$

Let V be any tangent vector to $\mathbb{D}^2$. The value of the Finsler function F on V is $F(V) = \sqrt{a(V,V)} + b(V)$. Since $\|b\| = \sqrt{a^{ij}b_ib_j} = \frac{4r}{4+r^2} < 1$. Since $0 \le r < 2$ we have $\|b\| < 1$. Let $V = p\partial_r + q\partial_\theta$ then

$$F(V) = \frac{\sqrt{p^2 + r^2q^2}}{1-\frac{r^2}{4}} + \frac{pr}{\left(1-\frac{r^2}{4}\right)\left(1+\frac{r^2}{4}\right)}. \tag{4.2}$$

If the Finsler metric is absolutely homogeneous, then the reverse of a geodesic is again a geodesic; such a conclusion does not hold for generic F. Our Finsler metric F is of Randers type which is not absolutely homogeneous, so we do not expect the reverse of its geodesic to be a geodesic. However, this is valid at trajectory level. Namely, if we reverse any given geodesic (say from P to Q) in our Example, then the trajectory curve from Q to P coincides with that of a geodesic.

Proposition 4.1. *If P is a point inside the Finsler-Poincaré disc with centre O, then the Finslerian arclength from O to P along any curve c is bounded below by that of the straight ray from O to P and the Finslerian arclength from P to O along any curve c is bounded below by that of the straight ray from P to O. Moreover, in general, the distance function defined here is not symmetric, in particular, $d(O, \partial M) = +\infty$ whereas $d(\partial M, O) = \log 2$.*

Proof. Fix a point P on the Euclidean circle of radius ϵ ($\epsilon < 2$) and center at origin O. Let its polar coordinate be $r = \epsilon$ and $\theta = \xi$. Let c be an arbitrary curve from O to P given by, $c(t) = (r(t)\cos\theta(t), r(t)\sin\theta(t)), 0 \le t \le \epsilon$. Here $r(0) = 0$ and $r(\epsilon) = \epsilon$ and $\theta(\epsilon) = \xi$. Let σ_1 be the straight ray joining O to P, given by $\sigma_1(t) = (t\cos\xi, t\sin\xi), 0 \le t \le \epsilon$. Since $\frac{\partial c}{\partial r} = (\cos\theta, \sin\theta) := \frac{\partial}{\partial r}$ and $\frac{\partial c}{\partial \theta} = r(-\sin\theta, \cos\theta) := \frac{\partial}{\partial\theta}$, we have $\dot{c} = \frac{dc}{dt} = \frac{\partial c}{\partial r}\frac{dr}{dt} + \frac{\partial c}{\partial\theta}\frac{d\theta}{dt} = \dot{r}\frac{\partial}{\partial r} + \dot{\theta}\frac{\partial}{\partial\theta}$. Hence in view of equation (4.2) we have

$$F(\dot{c}) = \frac{\sqrt{\dot{r}^2 + r^2\dot{\theta}^2}}{1-\frac{r^2}{4}} + \frac{\dot{r}r}{\left(1-\frac{r^2}{4}\right)\left(1+\frac{r^2}{4}\right)}$$
$$\geq \frac{1}{1-\frac{r^2}{4}}|\dot{r}| + \frac{\dot{r}r}{\left(1-\frac{r^2}{4}\right)\left(1+\frac{r^2}{4}\right)}$$
$$\geq \left(\frac{\pm 1}{1-\frac{r^2}{4}} + \frac{r}{\left(1-\frac{r^2}{4}\right)\left(1+\frac{r^2}{4}\right)}\right)\dot{r}. \quad (4.3)$$

Therefore the length of the arc c from O to P is given by

$$\int_0^{\epsilon} F(\dot{c}) \geq \int_0^{\epsilon}\left(\frac{\pm 1}{1-\frac{r^2}{4}} + \frac{r}{\left(1-\frac{r^2}{4}\right)\left(1+\frac{r^2}{4}\right)}\right)dr. \quad (4.4)$$

In case of going from O to P, considering plus sign in equation (4.4), we have $r(0) = 0, r(\epsilon) = \epsilon$. Thus

$$L_F(c) := \int_0^{\epsilon} F(\dot{c}) \geq \int_0^{\epsilon}\left(\frac{1}{1-\frac{r^2}{4}} + \frac{r}{\left(1-\frac{r^2}{4}\right)\left(1+\frac{r^2}{4}\right)}\right)dr. \quad (4.5)$$

Now the velocity of the Euclidean straight ray σ_1 from O to P is $\frac{\partial}{\partial r}$. Hence its Finslerian length is

$$L_F(\sigma_1) := \int_0^{\epsilon} F(\dot{\sigma_1})$$
$$= \int_0^{\epsilon}\left(\frac{1}{1-\frac{t^2}{4}} + \frac{t}{\left(1-\frac{t^2}{4}\right)\left(1+\frac{t^2}{4}\right)}\right)dt. \quad (4.6)$$

This integral is identical to the lower bound of $L_F(c)$; i.e., σ is the shortest among all curves from O to P. Hence

$$d(O, P) = L_F(\sigma_1) = \log\left(\frac{4+\epsilon^2}{(2-\epsilon)^2}\right). \quad (4.7)$$

In particular,

$$d(O, \partial M) = L_F(\sigma_1) = \lim_{\epsilon\to 2^-}\log\left(\frac{4+\epsilon^2}{(2-\epsilon)^2}\right) = +\infty. \quad (4.8)$$

Thus the Finslerian distance from the origin to the rim of the Poincaré disc is infinite.

Now we consider the case of going from P to O. Consider negative sign in equation (4.4); also since the curve goes from P to O, we have $r(0) = \epsilon, r(\epsilon) = 0$. Thus

$$L_F(c) := \int_0^{\epsilon} F(\dot{c}) \geq \int_{\epsilon}^{0}\left(\frac{-1}{1-\frac{r^2}{4}} + \frac{r}{\left(1-\frac{r^2}{4}\right)\left(1+\frac{r^2}{4}\right)}\right)dr, \quad (4.9)$$

that is

$$L_F(c) := \int_0^{\epsilon} F(\dot{c}) \geq \int_0^{\epsilon}\left(\frac{1}{1-\frac{r^2}{4}} - \frac{r}{\left(1-\frac{r^2}{4}\right)\left(1+\frac{r^2}{4}\right)}\right)dr. \quad (4.10)$$

Now the velocity of the Euclidean straight ray σ_2 from P to O is $-\frac{\partial}{\partial r}$. Hence its Finslerian length is

$$L_F(\sigma_2) := \int_0^{\epsilon} F(\dot{\sigma_2})$$
$$= \int_0^{\epsilon}\left(\frac{1}{1-\frac{(\epsilon-t)^2}{4}} + \frac{t}{\left(1-\frac{(\epsilon-t)^2}{4}\right)\left(1+\frac{(\epsilon-t)^2}{4}\right)}\right)dt$$
$$= \int_0^{\epsilon}\left(\frac{1}{1-\frac{t^2}{4}} + \frac{t}{\left(1-\frac{t^2}{4}\right)\left(1+\frac{t^2}{4}\right)}\right)dt. \quad (4.11)$$

This integral is identical to the lower bound of $L_F(c)$; i.e., σ_2 is the shortest among all curves from P to O. Hence

$$d(P, O) = L_F(\sigma_2) = \log\left(\frac{(2+\epsilon)^2}{4+\epsilon^2}\right). \quad (4.12)$$

In particular,

$$d(\partial M, O) = \lim_{\epsilon\to 2^-}\log\left(\frac{(2+\epsilon)^2}{4+\epsilon^2}\right) = \log 2. \quad (4.13)$$

Thus the Finslerian distance from rim to the origin of the Poincaré disc is $\log 2$.

A Physical Model: Here we describe a physical model, due to [1], of a Finsler-Poincaré disc. Suppose water in a kitchen sink is draining towards a sink-hole located at O, wall of the kitchen sink is at ∂M. Suppose a tiny bug is swimming either from O to ∂M or from ∂M to O along the straight ray, what we called σ_1 and σ_2 respectively. The mathematical Finslerian arc length is actually measuring the total physical time of the journey. It is now conceivable that, swimming against a current, the bug might need an eternity to reach ∂M, while aided by a current it can easily reach O in finite time.

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Calculating 10,000 Places of π Using Basic Calculus

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Abstract. This short paper describes an effective way to compute the value of π using mathematical techniques that should be familiar to any student with an introductory course in calculus. The procedure is iterative with each iteration doubling the number of digits of π. On a computer using an Intel i5 processor running at 2.6 GHz, an implementation of the method described here computed π to 1,500 places in 15 seconds and over 12,000 places in 70 minutes. These timings are based on an implementation using *Maxima*. An implementation in C++ using the Gnu GMP library cuts down the time by nearly 50% so that π is calculated to 12,000 places in approximately 35 minutes. Source code for both implementations are provided.

Keywords and Phrases. pi, calculus

1. Background

The method described here was developed to improve on a problem that appeared in the JEE-2010 exam [1]. This is the exam taken by students entering college, and admission to various Indian Institutes of Technologies is based on the performance in this exam. Problem 41 in the exam asks the prospective students to evaluate

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx \tag{1}$$

The correct answer is $\frac{22}{7} - \pi$. Note that this gives the classic approximation $\pi \approx 22/7$. An attempt to improve on this approximation resulted in the method described in this paper. It should be emphasized that this method is not the fastest method to calculate π. The fast methods for computing π are based on very advanced topics in mathematics[1]. The method described here can be understood by students who have had an introductory course in calculus.

2. Method to calculate π

First the solution to the problem in (1): By expanding the numerator and using long division we have

$$\frac{x^4(1-x)^4}{1+x^2} = \frac{x^4(x^4-4x^3+6x^2-4x+1)}{1+x^2}$$
$$= \frac{(x^8-4x^7+6x^6-4x^5+x^4)}{1+x^2}$$
$$= x^6-4x^5+5x^4-4x^2+4-\frac{4}{1+x^2} \quad (2)$$

Integrating we get

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2} = 1/7-4/6+5/5-4/3+4-\pi$$
$$= \frac{22}{7}-\pi \quad (3)$$

where we have made use of the standard result that is covered in basic calculus (as an example of change of variables)

$$\int \frac{1}{1+x^2}dx = \arctan(x) \quad (4)$$

$$\pi/4 = \arctan(1) \quad (5)$$

2.1 Classical methods based on Taylor series

Almost all classical results for calculating π use Taylor series expansion of $1/(1+x^2)$ and then integrating the series term by term. The Taylor series expansion gives

$$\arctan x = \int_0^x \frac{1}{1+u^2}du$$
$$= \int_0^x (1-u^2+u^4-u^6+u^8\cdots)du$$
$$= x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}\cdots$$
$$= x\left(1-\frac{x^2}{3}+\frac{x^4}{5}-\frac{x^6}{7}+\frac{x^8}{9}\right) \quad (6)$$

By substituting convenient values for x in the above series we get the following classical results that date back to Euler.

$$\frac{\pi}{4} = \arctan(1) = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}\cdots \quad (7)$$

$$\frac{\pi}{6} = \arctan\left(\frac{1}{\sqrt{3}}\right)$$
$$= \frac{1}{\sqrt{3}}\left(1-\frac{1}{3\times 3}+\frac{1}{3^2\times 5}-\frac{1}{3^3\times 7}\cdots\right) \quad (8)$$

While the above formulas are easy to program, they are too slow to be of practical use. In the case of the first series, the error after including n terms is approximately $1/n$. In the case of the second series it is of the order of $1/3^n$. Thus if one requires 10,000 digits of π, we need the error to be less than 10^{-10000}. Thus the first series would require an astronomical number of terms while the second would require about $10000/\log_{10}(3)$ or close to 21,000 terms. There are ways to speed up the calculations but the fundamental fact is that the methods that are based on Taylor series are too slow to be of practical use.

2.2 Approximating $1/(1+x^2)$: Newton's Method

Newton developed an alternative to Taylor series that is in general substantially faster. The idea is as follows:

Iterative Improvement

(1) Begin with some approximate answer (this could be a truncated Taylor series but it does not have to be).

(2) Based on the problem at hand, and the approximate answer, *improve* the answer. If done right, such an improvement will reduce the error in the answer.

(3) Repeat the previous step as needed.

The equation 2 can be reinterpreted as an approximation to $1/(1+x^2)$:

$$\frac{1}{1+x^2} = \frac{x^6-4x^5+5x^4-4x^2+4}{4} - \frac{x^4(1-x)^4}{4(1+x^2)} \quad (9)$$

Thus equation (9) states that $(x^6-4x^5+5x^4-4x^2+4)/4$ is an approximation to $1/(1+x^2)$ with an error of $-x^4(1-x)^4/$

[1]For example the Ramanujan's formula for π [2] modified by the Chudnovsky brothers [3] or the modification of Gauss-Legendre algorithm by Kanada [4] who holds the current record for computing π. Ramunajan's series for π requires a deep understanding of the theory of elliptic functions while the Gauss-Legendre method requires background in elliptic integrals. Both these topics are beyond the scope of an introductory course in calculus.

$(4(1+x^2))$. Note that in the interval 0 to 1, the numerator of the error is maximum at $x = 1/2$ and the error is no more than $1/(2^8 \times 4) = 1/1024$. Thus 22/7 overestimates π by no more than 4 times $1/1024$, i.e no more than 0.004. We can now use Newton's idea to improve the approximation. By keeping track of the error, we can determine when to stop the approximation process and calculate π by integration.

Proposition 2.2 (Newton's formula for reciprocals). *Suppose that f is an approximation to $1/g$. Let the error in approximation be given by $e = 1 - fg$. Suppose that $|e| < 1$. Then $\hat{f} = 2f - gf^2$ is an improved approximation with its error $\hat{e} = 1 - \hat{f}g$ given by $\hat{e} = e^2$. Also, if f and g are polynomials, then so is $\hat{f}$.*

Proof. By direct verification we have

$$\begin{aligned}\hat{e} &= 1 - \hat{f}g \\ &= 1 - (2f - gf^2) \times g \\ &= (g^2 f^2 - 2fg + 1) \\ &= (1 - fg)^2 = e^2 \qquad (10)\end{aligned}$$

If one wants to derive the approximation, we start with the first approximation and substitute the approximation recursively as follows

$$\begin{aligned} e &= 1 - fg \\ \Rightarrow \frac{1}{g} &= f + \frac{e}{g} = f + e \times \frac{1}{g} \quad \text{Initial approximation} \\ &= f + e \times \left(f + \frac{e}{g}\right) \quad \text{Use initial approximation} \\ &= f + ef + \frac{e^2}{g} = f + (1 - fg)f + \frac{e^2}{g} \\ &= 2f - gf^2 + \frac{e^2}{g} = \hat{f} + \frac{e^2}{g} \end{aligned}$$

Thus $\hat{f} = 2f - gf^2$ is a better approximation with error e^2.

□

The above method to iteratively calculate reciprocals is used to implement division in many computer architectures. Division is performed by first computing the reciprocal of the divisor and then multiplying by the reciprocal [5, p 134].

2.3 Iterative Calculation of π

We start with $f_0(x)$, some initial *polynomial* approximation to $1/(1+x^2)$ as

$$\frac{1}{1+x^2} = f_0(x) + \frac{e_0(x)}{1+x^2}$$

and integrating from 0 to 1 after multiplying by 4, we get

$$\pi = 4\int_0^1 f_0(x)dx + 4\int_0^1 \frac{e_0(x)}{1+x^2}dx$$

and hence we get the approximation

$$\pi \approx 4\int_0^1 f_0(x)dx$$

and the error in the approximation is

$$\text{error} = 4\left|\int_0^1 \frac{e_0(x)}{1+x^2}dx\right|$$

We use two measures, α, β, to estimate the magnitude of the error:

$$\alpha_0 = \max_{0 \le x \le 1} |e_0(x)| \qquad (11)$$

$$\beta_0 = \int_0^1 |e_0(x)|\, dx \qquad (12)$$

Since $1 + x^2$ is always greater than 1, the error between π and $4\int_0^1 f_0(x)dx$ can be estimated as

$$\begin{aligned}\text{error} = 4\left|\int_0^1 \frac{e_0(x)}{1+x^2}dx\right| &\le 4\int_0^1 \frac{|e_0(x)|}{1+x^2}dx \\ &\le 4\int_0^1 |e_0(x)|dx = 4\beta_0\end{aligned}$$

Next we use Newton's formula to get a better approximation f_1 with error $e_1(x) = e_0(x)^2$. Thus the error between π and $4\int_0^1 f_1(x)dx$ is no more than $4\beta_1$ where α_1 and β_1 are defined analogous to α_0 and β_0. We can estimate these two as follows:

$$\alpha_1 = \max_{0 \le x \le 1} |e_1(x)| = \max_{0 \le x \le 1} \left|e_0(x)^2\right| = \alpha_0^2 \qquad (13)$$

$$\begin{aligned}\beta_1 &= \int_0^1 |e_1(x)|\, dx \\ &= \int_0^1 |e_0(x)|^2\, dx \\ &\le \alpha_0 \int_0^1 |e_0(x)|\, dx \\ &= \alpha_0 \beta_0 \qquad (14)\end{aligned}$$

If $4\beta_1$ is smaller than the desired accuracy of π, we can stop. If not, we repeat the procedure. Since α's and β's are extremely small numbers it is better to work with their logarithms as follows

$$\log_{10}\alpha_0 = 2\log_{10}\alpha_0 \tag{15}$$

$$\log_{10}\beta_1 \le \log_{10}\alpha_0 + \log_{10}\beta_0 \tag{16}$$

$$\log_{10}(4\beta_1) \le \log_{10}\alpha_0 + \log_{10}(4\beta_0) \tag{17}$$

Note that in general if we improve f_k to obtain f_{k+1} we have the following recursive formulas

$$f_{k+1}(x) = 2f_k(x) - (1+x^2)f_k(x)^2 \tag{18}$$

$$\log_{10}\alpha_{k+1} = 2\log_{10}\alpha_k \tag{19}$$

$$\log_{10}(4\beta_{k+1}) = \log_{10}\alpha_k + \log_{10}(4\beta_{k+1}) \tag{20}$$

where we overestimate β_k, meaning the actual error would be smaller.

2.4 A practical implementation

We start with the initial approximation that arises from the JEE exam

$$f_0(x) = \frac{x^6 - 4x^5 + 5x^4 - 4x^2 + 4}{4}$$

for which we have

$$|e_0| = \frac{(x-1)^4x^4}{4}$$

By direct computation

$$\log_{10}\alpha_0 = \log_{10}(1/1024) = -3.0103$$

$$\log_{10}\beta_0 = \log_{10}(1/2520) = -3.4014$$

$$\begin{aligned}\log_{10}4\beta_0 &= -3.4014 + \log_{10}4\\ &= -3.4014 + 0.6021 = -2.7993\end{aligned}$$

Thus the initial approximation $4\int_0^1 f_0(x)dx = 22/7$ is accurate to 2.7993 decimal places so the first two digits after the decimal point are correct. This gives

$$\pi \approx 3.14.$$

If we perform one step of Newton's iteration we get

$$\begin{aligned}f_1(x) &= 2f_0(x) - (1+x^2)f_0(x)^2\\ &= \frac{\begin{array}{c}-x^{14} + 8x^{13} - 27x^{12} + 48x^{11} - 43x^{10} + 8x^9\\ +15x^8 - 16x^6 + 16x^4 - 16x^2 + 16\end{array}}{16}\end{aligned}$$

with the following error estimates

$$\log_{10}\alpha_1 = 2\log_{10}\alpha_0 = -6.0206$$

$$\begin{aligned}\log_{10}(4\beta_1) &\le \log_{10}\alpha_0 + \log_{10}(4\beta_0)\\ &= -3.0103 + -2.7993 = -5.8096\end{aligned}$$

Thus $4\int_0^1 f_1(x)dx = 47171/15015$ is accurate to 5.8096 decimal places. Rounding down we have π to 5 decimal places. This gives

$$\pi \approx 3.14159$$

If we repeat this process once more, we get f_2 which is too long to write out! However, we can calculate its accuracy using

$$\log_{10}\alpha_2 = 2\log_{10}\alpha_1 = -12.0412$$

$$\begin{aligned}\log_{10}4\beta_2 &= \log_{10}\alpha_1 + \log_{10}4\beta_1\\ &= -6.0206 + -5.8096 = -11.8302\end{aligned}$$

Thus $\int_0^1 4f_2(x)dx$ is accurate to 11.8302 decimal places or we have π to 11 decimal places. Although it is not possible to *hand calculate* successive approximations f_2, f_3 etc. we can easily compute the error estimates as follows:

k	$\log_{10}\alpha_k$	$\log_{10}4\beta_k$
1	−6.0206	−5.8096
2	−12.0412	−11.8302
3	−24.0824	−23.8714
4	−48.1648	−47.9538
5	−96.3296	−96.1186
6	−192.6592	−192.4482
7	−385.3184	−385.1074
8	−770.6368	−770.4258
9	−1541.2736	−1541.0626
10	−3082.5472	−3082.3362
11	−6165.0944	−6164.8834
12	−12330.1888	−12329.9778

Thus, after 9 iterations we have over 1,500 places of π. After 12 iterations we have well over 12,000 places of π. Note that each application of Newton's method results in doubling the number of decimal places that we get. *This is a characteristic feature of the Newton's Method.*

3. A Computer Program

3.1 Using Maxima, a typical Computer Algebra Software

One can easily program the procedure using any software that can work with polynomials, usually referred to as symbolic mathematics programs. By far the most popular and powerful program is *Mathematica* from Wolfram. However, Mathematica is not free. *Maxima*, which is easy to use (once you get past the non-standard syntax) is free and can be obtained online [6]. The software package also includes *wxMaxima* an easy to use front end. A more powerful front end to Maxima is the Sage software [7]. A *wxMaxima* program is given in section 4.1 and is also available online [8]. A point worth mentioning is that it is always advantageous to work with integers. Hence, the function $f(x)$ is written as $n(x)/d$ where $n(x)$ is the numerator polynomial with integer coefficients and d is the denominator. Newton's formula can then be written as

$$\begin{aligned}2f(x) - (1+x^2)f(x)^2 &= 2\frac{n(x)}{d} - (1+x^2)\frac{n(x)^2}{d^2}\\ &= \frac{2n(x)d - (1+x^2)n(x)^2}{d^2}\end{aligned}$$

and hence the iterative improvement is written as

$$\begin{aligned}n(x) &\leftarrow 2n(x)d - (1+x^2)n(x)^2\\ d &\leftarrow d^2\end{aligned}$$

3.2 Programming in C++

There is considerable advantage in programming the calculations in a general purpose programming language such as C++. However, computations require representing integers of arbitrary size. The easiest and freely available library for true representation of integers is the GNU Multiple Precision Arithmetic Library [12]. Polynomials are represented using arrays, with the coefficient of x^i is the i-th element of the array. Denoting the coefficient of $n(x)$ as n_i and the degree of $f(x)$ as δ, the algorithm for computing π is as follows:

Algorithm:

Step 0. *Initialize:* $n(x) = 4 - 4x^2 + 5x^4 - 4x^5 + x^6$, $d = 4$, degree = 6.

- $n_0 = 4, n_1 = 0, n_2 = -4, n_3 = 0, n_4 = 5, n_5 = -4, n_6 = 1$
- $d = 4$
- $\delta = 6$

Step 1. *Newton's step*. Repeat as many times as needed. Uses temporary array t

- $t_i = n_i, i = 0 \ldots \delta$
- $n_i = 0, i = 0 \ldots 2\delta + 2$
- $n_i {+}{=} 2dt_i$, $n_{2i} {-}{=} t_i^2$, $n_{2i+2} {-}{=} t_i^2$, $n_{i+j} {-}{=} 2t_it_j$, $n_{i+j+2} {-}{=} 2t_it_j$, $i = 0 \ldots \delta$, $j = 0 \ldots (i-1)$
- $d = d^2$
- $\delta = 2\delta + 2$

Here the symbol $+=$ represents *addto* and $-=$ represents *subtract from.*

Step 2. *Integrate.* The value of π is kept as ratio of integers α/β

- $\alpha = 0, \beta = 1$
- $\alpha = \alpha \times (i+1) + \beta n_i$, $\beta = \beta * (i+1)$, $\gamma = \gcd(\alpha, \beta)$, $\alpha = \alpha/\gamma$, $\beta = \beta/\gamma$, $i = 0 \ldots \delta$

The program that implements the above algorithm can be found in section 4.2 and also online at [9].

3.3 Test Results

The program shown in section 4.1 was run on a personal computer running Windows 7 operating system with a Intel is processor running at 2.8 GHz and 6 GB ram. 12 iterations of the Newton's method took 4217 seconds (approximately one hour and ten minutes) to provide 12,329 digits of π. 9 iterations took just 15 seconds to produce more than 1500 digits of π. Attempting more than 12 iterations failed due to running out of memory. The result of 12 iterations is shown below

```
12327 digits in 4217.0600000000004
   seconds
pi=3.141592653589793238462643383279502
   88419716939937510....
    7850171807793068108546900094458995
    2794243981392135055864221964834915
    1263901280383200109773868066287792
    3971801461343244572640097374257007
    3592100315415089367930081699805365
    2027600
```

The final digit sequence matches with the published value of π. See for example [10] and [11].

The implementation in C++ and the GNU Multiple Precision Arithmetic Library [12] cuts the computation time in half and 12,329 digits of π were obtained in 2014 seconds (approximately 34 minutes).

4. Computer Implementation

4.1 wxMaxima Program to compute Pi

```
/* [wxMaxima batch file version 1]
   [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima
   version 12.01.0 ] */

/* [wxMaxima: input   start ] */
/***********************

  Author: N. Natarajan, University of
     Michigan-Dearborn
  Copyright: Attribution-NonCommercial-
     ShareAlike 3.0 Unported
  See  http://creativecommons.org/
     licenses/by-nc-sa/3.0/

***********************/

/* Change the following as appropriate */

/* Set the number of iterations */
niter: 9;
/* Set the output file */
outfile:printf(false, "C:/users/public/
   pi/pi~d-iterations.txt", niter);
/* [wxMaxima: input   end   ] */

/* [wxMaxima: input   start ] */
n:x^6-4*x^5+5*x^4-4*x^2+4$
d:4$
log_alpha:log(1/1024)/log(10)$
log_4beta:log(4/2520)/log(10)$

start_time:elapsed_run_time()$ /* Get
   some time statistics */
degree:6$
thru niter do(
        st:elapsed_run_time(),
        n: expand(2*d*n - (1+x^2)*n^2),
        d: d^2,
        log_4beta:log_4beta+log_alpha,
        log_alpha:2*log_alpha,
        degree:2*degree+2,
        disp([elapsed_run_time()-st,
           float(log_4beta), degree])
)$
 piv:4*integrate(n,x,0,1)/d$
disp(elapsed_run_time()-st)$

 duration:elapsed_run_time()-start_time$

/* Discard last two digits to account
   for any possible rounding*/numdigs:
   floor(-log_4beta)-2$
/* [wxMaxima: input   end   ] */

/* [wxMaxima: input   start ] */
/*
 * Now write a function to print a value
    to a file
 * given the number of digits after the
    decimal point.

 * We first peel off the integer part

 * Next we multiply the fractional part
    by 10
 * and peel off one digit and print it.
 *   -repeat till we have printed the
    desired number of digits
 */

num2f(x, ndigs, file):=
   block([v, f, s, npts],
    v:0,
  /* peel off the integer part */
  f:floor(x), s:printf(file,"~d.", f),
     x:x-f,

  /* keep track of the number of decimal
     places and print a new line every
     50 places*/
  npts:0,
  thru ndigs do(
       x:10*x,
```

```
        f:floor(x),
        x:x-f,
        s: printf(file, "~d",f),
        npts:npts+1,
        if npts=50 then (npts:0, s:printf
           (file, "~&     "))),
s)$
/* [wxMaxima: input   end   ] */

/* [wxMaxima: input   start ] */
/* Save the result to a file for
   verification */

printf(true, "~d digits in ~f seconds~&",
   numdigs, duration)$

out:openw(outfile)$
printf(out, "~d digits in ~f seconds~&",
   numdigs, duration)$
printf(out, "pi=")$ num2f(piv,numdigs,
   out)$
close(out)$
num2f(piv, 30, true)$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which
   end with a comment! */
"Created with wxMaxima"$
```

4.2 C++ Program to compute Pi using the GNU GMP library

```
/***********************

    Author: N. Natarajan, University of
       Michigan-Dearborn
    Copyright: Attribution-NonCommercial-
       ShareAlike 3.0 Unported
    See  http://creativecommons.org/
       licenses/by-nc-sa/3.0/

***********************/
static char ID[] =
    "$Id: pi.cc,v 1.1 2012/08/29 05:45:37
        Nattu Exp Nattu $" "\n";

#include <stdio.h>
#include <stdlib.h>
#include <string.h>
#include <assert.h>
#include <limits.h>
#include <time.h>
#include <iostream>
#include <fstream>
#include <math.h>
using namespace std;

#include <gmpxx.h>
typedef mpz_class Integer;

#define NUM_ITERS 12
#define HIGHEST_POWER (2<<(NUM_ITERS+2))
typedef struct {
    int degree;
    Integer coef[HIGHEST_POWER];
} POLY;
typedef struct {
    Integer num;
    Integer den;
} FRACTION;

*/ **************************************
   ************************
 * Functions
 ****************************************
    ************************/
void newton(POLY & num, Integer & den);
   // One step of Newton
Integer gcd(Integer x, Integer y);  //
   GCD of 2 numbers
void normalize(FRACTION & f);   // Factor
   out common terms
void addto(FRACTION & f, Integer num,
   Integer den); // f = f+ num/den
FRACTION integrate(POLY & p);   //
   Integrate a polynomial
void print(ostream & ost, FRACTION f, int
   numplaces);   // Print a fraction
```

```
/* *************************************
   ************************/
POLY num;
Integer den;
POLY temp;                         //
   Temporarily hold the numerator during
   newton step

main() {
    FRACTION pi;
    int i;
    float log_alpha, log_4beta;
    long tstart, tend;
    int numdigits;
    ofstream ost;
    char fname[100];
    time_t rawtime;
    struct tm *timeinfo;

    num.degree = 6;
    num.coef[0] = 4;
    num.coef[1] = 0;
    num.coef[2] = -4;
    num.coef[3] = 0;
    num.coef[4] = 5;
    num.coef[5] = -4;
    num.coef[6] = 1;
    den = 4;
    log_alpha = log10(1 / 1024.0);
    log_4beta = log10(4 / 2520.0);

    tstart = time(0);
    time(&rawtime);
    timeinfo = localtime(&rawtime);
    cout << asctime(timeinfo) << endl;
    for(i = 0; i < NUM_ITERS; i++) {
        log_4beta += log_alpha;
        log_alpha *= 2.0;
        newton(num, den);
        cerr << i << "  " << time(0) -
           tstart << "\t" << log_4beta <<
           endl;
    }
    pi = integrate(num);
    pi.den *= den / 4;
    normalize(pi);
    tend = time(0);
    numdigits = -log_4beta - 2;
    sprintf(fname, "pi-%d-places.txt",
       numdigits);
    ost.open(fname, ofstream::out);
    cout << numdigits << " digits of Pi
       computed in "
        << tend - tstart << " seconds\n\
           n";
    cout << "Output written to: " <<
       fname << endl;
    ost << numdigits << " digits of Pi
       computed in "
        << tend - tstart << " seconds\n\
           n";
    print(ost, pi, numdigits);
    print(cout, pi, 30);
    time(&rawtime);
    timeinfo = localtime(&rawtime);
    cout << asctime(timeinfo) << endl;
    cout << "\n\n";
}

void newton(POLY & num, Integer & den) {
    int i, j;
    Integer two_den, ci, cicj;

    // copy num to temp
    temp.degree = num.degree;
    for(i = 0; i <= temp.degree; i++)
        temp.coef[i] = num.coef[i];
    num.degree = 2 * num.degree + 2;
    for(i = 0; i <= num.degree; i++)
        num.coef[i] = 0;
    two_den = 2 * den;
    for(i = 0; i <= temp.degree; i++) {
        ci = temp.coef[i];
        if(ci != 0) {
            num.coef[i] += two_den * ci;
               // 2 f
            for(j = 0; j < i; j++) {
```

```
                cicj = 2 * ci * temp.
                   coef[j];
                num.coef[i + j] -= cicj;
                   // 2 c[i] c[j] x^(i+j)
                num.coef[i + j + 2] -=
                   cicj;    // 2 c[i]
                   c[j] x^(i+j) x^2
            }
            cicj = ci * ci;
            num.coef[2 * i] -= cicj;
               // c[i]^2 x^(2i)
            num.coef[2 * i + 2] -= cicj;
               // c[i]^2 x^(2i) x^2
        }
    }
    den *= den;
}

Integer gcd(Integer x, Integer y) {
    Integer z;
    mpz_gcd(z.get_mpz_t(), x.get_mpz_t(),
       y.get_mpz_t());
    return z;
}

void normalize(FRACTION & f) {
    Integer d;
    d = gcd(f.num, f.den);
    f.num /= d;
    f.den /= d;
}

void addto(FRACTION & f, Integer num,
   Integer den) {    // f += x/y
    f.num = f.num * den + f.den * num;
    f.den *= den;
    normalize(f);
}

FRACTION integrate(POLY & p) {
    FRACTION answer;
    int i;
    answer.num = 0;
    answer.den = 1;
    for(i = 0; i <= p.degree; i++)
        addto(answer, p.coef[i],
           (i + 1));  // answer = answer
           + c[i]/(i+1)
    return answer;
}

void print(ostream & ost, FRACTION f, int
   numplaces) {
    int i;
    Integer x, y, temp;
    x = f.num;
    y = f.den;
    temp = x / y;
    ost << temp;
    ost << ".";
    x -= y * temp;
    for(i = 0; i < numplaces; i++) {
        x = 10 * x;
        temp = x / y;
        ost << temp;
        x -= temp * y;
        if(i % 50 == 49)
            ost << "\n  ";
    }
    ost << endl;
}


/*
        Compile using the following make
           options for gcc compiler

        LDLIBS=-L/usr/local/lib -lgmpxx
           -lgmp
        CFLAGS=-O3
        CXXFLAGS=-O3
*/
```

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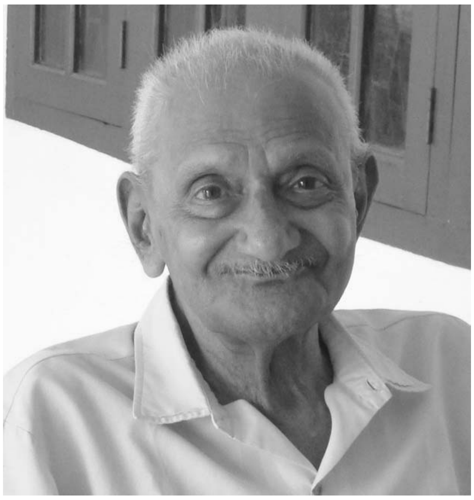
Balagangadharan Kopparambil – A Tribute to an Erudite Scholar

R. Sridharan
Chennai Mathematical Institute



K. Balagangadharan

8-10-1922 to 14-5-2012

grammata sola carent fato, mortemque repellunt
preterita renovant grammata sola biblis – Herbanus Maurus (776–856 CE)

The written word alone flouts destiny,
Revives the past and gives the lie to Death – Translation by Helen Waddel (1889–1965)

Balagangadharan passed away on 14th May 2012 in a hospital in Thrissur, due to a post-operative complication following surgery to set right his broken thigh bone. He was born on 8th October 1922, and hailed from a royal family, being the son of Shri Ramavarma Kuttan Thampuran (a prince of the royal family of Cochin) and Smt. Kalyanikutty Nethyaramma. Smt. Nethyaramma belonged to Kopparambil house at Avinissery near Thrissur. The surname Kopparambil was shortened to the letter K and he was called K. Balagangadharan officially. Balagangadharan had an elder brother K. Balaraman, who obtained his doctorate from University of Bombay (now called University of Mumbai) in English literature. He died in 2001.

I would like to include here, a brief summary of Balagangadharan's early life. After his private study at the palace residence of his father in Tripunithura, he had regular schooling for three years, in the fourth, fifth and sixth forms (1933–1936) in the same town. He then moved on to St. Thomas College, Thrissur, for the Intermediate and Bachelor's degree in Mathematics. His collegiate career was indeed academically excellent and for instance, he was awarded a gold medal for his outstanding performance in English at the Intermediate level. He went on then, to study at Madras Christian College and obtained his Master's degree in Mathematics (1940–43) from the University of Madras. He continued to stay at Madras Christian College to work as a tutor during 1943–45 and subsequently moved to Wadia College, Pune where he worked during the years 1945–51. (I had the occasion to be a member of the Board of Studies of Mathematics of the University of Bombay during 1964–67, and I remember Prof. Arjan Mirchandaney, who was Vice Principal of Jai Hind College, Bombay, who was also a member of the Board of Studies and who had been his colleague at Wadia college, telling me of the enormous respect and reputation that Balagangadharan had, among his colleagues and students, during this period, as a scholar of eminence).

It should be mentioned here that Prof. Balagangadharan was an erudite scholar of Sanskrit. Incidentally, his ancestors were adepts in the art of *Hastyayurveda* (expertise in the maintenance of the health of the elephants). Balagangadharan was always interested in Indian classical studies, particularly Sanskrit. Indeed, he was a scholar who not only was well versed in classical Sanskrit literature, but had special aptitude and deep knowledge of Sanskrit grammar, Nyaya philosophy and history of Indian mathematics). During his Wadia college days, Balagangadharan published in the Mathematics Student, 1947, a paper titled, "A consolidated list of Hindu mathematical works". An earlier work of his appears as an appendix to a paper of K. Mukunda Marar and C. T. Rajagopal (titled "On the Hindu quadrature of the circle") and contains Sanskrit slokas and English translations which constitute the main theorems used by Whish, the legendary official of the East India Company, who first brought out the contributions of the Kerala mathematicians. This work of Balagangadharan has been particularly highlighted by the reviewer of the Marar-Rajagopal paper in Mathematical Reviews.

One should remark that the work of C. T. Rajagopal and his school on History of Indian mathematics owes much to Balagangadharan. Balagangadharan is said to have even prepared an English translation of *Yuktibhasha* of Jyeshtadeva and given it to Prof. C. T. Rajagopal during their mathematical interactions. When Balagangadharan enquired about the manuscript with Rajagopal's wife shortly after his death, she apparently didn't know anything about the document. If this manuscript could be retrieved, it would be of great significance even now to the history of ancient Indian mathematics, since it would be the first translation of *Yuktibhasha* to English.

Balagangadharan was a voracious reader, and a scholar, whose main interest, when he was working at Pune, was to do research in mathematics and he was looking for a suitable research institute to join. Some of the well-known personalities who recommended Balagangadharan for such a position to research institutions like the TIFR, were Rao Saheb I. N. Menon, the then Director of Public Instruction, Cochin State, Shri C. T. Venugopal, a high official of the Railways, and the eminent Dr. P. Kesava Menon; Kesava Menon was Balagangadharan's teacher at Madras Christian College and his recommendation included the statement that Balagangadharan lacked the proper environment in Pune to continue his research interests and that he would shoot up as a great mathematician, if given the right milieu. Some of the other people who were also in high praise of Balagangadharan were Prof. Boyd and Prof. Kibble of Madras Christian College.

Following Balagangadharan's application for a research position at the TIFR in the fifties, Prof. K. Chandrasekharan, who was the head of the Mathematics School of the Tata Institute, recommended to Homi Bhabha that Balagangadharan be appointed as a research student (with a monthly stipend of Rs. 175), initially for one year. And indeed, Balagangadharan joined the Tata Institute on 13th July 1951. (It is amusing to note that his studentship amount exceeded the salary that he was getting in Wadia College as an assistant professor of Mathematics).

I shall now summarise the work and academic activities of Balagangadharan while he was at the Tata institute from 1951 till 1982, when he retired. Apart from his continued correspondence with Mukunda Marar and C. T. Rajagopal, which was a major influence on their work on history of Indian mathematics, Balagangadharan began to work with Chandrasekharan on Real analysis and published a paper titled "A quasi-Tauberian theorem on Fourier Series" in the Journal of Indian Mathematics Society in 1952, which deals with hard-core analysis and Tauberian type theorems. Incidentally, Balagangadharan was promoted as a research assistant in 1952 and as a research fellow in 1958. When Prof. K. Ito of Kyoto, Japan, visited the Tata Institute, Balagangadharan got interested in his lectures and this eventually led him to go over to Kyoto and spend a year or more there to work with Ito. His stay at Kyoto was indeed fruitful and resulted in a very interesting paper titled "The prediction theory of stationary random distributions" published in the Math. Coll. of Kyoto which uses many techniques like Schwartz theory of distributions, Wiener integrals, the Paley-Wiener theorem etc., and it has a very detailed review by P. Masani in Mathematical Reviews.

After Balagangadharan came back from Japan, he was awarded the PhD degree by the University of Bombay for his thesis titled "Fourier analysis and random distributions". It is interesting to note that there was a minor hitch in his synopsis being accepted by the University of Bombay since Balagangadharan got his Master's degree from some other University, but at the insistence of Prof. Chandrasekharan, the trouble was sorted out and Balagangadharan obtained his doctorate in 1962.

The UGC had decided to start centres of excellence in mathematics in some Indian universities and University of Bombay became one, when the eminent Prof. S. S. Shrikhande became the Head of the Department of Mathematics of the University of Bombay in 1963. Prof. Chandrasekharan played a decisive role in the creation of this centre in Bombay. Balagangadharan and I were deputed from the School of Mathematics, Tata Institute (we were only 'Fellows' of the Tata Institute at this point of time at the TIFR) to work at the centre as professors, and we, along with Prof. Shrikhande constituted the faculty of this newly formed centre during 1964–67. Balagangadharan's contributions to the centre, both academically and administratively were productive and crucial. It should be said to the credit of the centre that bright students did join. One of them was Patodi (who later became a famous mathematician) who, at that stage, was unaware of even the existence of the TIFR, joined the Centre and only after a year or two moved to TIFR. Patodi went on to do some outstanding work in Differential geometry but unfortunately his productive life was cruelly cut short by his death at a very young age.

I would like to reminisce at this stage a bit about an amusing incident which reflects the nature of academic administration which seems even now, to be the norm in this country. Within a short time after the establishment of the centre, a review committee from the UGC, which to the best of my knowledge consisted of no mathematician, came to assess our progress. One among the several criticisms that the committee raised was that our method of selection of research scholars to the centre was too strict. There was a long letter from the UGC to the university criticising our way of functioning. Balagangadharan, with his usual forthrightness was very angry with this 'unjust' criticism and prepared a rejoinder to be sent to the UGC as a reply. Prof. Shrikhande, the gentle person he was, did not object to it, but in any case he said that it had to get the approval of the Vice Chancellor of the University, who happened to be Dr. Gajendragadkar, the eminent retired Chief Justice of the Supreme Court of India. I remember the meeting in his chambers which the three of us attended. To say the least, Gajendragadkar was not amused by our intended reply and told us that: "I do not want the centre to be closed down during my regime, which the UGC would certainly do if they got such a reply". Needless to say, a suitably mollified reply was sent. In retrospect, it is interesting to note that though the centre wasn't closed then, the UGC decided (for reasons best known to them) at a very much later date, to close it down.

During 1975–80, Balagangadharan also worked at the newly formed TIFR Centre (in collaboration with the

Institute of Science) for Applicable Mathematics at Bangalore. Here too, Balagangadharan's influence was very substantial. To quote from the obituary notice issued by the Centre on Balagangadharan: "As far as the centre is concerned, Prof. Balagangadharan is considered to be a founding father..." . The Centre is also very appreciative of the meticulous efforts he took to make it a flourishing one. He was respected and sought after, not only by mathematicians, but also by the students and staff at large.

On the academic side, I want to mention two beautiful achievements of Balagangadharan, the first which was in collaboration with my colleague late Prof. M. K. V. Murthy. Prof. C. L. Siegel, the great German mathematician visited the Tata Institute three times, and during his last visit in the late sixties, he lectured on the singularities of the Three-Body Problem. Balagangadharan and Murthy wrote the notes of these lectures. It is to the great credit of these two scholars, that Siegel who, as is well-known, was a stickler for standards and could be quite critical in his comments, was in fact very happy with these notes. Incidentally, the notes have received a very detailed and appreciative review in Mathematical Reviews.

I finally come to focus on Balagangadharan's painstaking work translating the notes of Jacobi's lectures (in German) on Dynamics (delivered at the University of Königsberg in 1842–43), whose notes were prepared by Brockardt, and whose second edition was revised by Clebsch. I am rather happy that I was somewhat involved in the publication of these notes as a book by Hindustan Book Agency (HBA), Delhi. The HBA kindly agreed to publish these notes after clearing the copyright procedures. I sent the hand-written notes of Balagangadharan to the HBA (I don't know how many months or years of hard work it must have taken him to prepare this translation), and they sent me back many typed versions of the notes, which they wanted me to correct and return. I received immense help from Prof. Tulsi Das, who was then working in the CMI (like me) as an adjunct-professor, and we both tried to correct as many misprints as possible. However, it appears that there was still some hitch before the book could be published, probably because some German referees to whom the draft copy was sent were not happy with certain aspects of the translation. After a long delay, it was decided to publish the book and I was asked by the HBA for a biographical note on Balagangadharan to be included in the book. This, I sent, after getting the necessary details from Balagangadharan himself, who was then alive. Strangely enough, this note never appeared in the published volume. It seems that the translation itself was once again sent to a few others and, in the published version, after crediting Balagangadharan as the translator of the original German, the name of Prof. Viswarup Bannerjee (who was my colleague at TIFR, a theoretical physicist, now retired) appears as the "editor of the original translation". Unfortunately, the name of Prof. Tulsi Das, who did immense work to help with the proof- reading of the original translation, does not appear in the book (even though I had written to the HBA requesting them specifically that his contribution should be acknowledged in the book). However, on the brighter side, I should mention that the Bombay Math Colloquium, which received a donation of Rs. 5000 from Shri Harry Mirchandaney and his mother, in the memory of Harry Mirchandaney's father (about whom I mentioned earlier), sent it through me to Balagangadharan as a token of their appreciation of his translation of Jacobi's lectures.

But whatever be the strange history of the publication, I felt very proud when I read the highly laudatory and unusually lengthy review in Mathematical Reviews (in 2012), of this volume. The reviewer believes that the lectures of Jacobi are still very important for the important insights they offer. I only quote from the last paragraph. It says " ... No more words are necessary, in our opinion, to let the reader appreciate the scent of the 'true significance' of analytical mechanics, which is often hidden in modern courses on the subject. The lack, in the present edition, of any appropriate historical and technical introduction (as well as appropriate explicative footnotes) to Jacobi's lectures, in addition to the copious typos appearing throughout the text, is the only (and not essential) drawback of the book, which, however, will certainly bring a number of benefits to people interested in the science of one of the greatest minds of mathematical physics". These lines offered me considerable relief, since I was in doubt as to how the present mathematical community would receive this work.

Balagangadharan retired from the Tata Institute, as we said, in 1982. After his retirement, he was appointed as a professor of mathematics at the University of Calicut in 1984, and this was financially supported by the NBHM, UGC and other schemes of this University, during the five year plans, such as the unassigned grants and the like. The department of mathematics, University of Calicut, considers his stay at Calicut to be "a golden era". Balagangadharan, as usual made himself, the most useful, efficient and remarkable and very approachable academic to have around. He retired from the University in 1994. Prof. R. Sivaramakrishnan, to whom I am indebted for many of the early biographical details of Balagangadharan, likes to call him a "true professor" and has immense admiration for him and his contributions to the department. Similar views are expressed by his colleagues, Prof. Krishnakumar and many others who attended and benefitted by his many lectures at the department.

I would like to add a few personal remarks of my own, since I have known Prof. Balagangadharan, first as my teacher when I joined the School of Mathematics at TIFR in 1955, when he taught real analysis to our batch. (I wonder how many people now read the book 'Theory of the integral' of Saks' for learning Lebesgue integration, a beautiful book which he followed in his lectures). I also remember that, in my youthful enthusiasm, I have often gone to his hostel room (he lived just a floor above me in the hostel) to disturb him with mathematical doubts. Though Balagangadharan had the reputation of being a reserved person, to me, he was always extremely kind and took time to clarify any kind of doubts I had. Later on, I got to know him very well as a colleague, when we worked together at the University of Bombay, and then as a friend, as a teacher of Sanskrit, (we used to read *Meghadutam* together and even after his retirement, he used to send regular letters to me containing Sanskrit *slokas*, setting them as exercises for me to translate) and a true mentor (for instance, of my interest in Sanskrit Grammar, in particular, Panini's great work, and the remarkable commentary on it by Patanjali and Bhartrihari's *Vakyapadiya*, about all of which he used to have great regard and used to quote from them regularly). He was responsible also for awakening my interest in the study of history of Indian mathematics.

Though Balagangadharan acquired the reputation of being a reserved person, he was very communicative and he always had a fund of serious and non-serious things to talk about. He could quote, off-hand, from Shakespeare and Charles Dickens, in particular, from Pickwick Papers, David Copperfield, etc. I have also heard many interesting anecdotes from him of his Madras Christian College days (some of which involved also his friend, George Abraham, when they were both working as young tutors); for instance, the hilarious incidents during a meeting held in Madras Christian College when they were discussing why the bachelors should be discriminated against the married ones by giving them an increment of only Rs. 1.75 per month as against the Rs. 3.50 for the married ones, as proposed by the officials, etc. I could go on listing many such, but I will stop with only one more, with his imitation of how Prof. McPhail, who was the warden of the hostel during his stay, used to start the beginning of the week with a prayer which ends with "The end of all life is death. And the aim of life is to prepare for it", in McPhail's typical Scottish accent.

I would like to end this article quoting a few lines from one of his letters to a friend of mine in 1994. I should first remark, that when Balagangadharan retired, the pension scheme had not come into existence at TIFR. He retired with a provident fund, which of course did not last him long. The pension scheme did come a little later, and one had to apply for it within a specified period, but Balagangadharan being a typical academic, unacquainted with administrative nuances, missed the opportunity. Later on, he tried his best to induce the Tata Institute to include him in the pension scheme, but his efforts were of no avail. He wrote to me, and I tried to do my best. Prof. Seshadri and I wrote a letter to the Chairman of DAE seeking his help. I even made a personal request to the Chairman during one of his visits to Chennai, but eventually he also expressed his inability to be of help. Of course, obviously, deep-rooted bureaucracy can never be overcome by efforts of mere individuals. However, it should be mentioned that some of Balagangadharan's friends and colleagues in the mathematical community came forward, during this period, with very generous financial contributions to help him out, which incidentally shows the goodwill Balagangadharan had earned from his friends and admirers. I should mention that, after his retirement from Calicut University, Balagangadharan was living with his brother, and later when his brother died, with his brother's family, who all took care of him with great affection. However, he would naturally have felt much better if he had been financially independent.

During the years after his retirement from TIFR, Balagangadharan must have felt depressed and left out. He was a typical academic, for whom society does not care, unless one is famous otherwise. I quote a few sentences from a letter dated 31st August 1994, which he wrote to my friend, Prof. M. D. Srinivas, which measures the depth of his depression: "I am afraid that there is nothing much I can do to rectify matters; for one thing I have already become a back member, and even during my best days I have never been an 'establishment scientist' and I have been content to so remain. You know, the trouble in our country is that while an administrative scientist has come into his own and can throw his weight about, the creative and academic scientist is still reckoned as a lowly form of life. Of course, the basic reason for this is that in spite of all the lip-service paid to science, there is no general awareness among the educated public and the powers that be that scholarship and things of the mind are valuable in their own right and not for their social relevance. I am happy that I myself am an unrepentant adherent to older values, will soon bow out, but younger people like you will have to live with this; not a pleasant prospect".

In Balagangadharan's death, India has lost an erudite scholar, a great intellectual and a man of true integrity.[1]

[1]There are many people I would like to thank, from whom I have benefitted immensely, while writing this article. First, Aravinda, who in fact suggested that I write an article, Prof. R. Sivaramakrishnan who was of invaluable help in providing details, Prof. V. Thangaraj for his generous help by way providing me with photocopies of some of Balagangadharan's papers, Prof. Rani Siromoney for her kindness in sending me many details about MCC, Sri D. B. Sawant of TIFR for the trouble he took in providing me some details on Balagangadharan's association with the TIFR, Ravi Rao for helping me in various ways and Sri Madhuchandran, the nephew of Balagangadharan (son of Dr. Balaraman) and members of his family (particularly Anagha) for their valuable help in the preparation of this article, Prof. M. D. Srinivas for his ever willingness to help me in every possible way. I thank Vijayalaksmi who helped me in correcting the manuscript. As usual I find no words, to express my indebtedness to Nivedita but for whose help, this article would never have come into existence. May God bless her.

K. Balagangadharan: Some Reminiscences

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I had the privilege of knowing Balagangadharan very well, whose friendship I value and cherish most fondly. He was a scholar in the true sense of the word with interests ranging over mathematics, history of mathematics, literature (English, Malayalam and particularly Sanskrit), the art of Kathakali etc. He had also a great sense of humour. I understand that early in his life he had contemplated seriously of joining the Ramakrishna Mutt.

In the three decades of my life at the Tata Institute of Fundamental Research (TIFR), we used to meet very frequently, often on a daily basis. I remember our reading together (along with M. S. Narasimhan) Hadamard's book on "Partial differential equations". He would also talk about Wiener whose work he admired. Sridharan in his article mentions about Balagangadharan's writing the lecture notes of Siegel's course on Celestial mechanics. In fact, Balagangadharan also wrote the notes of Rademacher's lectures at the TIFR on "Analytic number theory".

Balagangadharan would quote passages with ease from several authors, especially Shakespeare, Dickens (he was particularly fond of "Pickwick papers"), Wodehouse etc. I learnt about Anatole France for the first time, from him; in fact, I read the beautiful book "The crime of Sylvester Bonnard" of Anatole France at his suggestion. Balagangadharan had profound interest in Sanskrit. A memorable experience of my early years at TIFR was our meeting every day after dinner for almost an entire month, when he would recount what he had read that day from the Mahabharata.

I have nostalgic memories of the many days when we used to have tea together at Wayside Inn (a restaurant near Kalagoda in Bombay), reading the column "Hare Bhai" of "The Current" magazine (edited by D. F. Karaka). This column featured a delightful parody of the way English was spoken by a section of the business community in Bombay (some examples: "why for not", "both the three of us").

After my marriage, Balagangadharan had visited my home many times and we used to have dinner together. My wife Sundari was quite fond of him. He would decline to have "ghee" and would often quote a statement in support attributed to the then well-known personality Mr. Nair of the Justice Party – "these Brahmins oozing ghee". Much to my envy, Balagangadharan could enjoy eating a whole raw green chili.

As mentioned by Sridharan, Balagangadharan played a crucial role in the setting up of the Department of Mathematics at the Bombay University, as well as the TIFR Centre for "Applicable mathematics" in Bangalore.

I visited him once after his retirement when he was working at the University of Calicut. I found that he was immensely respected and very popular with his colleagues and students.

I somehow have the feeling that Balagangadharan would have had a greater sense of fulfilment and recognition, had he been in a multi-disciplinary institution of excellence combining sciences and liberal arts. Unfortunately such institutions are very rare in India even now.

The photograph of Balagangadharan which appears here (in his old age) is indeed charming, but the picture which comes to my mind is a tall handsome man smoking a cigarette or a pipe.

M. S. Narasimhan
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Balagangadharan was already at TIFR when I joined there as a research student. His influence in guiding me and other students in our preliminary encounter with modern mathematics was considerable.

His role in educating young students in advanced mathematics and in helping them acquire a historical perspective of mathematics was important in those early days of TIFR. I still remember our reading together and discussing several mathematical topics and books: Petrovsky's and Hadamard's books on Partial differential equations, Zygmund's book on Trigonometric series, Saks and Zygmund on Complex variables, Theory of distributions of Laurent Schwartz. . . . The discussions were not confined only to topics in analysis; I still own the copy of "The topology of fibre bundles", by Steenrod, which he presented to me.

Apart from mathematics, many of his friends profited by his wide scholarship in English and Sanskrit literature. I remember vividly the intense discussions on mathematics and literature we had in our walks from Old Yacht Club (where TIFR was situated) to Wayside Inn or Strand Book Stall.

He played a crucial role in the formation of the TIFR Centre in Bangalore; he mentored several young mathematicians there too and also created a pleasant ambience to work. His passing away is a great loss and has formed a void that would be hard to fill.

Balagangadharan and the Bangalore Centre

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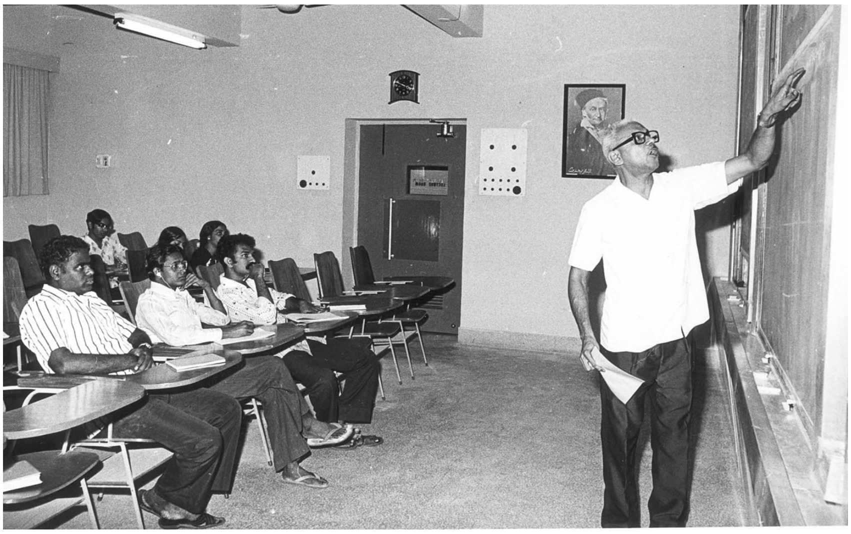


It was in September 1973 that the Joint Programme in Applications of Mathematics came about thanks to the efforts of many distinguished people like K. G. Ramanathan, K. B. Athreya,

*I wish to gratefully acknowledge Sahadevan's kind and generous help in getting some details as also the photograph that appears in the article.

M. G. K. Menon, B. V. Sreekantan and Satish Dhawan, all of whom took considerable interest in initiating this and running this as the TIFR-IISc Mathematics Program. The TIFR Mathematics Faculty and IISc Applied Mathematics Department welcomed this initiative and it was also in 1973 that K. Balagangadharan moved to Bangalore to run this programme jointly with IISc. Initially students were selected at Mumbai and then came to Bangalore for attending courses by eminent visiting professors in the field of Partial Differential Equations. From 1975 onwards, the students were selected at Bangalore for the joint programme. The programme continued in the Applied Mathematics Department in the initial years.

Once M. K. Venkatesha Murthy joined him, the two were an excellent team who worked together and ensured that all that was needed was done with great care and dedication. Balagangadharan was a towering personality in the Applied Mathematics Department, and when he and all of us moved to the TIFR Centre, he was the Planner, Architect, and the Executioner in everything that was happening there. To all of us who were a generation younger, we could not have found a better and more patient person – he was there to take care of everything. Some of us used to walk around with Balagangadharan seeing the ongoing construction; his observations and comments and his characteristic humorous punch lines were what we always looked forward to.

In 1977, the new building (TIFR Centre, next to ECE Department) was ready for the use of the Joint Programme in Mathematics and Radio Astronomy Group (which later moved to Pune and became part of the National Centre for Radio Astronomy). We moved to the bare building on the night of 31 March 1977 – moving most of our limited possessions by walk with Balagangadharan leading us – and everything else was moved from the Applied Mathematics Department immediately. We started functioning in the new building from 1 April.

Balagangadharan planned everything – the required items were bought for running the new Canteen and library (which was restricted to a cupboard full of books) acquired a respectable area and began to get well stocked, the seminar and lecture rooms were furnished with chairs he identified and specially designed by a local supplier. The interchangeable boards, a few tea poys were made at the TIFR Workshop in Mumbai at his request and installed in the new Centre.

The new Programme, TIFR Centre and all of us at Bangalore were fortunate to have Balagangadharan head this new initiative in Bangalore. He being the only permanent faculty at the centre, he was not only our teacher and mentor but also a friend, philosopher and guide. We remember very fondly the long discussions on history of mathematics and mathematicians lives as well as various other topics. E. T. Bell's book "For whom the bell tolls" and "Brighter than a thousand suns" are some of the master pieces that he introduced to the students. He had one of the best collections on Mathematicians, Scientists, History and Science. P. G. Wodehouse was one he often quoted in our discussions and walks.

Apart from his knowledge of mathematics and mathematicians, his knowledge of any subject was tremendous be it science, geography or history or on Bangalore. He knew when the Gulmohurs and Jacarandas in IISc would start blooming (and he noticed that the first Gulmohur to flower was one near the IISc Library and that it kept this date every year). The amount of details on Mathematicians, scientists, Bangalore, its weather, trees and plants that we learnt from him during our interactions was immense. Once the temperature rose, he used to correctly predict that it will rain. . . . Though not true in present-day Bangalore, it generally rained, once the temperature rose beyond a limit. Till mid-80s there were no fans in most places in IISc, as the temperature never went beyond a comfortable range.

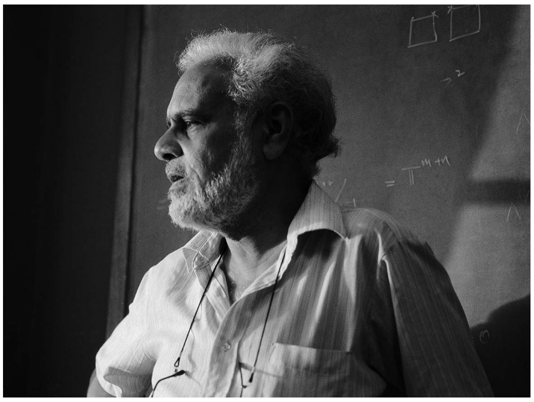
He was a strict disciplinarian and a proponent of hygienic way of life, even the plate in which he dined remained speck-free. It was common for him to go for shopping purchase items required for the Centre – especially after we moved to the TIFR Centre. And these included any special equipments, cultlery, crockery, when he also bought his cigars from Spencers on M. G. Road, peeping at the same time at the Premier Book Shop; the Institute did not have a Bank account, but transferred sufficient money to a new bank account which was opened in his personal name, and the accounts for this was rendered every month to Mumbai.

His personal requirements were meagre. After moving to the TIFR Centre, his lunch for many months was curd-rice (because the Canteen made only that) and later curd-rice and sambar-rice when the Canteen added that to its menu. He was simple and polite to the core, and was a very kind and caring person. At the Centre he was respected and sought after by not only mathematicians, but by the students and staff at large. He was an avid and patient listener and capable of understanding, and finding solutions to, so many problems where ordinary fellow human beings could succumb to irritation and impatience. A cigar smoker, which made his presence felt unseen, he was equally at ease while interacting with the Director of IISc, the students at the Centre, staff at all levels, or a visiting dignitary.

After retirement from TIFR, he joined the Calicut University as a guest faculty and helped the Mathematics Department there to grow. Most of us from the initial batches at the Bangalore Centre maintained our contact with him even till 8 months before his sad demise. Balagangadharan knew that we have spread our wings from the humble beginnings in 1973, that we are a much larger group who have moved to Yalahanka with wider interests and a strong Ph.D. programme. We last met him in his house in Trichur on 29 December 2011, when we promised to go back and meet him more frequently! Alas, that can't happen now and we will miss his inspiring personality and friendly disposition.

Remembering Daya-Nand Verma

Robert V. Moody



25-6-1933 to 10-6-2012

In 1966, I received a letter from Daya-Nand Verma. It was about my thesis on what are now called affine Lie algebras. My supervisor Maria Wonenburger had asked George Seligman at Yale University to be the external examiner of my thesis, and George was closely connected with Daya-Nand. It was never clear to me whether George or Nathan Jacobson was Verma's thesis advisor, though I doubt that it made any difference; Verma was extraordinarily brilliant and was always brimming with ideas. I am sure he made up his own program.

In any case, George showed Verma my thesis because he knew that he had also begun to develop a theory of affine Lie algebras. Luckily for me he had started quite a bit later than I had! Anyway, Verma's letter was very gracious and was more of a letter of introduction than anything else. He was never one of the highly competitive types. He respected mathematical ideas no matter where they came from.

Later that year he went to Las Cruces New Mexico to work with Louis Solomon (see Arun's reminiscences) and Edward Kobayashi. At that time Mew Mexico State University had an annual New Year's seminar, and they invited me down for that (Paul Cohen was speaking on his proof of the independence of the continuum hypothesis and the axiom of choice), mostly on the pretext of interviewing me for a job there the following year. So that was my first experience with Dayan (as he later asked me to call him), and this was my first opportunity to experience his intense and wide-ranging intellect. At the time I must say that I found this quite intimidating, but he also had a charming way about him and even when he disagreed with something, he could always do this in a way that was completely disarming.

He decided to take my wife and me around southern New Mexico, including Carlsbad Caverns, El Paso and the neighbouring Ciudad Juarez, (which was still a quiet Mexican border town at that time). We got introduced to a lot of Mexican food on that trip, the main thing being that it was sufficiently hot for Verma's latent Indian taste. That was to be the first of three travelling journeys I made with him, the other two being in India.

Not surprisingly the now famous Verma modules were a big part of his thesis, but what he set out to achieve, and did achieve, in that thesis is quite lovely and not often stated. He realized that the famous Weyl character formula could seen as a sort of alternating sum of characters of Verma modules (essentially free highest weight modules) with the weights running over a certain Weyl group orbit of weights, and this alternating sum was really a result of a Möbius inclusion-exclusion principle deriving from a certain partial order on the Weyl group. This same idea can be used to prove the character formula for the Kac-Moody algebras.

By the time I got to New Mexico State University in August 1967, Dayan had already gone back to India. I think that partially it was to get married. I don't know the details, but I think that he felt that it was the right time of life for marriage and that it was arranged. I only met his wife, Meena, on one occasion and never got to know her.

My next interaction with Dayan occurred when I spent some time at the Tata Institute of Fundamental Research (TIFR) in Bombay (as it still was then). We spent many hours together, talking mathematics, going to concerts of Indian music, which we both loved, at the National Centre for the Performing Arts at Nariman Point, and, most memorable of all, travelling to places in India. He loved to travel and loved to act as a travel guide. Travelling found him more relaxed, charming, and very kind. Still, he could get quite upset with incompetence. Our trip to Ellora and Ajanta involved taking an overnight train and detraining at something like 6:00 am at a small whistle stop along the way. The problem was that the beds in the sleeper cars had not been made up after the train had come in and we were expected to sleep in beds with soiled bed linen. Verma was greatly annoyed by this (rightfully so). Anyway, as we got off the train, he demanded to fill out complaint forms. With the long train anxious to get moving again, the conductor motioned us towards the station house. But no. Verma knew that the train carried its own forms and demanded to use these. So while the train stood restlessly in the station, the complaint book was duly fetched and he filled out the lengthy forms (in triplicate, needless to say). With the conductor anxiously waving us to finally leave, Verma said that I too wished to fill out a complaint form! Well, we didn't make any friends with the fellow passengers of that train, but I suppose the point was well and truly made. The caves at Ellora and Ajanta were fabulous, of course.

My last trip with him was to Pune where we visited the Inter-University Centre for Astronomy and Astrophysics. This is a very nice place, by the way, with a lot of very interesting sculptural art and tile work designs with mathematical and scientific emphasis. This was another train trip, this time up the very attractive narrow gauge railway from Mumbai.

Verma was a larger than life character. I think that he was a thorn in the side of administrators. He was not much cowed by authority! His interests were very wide-ranging. On my first visit to TIFR he was engaged in a long study of biology. This was long before mathematical biology became fashionable. I don't think it ever came to anything, but I remember that he was very good at identifying trees. At that time he also introduced me to Grunbaum and Shepherd's great book Tilings and Patterns, a book that was to be a great inspiration to me when I got interested in aperiodic order. At the time of my final visit Dayan was fascinated by the Collatz conjecture (or $3n + 1$) conjecture. Generally his power came from the depth of his thinking. He was a creator of ideas but, sadly, not one to follow them through. He wrote virtually no papers, although those he did write were very influential.

The photograph is one that I took of him at TIFR. I think that it was early 1995. It was late afternoon and the sunshine through the window was very beautiful. He was expounding some of his latest ideas on representation theory, but I got distracted by the light and grabbed my camera to take the shot. The picture brings back a flood of memories. Those days are gone forever, but they were good, and I shall always remember Verma as my friend and as my mentor too.

Robert V. Moody, September 2012

D.-N. Verma (1933–2012): A Memory

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Who was D.-N. Verma? A character certainly – even more so than most mathematicians. For those of us who dabble in Representation Theory he certainly has had a great effect on us for we cannot imagine a world without Verma modules and the rich theory and structures that they support. It is a great boon for me to know "Verma modules" and also to have known the man D.-N. Verma and the shimmering sea in his mind around these objects.

I can give only a very personal account. To do so I must go back to the year 1966.

D.-N. Verma completed his PhD in 1966 from Yale University under thev guidance of N. Jacobson. The title of his thesis was "Structures of Certain Induced Representations of Complex Semisimple Lie Algebras". These induced representations are what are now commonly called "Verma modules". According to my incomplete knowledge (the little I know comes, I think, from discussions long ago with Kostant), these modules had appeared previously in work of Harish-Chandra and Chevalley, but they solidly entered the community psyche after Vermas thesis came to the attention of Kostant and Dixmier. In Russia, at about the same time, the Gelfand school (particularly J. Bernstein, I. M. Gelfand and S. Gelfand) were intensely studying these modules and their work had a great influence in shaping the resulting theory.

I was born in October 1966. I like to say that in 1966–1967, R. Moody, L. Solomon, D.-N. Verma and I were all in Las Cruces, New Mexico USA working hard on research in Lie Theory. It is true that we were all in Las Cruces, New Mexico USA that year, but it is unlikely, given my age, that I was helping very effectively with the research. D.-N. Verma was a good friend of my father and so we certainly met that year. The team of R. Moody, L. Solomon and D.-N. Verma was probably the most promising trio of young Lie theorists of the time. By the time I properly entered Representation Theory in 1988 they had all become legends in the field.

D.-N. Verma spent the year 1967–1968 at the Institute of Advanced Study in Princeton and, in 1968 joined the Tata Insitute of Fundamental Research (TIFR) in Mumbai, where he remained, except for a few short periods as visiting Professor in Europe and the USA, until his retirement in 1993–1994. I remember that, as a child, every time that we were in India, we would, of course, spend some weeks in Mumbai to visit my Aunt who lived in Fort, near VT (now CST Station). An important part of our visit was our, usually daily, treks to the TIFR (we would walk across the maidan to catch the TIFR bus) where my father would visit his friends and I would, very happily, play on the beautifully manicured grass, and on the rocks along the ocean. A constant aspect of these outings was the company of Verma Uncle, who was, for me as a child, another one of those pleasant features of our visits to India and the TIFR.

In 1987, I entered graduate school at University of California, San Diego and was, not long after, taking a course in Representation Theory from N. Wallach. I remember talking casually on the phone with my father, who was asking me which courses I was doing. When I told him I was taking Representation Theory he asked me, "Did you learn Verma modules?" ... it clicked, at that moment I realised who Verma was indeed "Verma Uncle" was the Verma of Verma modules.

In 1991, I completed my PhD at San Diego, a new, fresh, uncertain, Representation Theorist. That summer my father and I visited family in India, including our usual trip to Mumbai. It was a wonderful and inspiring visit for me. Vermaji took me under his wing for a few weeks and, while my father was off talking physics with his colleagues, Vermaji began to teach me: his picture, answers to my questions, many beautiful vistas of the field that I hadnt imagined.

It must have been wonderful also for Vermaji that summer, as he had too few disciples that could process the flood of haphazard observations and relationships between structures. As I matured, it was also difficult for me in later conversations, as we students become rigid as we get older and dont listen so well. But at that moment, it was ideal, and there was no faster way for me to learn the depths and intricacies of the structures behind BGG resolutions, Jacobi-Trudi formulas and special values of Kazhdan-Lusztig polynomials. And learn

I did, fast, and it has had a great influence on all my future work.

From that time D.-N. Verma and I had two relationships: a familial one, as I was the son of his close friend, and a mathematical one. I have had consistent sporadic mathematical contact with him since 1991. As many of his friends know, one will, at periodic intervals, receive a long E-mail and a rambling preprint with many observations and partial theorems and not quite finished connections between important structures. The most recent of these arrived in my Inbox on 14 March of this year.

Looking back at this E-mail I am reminded of discussions with I. M. Gelfand in the late 1990s, which sometimes seemed to me to require infinite patience as he went on rambling about something that I couldnt focus on very well. However, on those few occasions when my willpower was great enough to force myself to focus for the complete story, I was always amazed afterwards at what a treasure of a piece of knowledge I had been given insights far beyond those occurring in ordinary months of work and learning. Verma was similar. If you had the patience and ability to wade through and parse it, you could be certain there would be a gem there. I remember sitting with Verma on the bus during an excursion on the free afternoon at a conference in Magdeburg in 1998 when he explained to me how the Pittie-Steinberg-Hulsurkar basis for K-theory of flag varieties was the same as the Shi arrangement. This is another example that has powerfully shaped my view of mathematics (the picture of the Shi arrangement exactly as Verma told it to me appears in my paper in the volume in honour of Steinbergs 80th birthday).

Vermas final E-mail to me was stimulated by the recent passing of our mutual good friend Harsh Pittie. This had motivated him to think again about the Pittie-Steinberg-Hulsurkar picture and the Shi arrangement and its relationships to various structures. His extensive E-mail has many paragraphs on this. Of course he is right, this is fundamentally connected to the Kazhdan-Lusztig theory of affine Weyl groups, cohomology and quantum cohomology of flag varieties, the moduli of stable vector bundles, conformal blocks, The Chevalley-Shephard-Todd theorem, the Knizhnik-Zamolodchikov connection, the moduli space of Riemann surfaces with marked points, the Verlinde formulas, and, in his words, the "Magical Expansion Formulae" (by which he means the formulas (7.1), (5.5), (2.3) and the formula in footnote 2 of his paper, "The role of affine Weyl groups in the representation theory of algebraic Chevalley group and their Lie algebras", in the Proceedings of the 1971 Summer School at Budapest edited by I. M. Gelfand).

However, I heartily admit that neither he nor I were ever capable of shaping all these connections into a coherent mathematical framework for easy processing by the community. In his E-mail, Verma suggests looking at his Budapest paper (certainly my favourite paper of his for its myriad of realisations). I concur with his suggestion, particularly after having spent a couple of afternoons this past week rereading bits and pieces of this paper. Many stimulating intricacies of the beautiful patterns of crystallographic symmetry and the gems around mathematics that are controlled by it are to be found here – for anyone willing to don their mask and snorkel and swim in the shimmering sea.

Obituary for Prof. D.-N. Verma

Shrawan Kumar

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I am truly saddened to learn the sudden passing away of Prof. Daya Nand Verma. It is a great personal loss to me and it is a loss to the mathematical community.

Even though his list of published works is small, he has had tremendous impact in the development of Lie theory.

Of course, anyone working in any aspect of Lie theory must know Verma modules, whose study was initiated by him in his PhD thesis (1966). His other influential work lies in modular representation theory. Verma conjectured that the category of algebraic representations of a semisimple simply-connected algebraic group G over an algebraically closed field k of characteristic $p > 0$ is equivalent to the category of locally-finite representations of the corresponding hyper-algebra U_k, which is an analogue of the enveloping algebra in characteristic p. Moreover, he went on to conjecture in

1971 the "linkage principle," on the composition factors of the Weyl module of G bringing in the role of the affine Weyl group, which was a very novel idea at the time. Both of these conjectures, which were subsequently proved by various mathematicians (notably Andersen, Cline-Parshall-Scott, Jantzen and J. Sullivan in the 1970s) have played a fundamental role in the development of modular representation theory. His long review of Macdonald's paper was very insightful and instrumental in future developments of Kac-Moody theory.

In addition, I would like to mention his work on Mobius functions, where he determined the Mobius function of a Coxeter group (1971); and his work with J. E. Humphreys on projective modules for finite Chevalley groups in defining characteristic (1973). Hulsurkar proved an important conjecture due to Verma on Weyl's dimension polynomials (1974). Recently, Verma explicitly determined the Hilbert syzygies for the homogeneous coordinate ring of the Grassmannians.

Whenever I got an opportunity, I often discussed mathematics with him, and his insightful observations were valuable to me. To cite one such instance: I had proved the Parthasarathy-Ranga Rao-Varadarajan conjecture in 1987. Verma, while attempting to give a purely algebraic proof, came up with a very interesting refinement of the conjecture, which I managed to prove subsequently.

During my recent visit to TIFR (in May this year), Verma explained to me some of his thoughts on the modular representation theory and the role Hulsurkar's work should play. I wish we had had more time to discuss his ideas, which was postponed to our next meeting. Who knew at the time that it was never going to be!

Apart from his devotion to mathematics, he had various other interests, including cooking and watching movies and plays. As far as I know, he managed to do mathematics and watch movies even on the very last day of his life.

I had the privilege to know him for more than 30 years and I will greatly miss him, especially during my visits to TIFR.

Prof. B. L. Sharma (1935–2012)

Satya Deo
HRI, Allahabad

The mathematics community was shocked to learn that Prof. B. L. Sharma passed away on September 26, 2012 suddenly while he was in Chandigarh in connection with some social work. Though he was almost 78, he was very active physically as well as mentally. He leaves behind his wife, son and numerous friends who will have to cope with this tragic news.

Prof. Banwari Lal Sharma was born in Firozabad district of western U.P. and completed his B. Sc. and M. Sc. degrees in mathematics from the university of Allahabad. He had a brilliant career securing first class with distinction in all of his exams and was appointed lecturer in Mathematics at the University of Allahabad immediately after his M. Sc.

After serving in the University of Allahabad for a couple of years, he went to Paris for his doctorate in mathematics. He worked under the guidance of Prof. H. Cartan, a famous algebraic topologist of international repute, and got his Sc. D. degree in Algebraic and Differential topology. His thesis dealt with the question of topological invariance of integral Pontrjagin characteristic classes of piecewise linear manifolds, and the results were published in Compte Rendus Acad Sc. Paris. He learnt everything on this subject at Paris starting from scratch; he did not know what is a map or a group before going to France, as until 1963 Abstract Algebra was not being taught in most of the Indian Universities including Allahabad. Even then Prof. Cartan accepted him as his student for his characteristic simplicity making honest efforts to learn the new subject. He learnt French language and became fluent in it. He visited France and other countries during winter season wearing only his khadi dress with a Indian sweater confronting the coldest weather conditions of European countries.

When he returned to India, he was offered jobs in several academic places in India including IIT, Kanpur, but he preferred to join back Allahabad University simply because of his principles. Prof. Sharma always kept his principles above all material gains throughout his life including name and fame. He played a crucial role in upgrading the syllabi of Mathematics at Allahabad University while he was only a lecturer and had officially no power at all. He was so contented with his mathematical work that he never applied for Readership or Professorship even at the University of Allahabad let alone speak of outside institutions; I am witness to this well-known fact. He was offered these positions by the authorities of Allahabad University without applying, and this was, of course, much later than he merited and only after his seniors were promoted. He was persuaded to accept them by his seniors and students at the University of Allahabad. He guided several research scholars for their doctorate degrees on various topics of Algebraic and Differential Topology at the university of Allahabad. As a man, he was a highly respected selfless person helping everyone all the time including the students as well as his colleagues.

Being an excellent teacher of Mathematics, capable of teaching any topic of pure or applied mathematics, he was well known to the university fraternity as a committed teacher and brilliant scholar with amazing memory. His simple dress, always wearing Khadi (white Kurta, dhoti and a chappal) was characteristic of his personality as a uniquely popular mathematics lecturer.

Prof. Sharma started a "topology seminar" at the University of Allahabad after returning to India from Paris, in the late evening hours outside office hours, which was attended by many Mathematics as well as Physics teachers of Allahabad University and its colleges. This seminar became a very successful and popular activity of Mathematics Department. It also resulted in writing of some text books for local use for graduate students (on the pattern of N. Bourbaki books), and were known for their originality. Many of us got our own initial training of basic mathematics only from these seminars. In due course of time, this seminar was converted, under the patronage of Prof. B. L. Sharma himself, into a series of "Topology Conferences" in the whole of India. Several topologists from research institutes and universities in the country became themselves involved in these conferences and they began to host them on an annual basis in various parts of the country. Many people know that, once announced, the conference will go on according to the schedule even if there is a curfew in the city. These conferences became stimulus to conferences of this nature in other subjects in the whole country, and the current ATM schools conducted by the NBHM are somewhat refined form of those conferences. A large number of mathematics research scholars have immensely benefitted from these conferences.

It is difficult to forget some of the events associated with his "principles" and "moral character." Prof. Sharma used to travel a lot on official works, but he would always travel in a sleeper class of the train. He stuck to this principle for his whole life. Even though he was entitled for travel by air or AC first class, he never claimed it. After reaching his destination when the host would request him to sign a TA bill form, he will take the form in his own hand and make sure that the host writes the actual fare of sleeper class and nothing more like taxi etc.

He had once fixed a particular date for the viva-voce exam of a PhD candidate in Jammu. Just a few days before he was to travel, his elder brother died. The date fell in between the last rites of his brother's death, but he didn't cancel his programme and reached Jammu with his head shaven. We were all surprised, but helpless to do anything in the matter. He conducted the exam and when we indicated that he should have cancelled his program, he said that the student should not suffer because of him.

Prof. Sharma had high regard for the cause for which the late Jai Prakash Narayan was known at that time. As a gesture of assistance, Prof. Sharma organized a mammoth rally in Allahabad for JP movement while most of the teachers of Allahabad were afraid even to attend that rally. As a result, he was jailed during emergency days for the full duration of 19 months. While in Naini jail, he wrote a book on Topology in Hindi and also translated "Riemannian Geometry" by C. F. Weatherburn in Hindi. In that jail he was lodged in a room with a person who developed great respect for Sharma Ji and treated him like his elder brother. However, that man (Veer Bahadur Singh) became chief minister of UP after the emergency was lifted and elections were held. Then, someone in the Mathematics Department of Allahabad University casually asked him, "Sharma Ji, you should also have become a minister in his cabinet". His answer was: I am very happy to remain only a citizen of this free country and wish the chief Minister to perform well.

In meetings and conferences, always addressed the audience with exemplary conviction regarding the state of affairs regardless of the topic particularly in mathematical teaching and research in this country. Once, while he was the chief guest in the opening ceremony of the annual conference of Ramanujan Mathematical Society (RMS) at Allahabad, he put fourth his frank views and raised questions (uncomfortable for all of us) about the achievements of the mathematicians of our country. He started by praising the beginning works of Ramanujan published in the Journal of Indian Mathematical Society and asked, in particular, as to why, in spite of all the facilities that we have in our country for doing research in mathematics, we have not been able to produce even one Fields Medalist so far.

While serving the cause of Mathematics, Prof. B. L. Sharma was also deeply involved in Social Service with national concern. Inspired by Vinoba Bhave, he had a mission to do something for the society and the country in the spirit of Gandhi Ji. This mission of Prof. Sharma was so dear to him that after retirement from Mathematics, he devoted himself as a full time social worker fighting against the multinationals who were obstructing the development of small scale industries in this country. Even while he was in service at Allahabad University, he was the Director of Gandhi Peace Foundation (Gandhi Bhawan) of Allahabad University and had organized numerous activities for the benefit of our society at large. He started an NGO called "Azadi Bachao Andolan" at Allahabad and published regularly a popular monthly newsletter to promote these activities. He even established a university called "Swaraj Vidya Peeth" in which the students, equipped with a strong moral character, are trained to serve our nation. This Vidyapeeth has never sought any financial help from any government body or organization; it is running from voluntary donations from the friends of Prof. B. L. Sharma. These days while he was in his late seventies, he had been travelling across the country to spread his mission of service to the nation. He had all the energy to keep himself always busy. His willpower and conviction of his mission was so strong that he never felt that there was any problem with his physique. Even if there was any problem, no one could know it because he was so active that there were no symptoms at all. We will always miss him and remember him as a great man, having unbelievable courage, always following the motto of "simple living and high moral thinking." He was very proud of his country and its youth.

Problems and Solutions

Edited by Amritanshu Prasad

The section of the Newsletter contains problems contributed by the mathematical community. These problems range from mathematical brain-teasers through instructive exercises to challenging mathematical problems. Contributions are welcome from everyone, students, teachers, scientists and other maths enthusiasts. We are open to problems of all types. We only ask that they be reasonably original, and not more advanced than the MSc level. A cash prize of Rs. 500 will be given to those whose contributions are selected for publication. Please send solutions along with the problems. The solutions will be published in the next issue. Please send your contribution to `problems@imsc.res.in` with the word "problems" somewhere in the subject line. Please also send solutions and comments on these problems to the same e-mail address with the word "solutions" somewhere in the subject line. Selected solutions will be featured in the next issue.

1. **Apoorva Khare, IMSc** Find the remainder when $1^{16} + 2^{16} + \cdots + 22^{16}$ is divided by 23.
2. **Anuj V.T., IMSc** Consider a decomposition $n = a_1 + \cdots + a_l$ of a positive integer as a sum of distinct positive integers $a_1 < \cdots < a_l$ such that the product $a_1 \dots a_l$ is maximized. If $8n + 25$ is a perfect square then $a_i - a_{i-1} = 1$ for each $i \in \{2, \dots, l\}$.
3. **Partha Sarathi Chakraborty, IMSc** For a maths quiz the teacher forms a random team from a class of 40 as follows:

   (1) Tickets numbered 1–40 are distributed randomly among the students.

   (2) A fair coin is tossed 6 times and the number of heads are counted, say k.

   (3) Those with tickets numbered $1, 2, \dots, k$ are selected.

   (4) A random team of extras are selected from those holding tickets numbered $7, \dots, 40$.

   (5) (3) and (4) together constitute the final team (here random means that each subset of students holding tickets numbered 7 to 40 has probability 2^{-34} of being selected).

What is the probability that the final team consists of (a) Judhisthir, Bheema, Arjuna, Nakula and Sahadeva. (b) Duryodhana, Duhshasana and Karna.

4. **Anilkumar C.P., IMSc** Given an integer n, consider the convex hull P of the set of points $(d, n/d)$, where d runs over the set $1 = d_1 < d_2 < \cdots < d_k = n$ of divisors of n (P is a polygon in the Cartesian plane). In terms of $d_1, \cdots, d_k$ find an expression for

   (1) The area of P.

   (2) The number of points on the boundary of P with integer coordinates.

5. **B. Sury, ISI Bengaluru** If we take any quadrilateral and join the midpoints of the sides,we can easily see that we obtain a parallelogram. More generally, take any polygon of $2n$ sides and consider the centroids of the sets of n consecutive vertices. Prove that the polygon of $2n$ sides so formed has opposite pairs of sides equal and parallel.

6. **B. Sury, ISI Bengaluru** Evaluate

$$1 - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \frac{1}{23} + \frac{1}{25} - \cdots.$$

Solutions to problems from the June issue

1. **Raman V., IMSc**

   (1) $-1 + 2^7$

   (2) $71 \times 2 - (17 - 2)$

   (3) $\frac{n-1}{2}(1) - \frac{n-1}{2}(1) + (1)$ if n is odd; $\frac{n-2}{2}(1) - \frac{n-2}{2}(1) + (2)$ if n is even.

2. **Balachandran Sathiapalan, IMSc** Consider

$$N(z) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ z & 0 & \cdots & 0 & 0 \\ 0 & z & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & z & 0 \end{pmatrix}$$

   Then $C(z) = 1 + N(z) + N(z)^2/2! + N(z)^3/3! + \cdots + N(z)^9/9! = e^{N(z)}$ (note that $N(z)^{10} = 0$). Therefore, $C(z)^{-1} = e^{-N(z)} = C(-z)$ (which is again a finite sum).

3. **S. Viswanath, IMSc** Take the n-dimensional vector subspace spanned by x^{a_i} inside $\mathbb{R}[x]$. Equip this with the inner product $\langle f, g\rangle := \int_0^1 fg\,dx$. Then A is just the Gramian matrix $(\langle x^{a_i}, x^{a_j}\rangle)$ of this inner product with respect to the basis $\{x^{a_i} : i = 1 \cdots n\}$, and is thus positive definite (see, for example, Chapter 8: Volumes of Parellelograms of *Honors Calculus III/IV* by Frank Jones, `http://www.owlnet.rice.edu/~fjones`).

B. Sury (ISI Bengaluru) points out that the determinant in the problem is a special case of a Cauchy matrix (see `http://en.wikipedia.org/wiki/Cauchy_matrix`), for whose determinant there is a well-known formula, thereby giving a different solution: for the $n \times n$ matrix A whose (i, j)th entry is $\frac{1}{a_i + b_j}$ for some real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ such that $a_i + b_j$ is never zero,

$$\det A = \frac{\prod_{i<j}(a_i - a_j)\prod_{i<j}(b_i - b_j)}{\prod_{i,j}(a_i + b_j)}.$$

Indeed, taking $b_i = a_i + 1$ gives the determinant of the problem. In this special case, the right hand side of the above formula has numerator $\prod_{i<j}(a_i - a_j)^2$, which is always positive, and denominator $\prod_{i,j}(a_i + a_j + 1)$ which is positive when the a_i's are. Sury has sent us a simple proof of the formula which is reproduced below:

Proof of the determinant formula

Consider the matrix $A_n = \left(\frac{1}{a_i + b_j}\right)_{i,j}$. Subtracting the first row from the other rows, the determinant does not change and equals

$$\det \begin{pmatrix} \frac{1}{a_1+b_1} & \frac{1}{a_1+b_2} & \cdots & \frac{1}{a_1+b_j} & \cdots & \frac{1}{a_n+b_n} \\ \frac{a_1-a_2}{(a_1+b_1)(a_2+b_1)} & \cdots & \cdots & \frac{a_1-a_2}{(a_1+b_j)(a_2+b_j)} & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \frac{a_1-a_n}{(a_1+b_1)(a_n+b_1)} & \cdots & \cdots & \frac{a_1-a_n}{(a_1+b_j)(a_n+b_j)} & \cdots & \cdots \end{pmatrix}$$

Taking out the factor $(a_1 - a_i)$ from the i-th row, for every $i > 1$ and, taking out the factor $\frac{1}{a_1+b_j}$ from the j-th column for each $j = 1, \cdots, n$ we get

$$\det A_n = \frac{\prod_{i>1}(a_1 - a_i)}{\prod_{j\le n}(a_1 + b_j)} \times \det \begin{pmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ \frac{1}{a_2+b_1} & \cdots & \cdots & \frac{1}{a_2+b_j} & \cdots & \frac{1}{a_2+b_n} \\ \vdots & & & & & \vdots \\ \frac{1}{a_i+b_1} & \cdots & \cdots & \frac{1}{a_i+b_j} & \cdots & \frac{1}{a_i+b_n} \\ \vdots & & & & & \vdots \\ \frac{1}{a_n+b_1} & \cdots & \cdots & \frac{1}{a_n+b_j} & \cdots & \frac{1}{a_n+b_n} \end{pmatrix}$$

Subtracting the first column of the latter matrix from the other columns and again taking common factors out, we have

$$\det A_n = \frac{\prod_{i>1}[(a_1-a_i)(b_1-b_i)]}{\prod_{i>1}(a_i+b_1)\prod_j(a_1+b_j)}$$
$$\times \det \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 1 & \frac{1}{a_2+b_2} & \cdots & \frac{1}{a_2+b_j} & \cdots & \frac{1}{a_2+b_n} \\ \vdots & & & & & \vdots \\ 1 & \frac{1}{a_n+b_2} & \cdots & \frac{1}{a_n+b_j} & \cdots & \frac{1}{a_n+b_n} \end{pmatrix}$$
$$= \frac{\prod_{i>1}[(a_1-a_i)(b_1-b_i)]}{\prod_{i>1}(a_i+b_1)\prod_j(a_1+b_j)} \det A_{n-1}.$$

Inductively, $\det A_{n-1} = \frac{\prod_{1<i<j}(a_i-a_j)\prod_{1<i<j}(b_i-b_j)}{\prod_{i,j>1}(a_i+b_j)}$, which gives the asserted determinant for A_n.

4. **Apoorva Khare, Stanford University** For each $r \in R$, the F-linear map $R \to R$ given by $x \mapsto rx$ is injective (since r is not a zero divisor). Since every injective linear endomorphism of a finite dimensional vector space is surjective, there exists $x \in r$ such that $rx = 1$.
5. **Apoorva Khare, Stanford University** Consider

$$f(t) = 1 - \sum_{i=1}^{k} \prod_{j=1, j\neq i} \frac{a_j - t}{a_j - a_i},$$

a polynomial in t of degree less than k. Since f vanishes for k distinct values of t (namely $a_1, \ldots, a_k$), f must be identically zero. In particular, $f(0) = 0$, which is the required identity.

B. Sury (ISI Bengaluru) writes

In fact, this problem has many more general versions. For instance, if f, g are polynomials over any field F such that all the roots $\alpha_1, \cdots, \alpha_n$ of g (in an algebraic closure of F) are distinct and, if the degree of f is $< n-1$, then $\sum_{i=1}^{n} \frac{f(\alpha_i)}{g'(\alpha_i)} = 0$.

Here, if $g(x) = b_0 + b_1 x + \cdots + b_n x^n$, then

$$g'(x) := b_1 + 2b_2 x + \cdots + n b_n x^{n-1}$$

When we are over a subfield of $\mathbf{C}$, the assertion $\sum_{i=1}^{n} \frac{f(\alpha_i)}{g'(\alpha_i)} = 0$ when $\deg f < n-1$, is also a consequence of the fact that the sum of the residues (including ∞) of the meromorphic function $f(z)/g(z)$ is 0. If deg f =deg $g-1$, the above sum $\sum_{i=1}^{n} \frac{f(\alpha_i)}{g'(\alpha_i)} = a/b$, where a, b are the top coefficients of f, g respectively.

Using this generalization, some results such as the following have been obtained in *Abel's theorem and a Gaussian determinant*, Expositiones Mathematicae 14 (1996), pages 85–91.

If $0 < a_1 < a_2 < \cdots < a_n$ are integers, then consider the matrix A whose (i, j)-th entry is the Gaussian polynomial $G_x(a_i, j-1)$ where

$$G_x(n, r) = \frac{(x^n - 1)(x^{n-1} - 1)\cdots(x^{n-r+1} - 1)}{(x^r - 1)(x^{r-1} - 1)\cdots(x - 1)}$$

for $n \geq r > 0$ and $G_x(n, 0) = 1$.

Then, $\det A = x^{n(n-1)/2} \prod_{i>j} \frac{x^{a_i} - x^{a_j}}{x^i - x^j}$, a polynomial with integer coefficients.

In particular, taking $x = 1$, we get $\prod_{i>j} \frac{a_i - a_j}{i-j}$ is an integer. Indeed, this number also occurs as the dimension of a representation of the group $SU(n)$.

6. **Apoorva Khare, Stanford University** We shall show that if g has a pole, then $f^n - g^n$ has a pole for infinitely many n.

We have

$$f^n - g^n$$
$$= (f-g)(f^{n-1} + f^{n-2}g + \cdots + fg^{n-2} + g^{n-1}).$$

We know that $f = g + h$ for some holomorphic h. Therefore,

$$h[(g+h)^{n-1}g + (g+h)^{n-2}g^2 + \cdots + g^{n-1}]$$

Suppose b has a pole of degree d at a point z. The second factor of the above factorization is of the form $ng^{n-1} + h_1$, where h_1 has a pole of order not more than $d(n-2)$. Therefore, the second factor has a pole of order $d(n-1)$ at z. As n increases, the order of the pole of the second factor at z will increase to infinity. Therefore, no matter what r is the product can not be holomorphic for infinitely many n.

Then $b = x/y$ for some $x, y \in A$ with $y \neq 0$, with x and y having no prime factors in common, and with y not a unit. Suppose that $y \in (p)$ for some prime ideal (p). Then

7. **C. P. Anilkumar and Kamalakshya Mahatab, IMSc** Suppose we have a factorization

$$1 - x_1 \ldots x_k = f(x_1, \ldots, x_k) g(x_1, \ldots, x_k). \quad (*)$$

Setting $x_1 = 0$ in the above identity gives $1 = f(0, x_2, \ldots, x_k) g(0, x_2, \ldots, x_k)$. Therefore, $f(0, x_2, \ldots, x_k)$ and

$g(0, x_2, \ldots, x_k)$ are constant. By scaling them, we may assume they are identically 1. In other words,

$$f(x_1, \ldots, x_k) = 1 + x_1 f_1(x_2, \ldots, x_k),$$
$$g(x_1, \ldots, x_k) = 1 + x_1 g_1(x_2, \ldots, x_k).$$

for polynomials f_1 and g_1 in $k-1$ variables. If both f_1 and g_1 are non-zero, then the product will have a monomial with x_1^2 in it, which contradicts $(*)$. In other words, either f or g is identically 1, which shows that $1 - x_1 \ldots x_k$ is irreducible.

8. **Apoorva Khare, Stanford University** Say $C_1, C_2, \ldots$ are disjoint closed subsets of the reals contained in (0,1). We produce $x_0 \in (0, 1)$ not in any C_n.

   Set $a_0 = 0$ and $b_0 = 1$. If all C_n are empty then we are done. If not, set $a_1 = \sup C_1$.

   If no C_n intersects (a_1, b_0), we can choose $x_0 = (a_1+b_0)/2$. Otherwise, let $n_1 > 1$ be the least integer such that C_{n_1} intersects $(a_1, 1)$. Let $b_1 = \inf[C_{n_1} \cap (a_1, 1)]$.

   If no C_n intersects (a_1, b_1), then we can choose $x_0 = (a_1 + b_1)/2$. Otherwise, let $n_2 > n_1$ be the least integer such that C_{n_2} intersects (a_1, b_1). Let $a_2 = \sup[C_{n_2} \cap (a_1, b_1)]$.

   If no C_n intersects (a_2, b_1), then we can choose $x_0 = (a_2 + b_1)/2$. Otherwise, let $n_3 > n_2$ be the least integer such that C_{n_3} intersects (a_2, b_1). Let $b_2 = \inf[C_{n_3} \cap (a_2, b_1)]$, and so on.

   Either this process stops, giving us x_0 as required, or we get a nested sequence of open intervals (a_n, b_n), whose intersection (which is also non-empty) is disjoint from all C_n. In the latter case, we may then take x_0 to be any point of this intersection.

Details of Workshops/Conferences in India

For details regarding Advanced Training in Mathematics Schools
Visit: `http://www.atmschools.org/`

Name: International conference on Reliability, Infocom Technologies and Optimization (ICRITO 2013)
Location: Noida, Uttar Pradesh, India
Date: January 29–31, 2013
Visit: `http://www.amity.edu/aiit/icrito2013/`

Name: International Conference on Mathematics and Information Technology
Location: Chennai, Tamilnadu, India
Date: March 15–16, 2013
Visit: `http://www.worldairco.org/ICMIT%20Mar%202013/ICMIT.html`

Name: National Conference on Recent Trends in Mathematical Sciences and their Applications (NCRTMSA, 2012)
Location: Mody Institute of Technology and Science, Lakshmangarh, Rajasthan, India
Date: November 5–6, 2012
Visit: `http://www.mitsuniversity.ac.in/`
Link: National conference on Mathematics (NCRTMSA, 2012)

Name: National Symposium on Mathematical Methods and Applications (NSMMA-2012)
Location: Indian Institute of Technology Madras, Chennai, India
Date: 22nd December 2012
Visit: `http://mat.iitm.ac.in/nsmma/`

Name: National Conference on High Performance Computing and Simulation
Location: National Institute of Science and Technology, Orissa, India
Date: January 18–19, 2013
Visit: `http://www.nist.edu/nchpcs/index.php`

Name: National Conference on Computational and Applied Mathematics in Science and Engineering (CAMSE-2012)
Location: Visvesvaraya National Institute of Technology Nagpur, Maharashtra, India
Date: December 21 – 22, 2012
Visit: `http://vnit.ac.in/`
Link: Conference on CAMSE – 2012

Name: International Conference on Mathematical Sciences (ICMS-2012)
Location: S.S.E.S. Amt's Science College, Nagpur, Maharashtra, India
Date: December 28 – 31, 2012
Visit: `www.icms2012.org`

Name: Advanced School on Analysis and International Workshop on Mathematical Modeling and Dynamical Systems
Location: Gandhigram Rural Institute – Deemed University, Dindigul, Tamil Nadu, India
Date: February 27 – March 3, 2013
Contact: Dr. P. Balasubramaniam
Department of Mathematics, Gandhigram Rural Institute – Deemed University, Gandhigram, Dindigul, Tamil Nadu 624 302
E-mail: `grumath@gmail.com` Tel: 91-451-2452371

Name: National Conference on Analysis and Applications
Location: National Institute of Technology, Srinagar, Kashmir, India
Date: November 19 – 20, 2012
Visit: `http://www.nitsri.net/conf_math.pdf`

Name: National Conference on Recent Trends in Mathematical and Computational Sciences
Location: Institute of Advanced Study in Science and Technology, Boragaon, Guwahati, Assam
Date: November 28 – 30, 2012
Contact: Dr. Lipi B. Mahanta (Convener – National Conference)
CCNS, IASST, Vigyan Path, Paschim Boragaon, Ghy – 35, Assam
E-mail: `ccnsdept@gmail.com` Cell: 97060-41892, 98640-22447

Details of Workshops/Conferences in Abroad

Name: ACM-SIAM Symposium on Discrete Algorithms (SODA13)
Location: Astor Crowne Plaza Hotel, New Orleans, Louisiana
Date: January 6–8, 2013
Visit: `http://www.siam.org/meetings/da13/`

Name: Iwasawa Theory, Representations, and the p-adic Langlands Program
Location: University of Münster, Münster, Germany

Date: January 7–12, 2013
Visit: `http://wwwmath.uni-muenster.de/sfb878/activities/`

Name: AIM Workshop: Modeling problems related to our environment
Location: American Institute of Mathematics, Palo Alto, California
Date: January 14–18, 2013
Visit: `http://www.aimath.org/ARCC/workshops/modelenvironment.html`

Name: Noncommutative Algebraic Geometry and Representation Theory
Location: Mathematical Sciences Research Institute, Berkeley, California
Date: January 14 – May 24, 2013
Visit: `http://www.msri.org/web/msri/scientific/programs/show/-/event/Pm145`

Name: Winter School on Mathematical Physics
Location: Universidad Nacional Autonoma de Mexico, Mexico City, Mexico
Date: January 14 – February 1, 2013
Visit: `http://leibniz.iimas.unam.mx/~wsmph/index.html`

Name: Mathematics of Planet Earth 2013 – Pan-Canadian Thematic Program – Disease Dynamics 2013: Immunization, a True Multi-Scale Problem
Location: PIMS, Vancouver, British Columbia, Canada
Date: January 17–19, 2013
Visit: `http://www.crm.umontreal.ca/act/theme/theme_2013_1_en/disease_dynamics13_e.php`

Name: AIM Workshop: Online databases – from L-functions to combinatorics
Location: International Centre for Mathematical Sciences Edinburgh, Scotland, United Kingdom
Date: January 21–25, 2013
Visit: `http://www.aimath.org/ARCC/workshops/onlinedata.html`

Name: Connections for Women: Noncommutative Algebraic Geometry and Representation Theory
Location: Mathematical Sciences Research Institute, Berkeley, California
Date: January 24–25, 2013
Visit: `http://www.msri.org/web/msri/scientific/workshops/all-workshops/show/-/event/Wm9061`

Name: International Conference on Fluids And Variational Methods
Location: University of Leipzig, Leipzig, Germany
Date: January 28 – February 1, 2013
Visit: `http://www.math.uni-leipzig.de/~fluids2013/`

Name: Introductory Workshop: Noncommutative Algebraic Geometry and Representation Theory
Location: Mathematical Sciences Research Institute, Berkeley, California
Date: January 28 – February 1, 2013
Visit: `http://www.msri.org/web/msri/scientific/workshops/all-workshops/show/-/event/Wm9062`

Name: ICERM Semester Program: Automorphic Forms, Combinatorial Representation Theory and Multiple Dirichlet Series
Location: ICERM, Providence, Rhode Island

Date: January 28 – May 3, 2013
Visit: `http://icerm.brown.edu/sp-s13`

Name: The Third International Conference on Digital Information Processing and Communications (ICDIPC2013)
Location: Islamic Azad University (IAU), Dubai, UAE
Date: January 30 – February 1, 2013
Visit: `http://sdiwc.net/conferences/2013/Dubai/`

Name: 2nd Annual International Conference on Computational Mathematics, Computational Geometry & Statistics (CMCGS 2013)
Location: Hotel Fort Canning, Singapore, Singapore
Date: February 4–5, 2013
Visit: `http://www.mathsstat.org/`

Name: AIM Workshop: Stochastics in geophysical fluid dynamics
Location: American Institute of Mathematics, Palo Alto, California
Date: February 4–8, 2013
Visit: http://www.aimath.org/ARCC/workshops/stochasticfluid.html

Name: The Second Biennial International Group Theory Conference
Location: Dogus University, Istanbul, Turkey
Date: February 4–8, 2013
Visit: `http://istanbulgroup2013.dogus.edu.tr`

Name: Doc-course on Complex Analysis and Related Areas
Location: Universities of Seville and Málaga, Spain
Date: February 4 – March 15, 2013
Visit: `http://www.imus.us.es/CARA13/`

Name: International Conference on Mathematical Sciences and Statistics (ICMSS2013)
Location: Kuala Lumpur, Malaysia
Date: February 5–7, 2013
Visit: `http://math.upm.edu.my/icmss2013`

Name: Mathematics of Planet Earth 2013 – Pan-Canadian Thematic Program – Models and Methods in Ecology and Epidemiology
Location: CRM, Montréal, Canada
Date: February 6–8, 2013
Visit: `http://www.crm.umontreal.ca/act/theme/theme_2013_1_en/epidemiology13_e.php`

Name: The AMSI Workshop on Graph C^*-algebras, Leavitt path algebras and symbolic dynamics
Location: University of Western Sydney, Australia
Date: February 11–14, 2013
Visit: `http://sites.google.com/site/amsiuws2012/`

Name: ICERM Workshop: Sage Days: Multiple Dirichlet Series, Combinatorics, and Representation Theory
Location: ICERM, Providence, Rhode Island
Date: February 11–15, 2013
Visit: `http://icerm.brown.edu/sp-s13-w1`

Name: Representation Theory, Homological Algebra, and Free Resolutions
Location: Mathematical Sciences Research Institute, Berkeley, California
Date: February 11–17, 2013
Visit: `http://www.msri.org/web/msri/scientific/workshops/all-workshops/show/-/event/Wm8999`

Name: DIMACS Workshop on Energy Infrastructure: Designing for Stability and Resilience
Location: DIMACS Center, CoRE Building, Rutgers University, Piscataway, New Jersey
Date: February 20–22, 2013
Visit: `http://dimacs.rutgers.edu/Workshops/Infrastructure/`

Name: AIM Workshop – Brauer groups and obstruction problems: Moduli spaces and arithmetic
Location: American Institute of Mathematics, Palo Alto, California
Date: February 25 – March 1, 2013
Visit: `http://www.aimath.org/ARCC/workshops/brauermoduli.html`

Name: 36th Annual Texas Partial Differential Equations Conference
Location: The University of Texas at El Paso, El Paso, Texas
Date: March 2–3, 2013
Visit: `http://math.utep.edu/texaspde`

Name: The First International Conference on Green Computing, Technology and Innovation (ICGCTI2013)
Location: The Asia Pacific University of Technology and Innovation (APU), Kuala Lumpur, Malaysia
Date: March 4–6, 2013
Visit: `http://sdiwc.net/conferences/2013/Malaysia4/`

Name: The Second International Conference on e-Technologies and Networks for Development (ICeND 2013)
Location: The Asia Pacific University of Technology and Innovation (APU), Kuala Lumpur, Malaysia
Date: March 4–6, 2013
Visit: `http://sdiwc.net/conferences/2013/Malaysia2/`

Name: Forty-Fourth Southeastern International Conference on Combinatorics, Graph Theory and Computing
Location: Florida Atlantic University, Boca Raton, Florida
Date: March 4–8, 2013
Visit: `http://www.math.fau.edu`

Name: ICERM Workshop: Whittaker Functions, Schubert Calculus and Crystals
Location: ICERM, Providence, Rhode Island
Date: March 4–8, 2013
Visit: `http://icerm.brown.edu/sp-s13-w2`

Name: Interactions between Analysis and Geometry
Location: Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, California
Date: March 11–14, 2013
Visit: `http://www.ipam.ucla.edu/programs/iag2013/`

Name: IAENG International Conference on Operations Research 2013 (ICOR'13)
Location: Royal Garden Hotel, Hong Kong, China
Date: March 13–15, 2013
Visit: `http://www.iaeng.org/IMECS2013/ICOR2013.html`

Name: 29th Southeastern Analysis Meeting (SEAM 2013)
Location: Virginia Tech, Blacksburg, Virginia
Date: March 15–16, 2013
Visit: `http://www.math.vt.edu/seam2013/`

Name: 84th Annual Meeting of GAMM (The International Association of Applied Mathematics and Mechanics)
Location: University of Novi Sad, Novi Sad, Serbia
Date: March 18–22, 2013
Visit: `http://www.dmi.uns.ac.rs/gamm2013`

Name: 17th International Conference on Discrete Geometry for Computer Imagery (DGCI 2013)
Location: Escuela Tecnica Superior de Ingenieria Informatica, Univ. de Sevilla, Seville, Spain.
Date: March 20–22, 2013
Visit: `http://dgci2013.us.es/`

Name: 7th IMA Conference on Quantitative Modelling in the Management of Health and Social Care
Location: Central London College, London, United Kingdom
Date: March 25–27, 2013
Visit: `http://www.ima.org.uk`

Name: AIM Workshop: Mathematical problems arising from biochemical reaction networks
Location: American Institute of Mathematics, Palo Alto, California
Date: March 25–29, 2013
Visit: `http://www.aimath.org/ARCC/workshops/biochemnet.html`

ICTP organizes numerous international conferences, workshops, seminars and colloquiums every year
For Details Visit: `http://www.ictp.it/scientific-calendar.aspx`

ICTS International Centre for Theoretical Sciences

Tata Institute of Fundamental Research

Winter School on Stochastic Analysis and Control of Fluid Flow

03–20 December, 2012

Venue: Indian Institute of Science Education and Research, Thiruvananthapuram, Kerala India.

Objective: The aim of the school is to make students and researchers across various organizations working in fluid flow problems, well acquainted with the basic and advanced topics in control of partial differential equations (pdes) arising from fluid dynamics with special emphasis on Navier-Stokes equations (NSE) in both deterministic and stochastic settings. In the first week, basics of NSE, stochastic integration and stochastic differential equations, pdes and controlled diffusion processes will be introduced. This would help in building the background for the advanced topics to be covered later on. In the following two weeks solvability and control in deterministic and stochastic settings as well as ergodicity and large deviation of NSE will be covered. Topics on stochastic Landau-Lifschitz-Gilbert equation on manifolds will be covered using tools from differential geometry and stochastic analysis. The analysis of numerical methods for control of non stationary NSE will be covered. Mainly finite element based methods and POD-Galerkin model reduction methods will be discussed. The winter school will also comprise of one day discussion meeting where a number of Indian senior experts will be invited to present their recent works related to the theme of the school. The discussion meeting aims to overview open problems and possible pathways to tackle them. This winter school will serve as a platform for the researchers from India and abroad to exchange their Ideas and enhance the state of the art research in this field. This is also a golden opportunity for the young researchers to get into the broad research area of control of pdes.

Important Dates:

Registration begins : 1st July, 2012
Registration closes : 30th September, 2012

Speakers for the Winter School:

V. Barbu (Academy of Romania, Romania)
Z. Brzezniak (University of York, UK)
M. K. Ghosh (IISc, Bangalore, India)
K. Suresh Kumar (IIT Bombay, India)
A. K. Nandakumaran (IISc, Bangalore, India)
S. S. Ravindran (University of Alabama, USA)
J. P. Raymond (University of Toulouse, France)
M. Romito (University of Pisa, Italy)
B. Ruediger (University of Wuppertal, Germany)
A. J. Shaiju (IIT Madras, India)
S. S. Sritharan (Naval Postgraduate School, USA)
P. Sundar (Louisiana State University, USA)

Who Can Apply? Ph.D. students, post-doctoral fellows and junior faculty with sound knowledge in PDE theory and non-linear functional analysis are encouraged to apply. Applications from talented and motivated M.Sc. students will also be considered.

Financial Support: Accommodation, local hospitality and partial travel support is available to all selected participants.

Organizing Committee:

Sheetal Dharmatti (IISER Thiruvananthapuram)
Raju K. Georage (IIST, Thiruvananthapuram)
Utpal Manna, Convener (IISER Thiruvananthapuram)
A. K. Nandakumaran (IISc, Bangalore)
M. P. Rajan (IISER Thiruvananthapuram)

Further Information Visit:

`http://www.icts.res.in/program/control2012`

or write to: `control2012@iisertvm.ac.in`

Indo-Slovenia Conference on Graph Theory and Applications (Indo-Slov-2013)

February 22–24, 2013, Thiruvananthapuram, India

Organized by Research Groups in Discrete Maths from Maribor, Slovenia and Trivandrum-Cochin, India

We are pleased to announce an Indo-Slovenian Conference on Graph Theory and Applications (Indo-Slov-2013) jointly organized by the research groups in discrete mathematics of University of Kerala and Cochin University of Science and Technology from India and the Maribor research group from Slovenia during February 22–24, 2013 at Thiruvananthapuram, India. This is to celebrate our fruitful research collaboration for the last seven years. The aim of the conference is to strengthen the collaboration between discrete mathematicians in India and Slovenia. The conference will be on graph theory in general, but emphasis will be on metric aspects, graph products and graph algorithms.

Call for Papers: Authors are invited to submit papers presenting original and unpublished research in any area of graph theory, especially on metric graph theory, graph products and graph algorithms. Authors may submit drafts of full papers or extended abstracts in LaTeX style.

Registration: The participants can register for the conference by sending the registration fee in favour of the Organising Secretary, Indo-Slov-2013 payable at Trivandrum. The registration fee for Indian participants is Rs. 3000/ which includes accommodation and Local Hospitality and that of the foreign participants is $200 which includes local hospitality. The deadline for registration is December 20, 2012.

For more Information, Contact

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