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Ordinals

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This is an expository article on ordinals presented with a pedagogical viewpoint. First we give a brief on well-ordered sets for the reader to get an insight into the structure of ordinals. We lay emphasis on the topology of ordinals.

1.1 Definition. A partially ordered set in which every nonempty set has a least element is called a well ordered set. (henceforth abbreviated as w.o set)

First and foremost we have Zermelo's axiom which is equivalent to the Axiom of choice: Every set can be well ordered.

For 'a' in a w.o set X, by the initial segment determined by a, we mean the set $s(a) = \{x \in X : x < a\}$.

We keep in record the following basic principle in w.o sets.

1.2 Principle of transfinite induction. In a w.o set X, let $A \subseteq X$ with the property: $x \in A$ whenever $s(x) \subseteq A$. Then $A = X$.

Proof. If $A \neq X$, let x_0 be the least element of $X \setminus A$. Then $s(x_0) \subseteq A$ and $x_0 \notin A$, contradicting the hypothesis. □

1.3 Definition. A bijection f of a w.o set X onto a w.o set Y is said to be a similarity between X and Y if a $a \leq b \Leftrightarrow f(a) \leq f(b)$.

For example, $f(n) = 2n$ is a similarity between positive integers and even positive integers with the usual order.

Clearly similarity is an equivalence relation in any set of w.o sets. W.o sets are identified up to similarity .

The following lemma is fundamental about w.o sets.

1.4 Lemma. *Let f be a similarity of a w.o set X onto itself. Then $a \leq f(a)$ for every $a \in X$.*

Proof. Otherwise, there is a least element $b \in X$ with $f(b) < b$. The element b being the least such, $f(f(b)) \geq f(b)$. From $f(b) < b$, we have by similarity, $f(f(b)) < f(b)$. The contradiction establishes the lemma. □

The following two results follow from the above lemma.

1.5 Proposition. *A similarity between two w.o sets, if it exists, is unique.*

Proof. If f, g are similarities from X onto Y, then $g^{-1}f$ is a similarity of X onto itself. By Lemma 1.4, $a \leq g^{-1}f(a)$ and so $g(a) \leq f(a)$ for every $a \in X$. By considering fg^{-1}, we get $f(a) \leq g(a)$. Hence $f(a) = g(a)$ for every $a \in X$ and the similarity is unique. □

1.6 Proposition. *No w.o set is similar to an initial segment of itself.*

Proof. If f is a similarity of X onto $s(a)$ for some $a \in X$, then $f(a) \in s(a)$ and so $f(a) < a$, contradicting the lemma. □

The above two facts are used in establishing the following. We present only a sketch of the proof.

1.7 Theorem. *For any two w.o set X, Y either X is similar to Y or one is similar to an initial segment of the other.*

Sketch of the Proof. Let $X_0 = \{a \in X : s(a)$ is similar to $s(b)$ for some $b \in Y\}$.

Case (i): $X_0 = X$. Then X is similar to Y or to an initial segment of Y.

Case (ii): $X_0 \neq X$. Then Y is similar to an initial segment of X. □

We leave the proof of these two cases as an exercise.

1.8 Definition. An initial segment of a w.o set is called an ordinal (or ordinal number).

1.9 Remarks.

(1) Since w.o sets are identified up to similarity, we may refer to an ordinal without any reference to the w.o set of which the ordinal is an initial segment.

(2) We may label an ordinal $s(\alpha)$ in X as α itself. Precisely, if X is a w.o set consider the set $\mathcal{A}$ of all initial segment of X (This is within the axioms of set formation). In this set, label $s(\alpha)$ as α. Define $\alpha \le \beta$ if $s(\alpha) \subseteq s(\beta)$. Then $\mathcal{A}$ is a w.o set with $s(\alpha) = \alpha$ for every $\alpha \in \mathcal{A}$. Also the following are equivalent

(i) $\alpha < \beta$ (ii) $\alpha \subset \beta$ (iii) $\alpha \in \beta$.

(3) In view of Theorem 1.7, we have the law of trichotomy: For any two ordinals α, β,

$$\alpha < \beta \text{ or } \alpha = \beta \text{ or } \beta < \alpha.$$

Also it is simple to see that any set of ordinals is well ordered by the above relation.

(4) For any w.o set X, consider $\tilde{X} = X \cup \{X\}$ with the extended order $\alpha < X$ for every $\alpha \in X$. Then the w.o set X is similar to $s(X)$ in $\tilde{X}$. Thus any w.o set is similar to an ordinal.

Now let us see for ourselves some ordinal numbers, at least the countable ones, to get a feel for these numbers. With this purpose in mind, let us give the following interpretations for some of the numbers we know:

$$
\begin{aligned}
0 &= \varnothing, \text{ the empty set.}\\
1 &= \{0\}\\
2 &= \{0, 1\} = 1 \cup \{1\}.\\
&\vdots\\
n &= \{0, 1, 2, \dots, n-1\} = (n-1) \cup \{n-1\}\\
&\vdots\\
w &= \{0, 1, 2, \dots\}, \text{ the first (countably) infinite ordinal.}\\
w+1 &= \{0, 1, 2, \dots, w\} = w \cup \{w\}.\\
w+2 &= \{0, 1, 2, \dots, w, w+1\} = (w+1) \cup \{w+1\}\\
&\vdots
\end{aligned}
$$

We may consider these as elements as well as initial segments of the w.o set

$$
\begin{aligned}
X = \{&0, 1, 2, \dots, w, w+1, \dots, w2, w2+1, \dots, w3, \dots\\
&w4, \dots, w^2, w^2+1, \dots, w^2+w, \dots, w^2+w2, \dots\\
&w^2+w3, \dots, w^2 2, \dots, w^2 3, \dots,\\
&w^3, \dots w^4, \dots w^w, \dots, (w^w)^w \dots\}.
\end{aligned}
$$

The order in X is as it is written.

1.10 Remarks.

(1) For every ordinal α, we have its immediate successor $\alpha + 1 = \alpha \cup \{\alpha\}$. An ordinal without an immediate predecessor is called a limit ordinal. For instance, in the above example, $w, w2, w3, \cdots, w^2, w^3$ etc., are limit ordinals.

(2) In the above example, all the elements of X are obtained through a countable process and so we have obtained only countable ordinals explicitly. What about uncountable ordinals? In this direction, we have the following.

1.11 Proposition. *There exists an uncountable w.o set Ω with largest elements w_1 such that for every $\alpha < w_1$, $s(\alpha)$ is countable.*

Proof. Let X be an uncountable w.o set. Let

$$A = \{x \in X \colon s(x) \text{ is uncountable}\}.$$

If $A \neq \varnothing$, take w_1 as the least elements of A and

$$\Omega = [0, w_1] = \{x \in X \colon x \le w_1\};$$

here we have denoted the least element of X by 0.

If $A = \varnothing$, take $\Omega = X \cup \{X\}$. Extend the order in X to Ω by defining $\alpha < X$ for every $\alpha \in X$. Take $w_1 = X \in \Omega$. □

w_1 is the first uncountable ordinal.

2. Ordinal Topology

The ordinal topology on $[0, w_1]$ is the one generated with sets of the form $[0, \alpha)$, $(\beta, w_1]$ as subbase members. Consequently we have the subspace topology on $[0, w_1)$.

For $\alpha < \beta$ in $[0, w_1]$, clearly $[0, \alpha+1)$ and $(\alpha, w_1]$ are separating open sets for α and β. So the ordinal topology on $[0, w_1]$, hence on $[0, w_1)$, is Hausdorff.

The following result is key to many of the topological properties in ordinals.

2.1 Proposition. *We have $w_1 \in \overline{[0, w_1)}$ in $[0, w_1]$. However, there is no sequence in $[0, w_1)$ converging to w_1.*

Proof. Every basic open set $(\alpha, w_1]$ intersects $[0, w_1)$. Hence $w_1 \in \overline{[0, w_1)}$.

If $\{\alpha_n\} \subseteq [0, w_1)$ and $\alpha_n \to w_1$, then $[0, w_1) = \bigcup_{n=1}^{\infty} [0, \alpha_n)$ will be countable. Hence no sequence in $[0, w_1)$ converges to w_1. □

By a similar argument, we have the following.

2.2 Proposition. *If* $\{\alpha_n\} \subseteq [0, w_1)$*, then* $\sup \alpha_n = \gamma < w_1$.

Proof. Follows from the observation

$$[0, \gamma] = \bigcup_{n=1}^{\infty} [0, \alpha_n). \qquad \square$$

2.3 Proposition. $[0, w_1]$ *is compact.*

Proof. Fist we prove that $[0, w_1]$ is Lindelöf. For any open cover $\{\mathcal{U}_\alpha\}$ of $[0, w_1]$, there is a basic open set $(\beta, w_1]$ such that $(\beta, w_1] \subseteq \mathcal{U}_{\alpha_0}$ for some α_0. Fix γ such that $\beta < \gamma < w_1$. Since $[0, \gamma)$ is countable, countably many $\mathcal{U}_\alpha$'s together with $\mathcal{U}_{\alpha_0}$ will cover $[0, w_1]$. Hence $[0, w_1]$ is Lindelöf. Now to prove compactness, it is enough to show that $[0, w_1]$ is countably compact. Let $\{\mathcal{U}_n\}$ be an open cover for $[0, w_1]$. If this does not have a finite subcover, fix $\alpha_n \notin \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_n$ for $n = 1, 2, \ldots$ with $\alpha_n < w_1$. Now $\alpha_0 = \sup \alpha_n \in \mathcal{U}_j$ for some j. Therefore $\mathcal{U}_j$ contains infinitely many α_n's. By construction, for every k, $\alpha_n \notin \mathcal{U}_k$ for $n \geq k$. This contradiction completes the proof. $\square$

2.4 Remark. The second part of the above proof also shows that $[0, w_1)$ is countably compact. However we leave it as an exercise to show that $[0, w_1)$ is not Lindelöf and hence not compact.

The following are some of the useful exercises in ordinal topology.

2.5 Exercises.

(i) $[0, w_1)$ is first countable, $[0, w_1]$ is not.
(ii) $[0, w_1)$ is not separable not Lindelöf; hence not second countable.
(iii) $[0, w_1)$ is countably compact.
(iv) $[0, w_1]$ is not separable.
(v) $\{w_1\} \subseteq [0, w_1]$ is an F_σ*-set* and not a G_δ*-set*.
(vi) A continuous real-valued function on $[0, w_1)$ must be a constant on a tail $[\beta, w_1)$.

Solution.

Exercises 2.5 (i)–(v) are easy whereas (vi) is not. So we shall provide.

Proof of (vi): Let $f\colon [0, w_1) \to \mathbb{R}$ be continuous. First we make the following

Claim ⊛: For every positive integer n, there exists $\alpha_n < w_1$ such that $|f(\alpha) - f(\alpha_n)| < \frac{1}{n}$ for $\alpha_n < \alpha < w_1$. Suppose otherwise, then there is a positive integer n_0 and a sequence $\{\xi_n\}$ with $0 = \xi_0 < \xi_1 < \cdots < \xi_n < \xi_{n+1} < \cdots < w_1$ such that $|f(\xi_n) - f(\xi_{n+1})| \geq \frac{1}{n_0}$. Take $\gamma = \sup \xi_n < w_1$. Then in every neighbourhood of γ, we have ξ_n and ξ_{n+1} such that at least one of $f(\xi_n)$ and $f(\xi_{n+1})$ stays away from $\left(f(\gamma) - \frac{1}{3n_0}, f(\gamma) + \frac{1}{3n_0}\right)$. This contradicts the continuity of f at γ, establishing claim ⊛ and providing the α_n's.

Take $\beta = \sup \alpha_n < w_1$. If $\alpha > \beta$, then $\alpha > \alpha_n$ and so $|f(\alpha) - f(\alpha_n)| < \frac{1}{n}$. Also $|f(\beta) - f(\alpha_n)| < \frac{1}{n}$; hence $|f(\alpha) - f(\beta)| < \frac{2}{n}$ for every n. Therefore, $f(\alpha) = f(\beta)$ for every $\alpha > \beta$. This completes the proof.

The above proof is by Vickery as presented in Topology by James Dugundji.

3. Algebra of Ordinals

A brief introduction of the algebra of ordinals is worthwhile.

(i) For ordinals α, β we define the operation of addition "+" $\alpha + \beta$ as follows: Consider two disjoint w.o sets X, Y, similar to α, β respectively. On $X \cup Y$, preserve the orders in X and Y; for $x \in X$, $y \in Y$ define $x < y$. Then $X \cup Y$ is a w.o set. The ordinal similar to it is defined as $\alpha + \beta$.

Note that $1 + w = w < w + 1$. Thus + is not commutative.

(ii) We define the operation of multiplication $\alpha\beta$ as follows: In $\alpha \times \beta$, define $(a, b) < (c, d)$ if $b < d$ or $b = d$ and $a < c$ (This is the reverse dictionary order). Then $\alpha \times \beta$ is a w. o set. The ordinal similar to it is defined as $\alpha\beta$.

Note that $2w = w < w2$. Thus ordinal product is not commutative. We have the identities $\alpha 0 = 0 = 0\alpha$, $\alpha 1 = \alpha = 1\alpha$; the associative law $\alpha(\beta\gamma) = (\alpha\beta)\gamma$; the left distributive law $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$; the right distributive law is obviously not valid:

$$(1 + 1)w = 2w \neq w + w = w2.$$

(iii) For any set X, the smallest ordinal equivalent to (i.e., in 1-1 correspondence with) X is called the cardinal number of X.

For further details on the algebra of ordinals, one is referred to Naive set theory by Paul R. Halmos.

Asymptotic Integration of Second Order Linear Differential Equations

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Introduction

The asymptotic integration method to find out the solutions of non-linear boundary layer equations is the corner-stone of applied mathematics. This is a method to find the approximate solutions of the non-linear boundary layer equations for very large values of the independent variables. One of the other corner-stones of applied mathematics is scientific computing and it is interesting to note that these two subjects have grown together. However, this is not unexpected given their respective capabilities. By using computer, one is capable of solving problems that are non-linear, non-homogeneous and multi-dimensional. Moreover, it is possible to achieve very high accuracy. The drawbacks are that the computer solutions do not provide much insight into the physics of the problems, particularly for those who do not have access to the approximate software or computer, and there is always the question as to whether or not the computed solution is correct. On the other hand, the asymptotic integration methods are also capable of dealing with non-linear, non-homogeneous and multi-dimensional problems. So, the main objective behind the use of the asymptotic integration method is to provide reasonably accurate expression for the solution for large values of the independent variables. By doing this, one is capable of deriving an understanding of the physics of the problem. Also, one can use the result in conjunction with the original problem, to obtain the more efficient numerical procedures for computing the solution.

As a matter of afore-mentioned facts, the method of asymptotic integration of second order linear differential equations was used by the workers [1–8] to study the asymptotic behaviours of the boundary layer equations of the Falkner-Skan [9] type.

The objective of this article is to discuss the asymptotic integration of second order linear differential equations for the two cases only (a) elliptic and (b) non-elliptic.

Elliptic and Non-elliptic Equations

A partial differential equation is called quasi-linear if it is linear in the highest derivatives. Hence a second-order quasilinear equation in two independent variables x, y can be written.

$$au_{xx} + 2bu_{xy} + cu_{yy} = F(x, y, u, u_x, u_y) \qquad \text{(A)}$$

where u is an unknown function. This equation is said to be of elliptic type if $ac - b^2 < 0$; parabolic type if $ac - b^2 = 0$ and hyperbolic if $ac - b^2 > 0$. An equation which is not elliptic is also called non-elliptic. Here the coefficients a, b, c may be the functions of x, y so that the type of (A) may be different in different regions of the xy-plane. This classification is not merely a formal matter but is of great practical significance because the general behaviour of solutions differs from type to type and so do the boundary and initial conditions that must be taken into account.

Applications involving elliptic equations usually lead to boundary value problems in a region R, called a first boundary value problem or Dirichlet problem if u is prescribed on the boundary curve C of R, a second boundary value problem or Neumann problem if normal derivative $\frac{\partial u}{\partial n}$ of u is prescribed on C, and a third or mixed problem if u is prescribed on a part of C and $\frac{\partial u}{\partial n}$ on the remaining part. C is usually a closed curve or sometimes consists of two or more such curves.

The Laplace equation $u_{xx} + u_{yy} = 0$ is elliptic, the heat equation $u_t = c^2 u_{xx}$ is parabolic, the wave equation $u_{tt} = c^2 u_{xx}$ is hyperbolic, and the Tricomi equation $yu_{xx} + u_{yy} = 0$

is of mixed type; elliptic in the upper half-plane and hyperbolic in the lower half plane.

Elliptic cases

Let us consider the problem of asymptotic integration of the equations

$$u'' + q(t)u = 0, \tag{1}$$

where $q(t)$ is continuous for large t.

Here we shall centre around the situations where the coefficient $q(t)$ is nearly a constant or (1) can be reduced to this case. When $q(t)$ is constant, say λ, and λ is real and positive, then the solutions are, roughly speaking, of the same order of magnitude. On the other hand, if λ is not real and positive, then there is essentially one small solution, as $t \to \infty$, and the other solutions are large. The facts indicate that different techniques will be needed when $q(t)$ is nearly a constant λ, and λ is or not real and positive. In this case, we will consider only the first case. The results will be given in the form of theorems.

Theorem 1. *In the differential equation (1) and*

$$w'' + q_0(t)w = 0, \tag{2}$$

let $q(t)$, $q_0(t)$ be continuous, complex-valued functions for $0 \le t < \infty$, satisfying

$$\int_0^\infty |w(t)|^2 |q_0(t) - q(t)| dt < \infty \tag{3}$$

for every solution $w(t)$ of (2). Let $u_0(t)$, $v_0(t)$ be linearly independent solutions of (2). Then to every solution $u(t)$ of (1), there corresponds at least one pair of constants α, β such that

$$u(t) = [\alpha + o(1)]u_0(t) + [\beta + o(1)]v_0(t), \tag{4}$$

$$u'(t) = [\alpha + o(1)]u_0'(t) + [\beta + o(1)]v_0'(t), \tag{5}$$

as $t \to \infty$; conversely, to every pair of constants α, β; there exists at least one solution $u(t)$ of (1) satisfying (4).

Corollary 1. *Let $q(t)$ be a continuous complex valued function on $0 \le t < \infty$, satisfying*

$$\int^\infty |1 - q(t)| dt < \infty.$$

Then if α, β are constants, there exists one and only one solution $u(t)$ of (1) satisfying the asymptotic relations

$$\left.\begin{aligned} u(t) &= [\alpha + o(1)]\cos t + [\beta + o(1)]\sin t, \\ u'(t) &= -[\alpha + o(1)]\sin t + [\beta + o(1)]\cos t. \end{aligned}\right\} \tag{6}$$

The proofs of Theorem 1 and Corollary 1 are given by Hartman [10, p. 170].

Exercise 1. If α, β are the constants, there exists a unique solution $v(t)$ of the Bessel equation $t^2 v'' + tv' + (t^2 - \mu^2)v = 0$ for $t > 0$ such that $u(t) = t^{1/2}v(t)$ satisfies (6) as $t \to \infty$.

Exercise 2. Let q(t) be a positive function on $0 \le t < \infty$ possessing a continuous second order derivative and such that

$$\int^\infty q^{1/2}(t)dt = \infty \quad \text{and} \quad \int^\infty \left| \frac{5q'^2}{16q^3} - \frac{q''}{4q^2} \right| q^{1/2}(t)dt < \infty.$$

Then the assertion of the corollary 1 remains valid if (6) is replaced by

$$\begin{aligned} q^{1/4}u &= [\alpha + o(1)]\cos\left(\int_0^t q^{1/2}(s)ds\right) \\ &\quad + [\beta + o(1)]\sin\left(\int_0^t q^{1/2}(s)ds\right), \\ (q^{1/4}u)'q^{-1/2} &= -[\alpha + o(1)]\sin\left(\int_0^t q^{1/2}(s)ds\right) \\ &\quad + [\beta + o(1)]\cos\left(\int_0^t q^{1/2}(s)ds\right). \end{aligned}$$

Exercise 3. If $f(t) = 1 - q(t)$ is a non-vanishing complex-valued function for $t \ge 0$ which is having a continuous derivative satisfying

$$\int^\infty \left| d\left(\frac{f'}{f^{3/2}}\right) \right| < \infty, \qquad \gamma = \lim_{t\to\infty} \frac{f'}{4f^{3/2}} \quad \text{and} \quad \gamma^2 \ne 1,$$

such that $\exp\left\{\pm i \int^t f^{1/2}\left(1 - \frac{f'^2}{16f^3}\right)^{1/2} dt\right\}$ are bounded as $t \to \infty$, then the differential equation $u'' + f(t)u = 0$ has a pair of solutions, as $t \to \infty$,

$$u \sim f^{-\frac{1}{4}}(t) \exp\left\{\pm i \int^t f^{1/2}\left(1 - \frac{f'^2}{16f^3}\right)^{1/2} dt\right\},$$

$$u' \sim [-\gamma \pm i(1 - \gamma^2)^{1/2}]f^{1/2}u.$$

Here all the powers of $f(t)$ that occur can be assumed to be integral (positive or negative) powers of a fixed continuous fourth root $f^{1/4}(t)$ of $f(t)$.

Non-elliptic cases

The asymptotic integrations of (1) where $q(t)$ is a real, but not positive constant, can be obtained in the following manner:

Let us take a special structure of the second order equation

$$(p(t)u')' + q(t)u = 0. \tag{7}$$

This equation is equivalent to a system of the form

$$v' = \beta(t)z, \quad z' = \gamma(t)v \tag{8}$$

in which the diagonal elements vanish. This system can not be reduced to an equation of the form (7) unless either $\beta(t)$ or $\gamma(t)$ does not vanish. The main results on (7) will be based on lemmas dealing with (8).

A system of the form (8) on $0 \le t < \omega(\le \infty)$ will be called of type Z at $t = \omega$ if $z(\omega) = \lim_{t\to\omega} z(t)$ exists for every solution $(v(t), z(t))$, and $z(\omega) \ne 0$ for some solution. It is easy to see that (8) is of type Z iff there exist linearly independent solutions $(v_j(t), z_j(t))$, $j = 0, 1$ such that $\lim z_0(t) = 0$ and $\lim z_1(t) = 1$.

Lemma 1. *Let $\beta(t)$, $\gamma(t)$ be continuous complex-valued functions for $0 \le t < \omega(\le \infty)$.*

Suppose that

$$\int^{\omega} |\gamma(t)|dt < \infty; \tag{9}$$

$$\int^{\omega} |\gamma(t)| \left(\int^{t} |\beta(s)|ds \right) dt < \infty \tag{9'}$$

or, more generally, that

$$\int^{\omega} \gamma(t)dt = \lim_{T\to\omega} \int^{T} \gamma(t)dt \tag{10}$$

exists and that

$$\int^{\omega} |\beta(s)|\Gamma(s)ds < \infty,$$

where

$$\Gamma(s) = \sup_{s\le t<\omega} \left| \int_{t}^{\omega} \gamma(r)dr \right|. \tag{11}$$

Then (8) is of type Z.

Proof. The two quadratures of (8) give

$$v(t) = \int_{T}^{t} \beta(s)z(s)ds + C_1, \quad C_1 = v(T), \tag{12}$$

$$z(t) = \int_{T}^{t} \gamma(s) \int_{T}^{s} \beta(r)z(r)drds + C_1 \int_{T}^{t} \gamma(s)ds + C_2, C_2 = z(T). \tag{13}$$

On changing the order of integration, the last formula becomes

$$z(t) = \int_{T}^{t} \beta(r)z(r) \int_{r}^{t} \gamma(s)dsdr + C_1 \int_{T}^{t} \gamma(s)ds + C_2. \tag{14}$$

If $t \ge T$, then $T \le r \le t$ and the definition of Γ in (11) imply that

$$\left| \int_{r}^{t} \gamma(s)ds \right| \le \left| \int_{r}^{\omega} \gamma(s)ds \right| + \left| \int_{t}^{\omega} \gamma(s)ds \right| \le 2\Gamma(r). \tag{15}$$

Consequently,

$$|z(t)| \le 2 \int_{T}^{t} |\beta(s)|\Gamma(s)|z(s)|ds + C,$$

where

$$C = 2|C_1|\Gamma(r) + |C_2|. \tag{16}$$

Now, for further manipulations, we shall base ourselves on Gronwall's Inequality. □

Gronwall's Inequality

This Inequality can be stated in terms of the following theorem:

Theorem. Let $u(t)$, $v(t)$ be non-negative, continuous functions on [a, b]; $C \ge 0$ a constant; and

$$v(t) \le C + \int_{a}^{t} v(s)u(s)ds \quad \text{for} \quad a \le t \le b.$$

Then $v(t) \le C \exp\left\{\int_a^t u(s)ds\right\}$ for $a \le t \le b$.

Based on the above Gronwall's Inequality, we have

$$|z(t)| \le C \exp\left\{2 \int_{T}^{t} |\beta(s)|\Gamma(s)ds\right\} \le C \exp\left\{2 \int_{T}^{\omega} |\beta(s)|\Gamma(s)ds\right\} \tag{17}$$

for $T \le t < \omega$. Hence (10) implies that $z(t)$ is bounded. The relations (14) and (11) then show that $z(\omega) = \lim_{t\to\omega} z(t)$ exists.

The limit $z(\omega)$ is obtained by writing $t = \omega$ in (14). In order to show that $z(\omega) \neq 0$ for some solution of (8), let us choose the initial conditions $C_1 = v(T) = 0$ and $C_2 = z(T) = 1$ in (12), (13). Thus, $C = 1$ in (16) and (17) and so (14), (15) and (17) give

$$|z(\omega) - 1| \le 2\left(\int_T^\omega |\beta(r)|\Gamma(r)dr\right) \exp\left\{2\int_T^\omega |\beta(s)|\Gamma(s)ds\right\}.$$

Since the right side tends to zero as $T \to \omega$, it follows that if T is sufficiently near to ω, then $z(\omega) \neq 0$. This proves the Lemma 1.

Lemma 2. *If $\beta(t)$ and $\gamma(t)$, where $0 \le t < \omega(\le \infty)$, are continuous real-valued functions which do not change a sign (i.e. $\beta \ge 0$ or $\beta \le 0$ and $\gamma \ge 0$ or $\gamma \le 0$) and if (8) is of type Z, then (9)–(9′) hold.*

Proof. Let $(v(t), z(t))$ be the solution of (8) such that $z(\omega) = 1$. It can also be supposed that $v(T) = 0$ for some T. Otherwise, it is possible to add to $(v(t), z(t))$ a suitable multiple of a solution $(v_0(t), z_0(t)) \neq 0$ for which $z(\omega) = 0$. In fact, $v_0(t) \equiv 0$ can not hold, for then (8) shows that $z_0(t) \equiv z_0(\omega) = 0$.

Thus $C_1 = 0$ in (13) and $z(\omega) = 1$ shows that (9′) holds since β, γ do not change the sign. If $\beta(t) \neq 0$ for t near ω, (9) follows. If, however, $\beta(t) \equiv 0$ and (8) is of type Z, then (9) holds when γ does not change signs. This completes the proof.

Lemma 3. *Let $\beta(t), \gamma(t)$ be as in lemma 1. In addition, suppose that $\beta(t) \ge 0$ and that*

$$\int^\omega \beta(t)dt = \infty. \tag{18}$$

Then (8) has a pair of solutions $(v_j(t), z_j(t))$ for $j = 0, 1$, satisfying, as $t \to \infty$,

$$v_0 \sim 1, z_0 = o\left(\frac{1}{\int^t \beta(s)ds}\right); \tag{19}$$

$$v_1 \sim \int^t \beta(s)ds, \quad z_1 \sim 1. \tag{20}$$

Proof. By Lemma 1, (8) has a solution $(v_1(t), z_1(t))$ such that $z_1(\omega) = 1$. Thus the first part of (20) follows from the first equation in (8).

Let us note that

$$\int^t \frac{\beta(s)ds}{\left(\int^s \beta(r)dr\right)^2} = \text{constant} - \frac{1}{\int^t \beta(s)ds} \tag{20′}$$

tends to const. as $t \to \omega$ by (18). Consequently, the integral $C_1 = \int_T^\omega \frac{\beta(s)ds}{v_1^2(s)}$ is absolutely convergent (for T near ω).

Let $(v(t), z(t))$, $(v_1(t), z_1(t))$ be solutions of (8). Then

$$z_1(t)v(t) - v_1(t)z(t) = C_0 \tag{21}$$

is a constant.

If $z_1(t) \neq 0$ and (21) is multiplied by $\frac{\gamma(t)}{z_1^2(t)}$, it is seen from (8) that $\left(\frac{z}{z_1}\right)' = \frac{C_0\gamma}{z_1^2}$, and hence there is a constant C_1 such that

$$z(t) = C_1 z_1(t) + C_0 z_1(t)\int_T^t \frac{\gamma(s)ds}{z_1^2(s)} \tag{22}$$

if $z_1 \neq 0$ on the interval $[T, t]$. Similarly, if $v_1 \neq 0$ for $[T, t]$ then $\left(\frac{v}{v_1}\right)' = -\frac{C_0\beta}{v_1^2}$ and

$$v(t) = C_1\, v_1(t) - C_0\, v_1(t)\int_T^t \frac{\beta(s)ds}{v_1^2(s)}. \tag{23}$$

It follows from (23) with the choice $C_0 = 1$, that (8) has a solution $(v, z) = (v_0, z_0)$ satisfying (21) with $C_0 = 1$ and

$$v_0(t) = v_1(t)\int_t^\omega \frac{\beta(s)ds}{v_1^2(s)}.$$

Then $v_0 \sim 1$ follows from the first part of (20) and (20′). Letting $(v, z, C_0) = (v_0, z_0, 1)$ in (21) and solving for z_0 gives the last part of (19). This completes the proof. □

For more detailed explanation, the readers may refer Hartman [10, page 377].

Exercise 4. Suppose that (8) is of type Z and that $(v_1(t), z_1(t))$ is a solution of (8) satisfying $z_1(\omega) = 1$. Then it shows that

$$\int^\omega \frac{\gamma(t)dt}{z_1^2(t)} = \lim_{T\to\omega}\int^T \frac{\gamma(t)dt}{z_1^2(t)} \quad \text{exists}$$

and that (8) has a solution $(v_0(t), z_0(t))$ in which

$$z_0(t) = z_1(t)\int_t^\omega \frac{\gamma(s)ds}{z_1^2(s)}. \tag{24}$$

for t near ω. If $(v(t), z(t))$ is any solution of (8), then

$$v(t) = o\left(1 + \int^{t} |\beta(s)|ds\right) \quad \text{as} \quad t \to \omega.$$

Some more theorems and lemmas pertaining to the asymptotic integrations are given in Hartman and Wintner [11,12], Massera and Schaffer [13], Olech [14], etc.

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Hahn Banach Theorems

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Abstract. We give an overview of Hahn-Banach theorems. We present the statements of these theorems alongwith some definitions that are required to understand these statements and make some comments about the relevance, applications etc. Proofs are not given.

1. Introduction

The Hahn-Banach theorems are one of the three most important and fundamental theorems in basic Functional Analysis, the other two being the Uniform Boundedness Principle and the Closed Graph Theorem. Usually Hahn-Banach theorems are taught before the other two and most books also present Hahn-Banach theorems ahead of Uniform Boundedness

Principle or the Closed Graph Theorem. This may be due to several reasons. The statements, proofs and applications of Hahn-Banach theorems are relatively easier to understand. In particular, the hypotheses do not include completeness of the underlying normed linear spaces and proofs do not involve the use of Baire Category Theorem. In this article,[1] we give an overview of Hahn-Banach theorems. We present the statements of these theorems alongwith some definitions that are required to understand these statements and make some comments about the relevance, applications etc. Detailed proofs are not given as these can be found in any introductory textbook on Functional Analysis for example [1, 4, 6].

There are two classes of theorems commonly known as Hahn-Banach theorems, namely Hahn-Banach theorems in the extension form and Hahn-Banach theorems in the separation form. All these theorems assert the existence of a linear functional with certain properties. Why is it important to know the existence of such functionals? In a large number of applications of practical importance, the objects of study can be viewed as members of a vector space. A study of such objects involves making various measurements/observations. These are functionals on that vector space. As the names suggest, Hahn-Banach theorems in the extension form assert that functionals defined on a subspace of a vector space (frequently with some additional structure, usually with a norm or topology) and having some additional properties (like linearity, continuity) can be extended to the whole space while retaining these additional properties. This is useful in asserting the existence of certain functionals and this in turn can be used in applications involving approximation of certain functions. These theorems and their proofs are analytic in nature. On the other hand, Hahn-Banach theorems in the separation form and also their proofs are geometric in nature. These theorems deal with the following question: Given two disjoint convex sets in a vector space, when is it possible to find a hyperplane such that the two given convex sets lie on the opposite sides of that hyperplane? This information is useful in answering several questions dealing with optimization problems, also known as Mathematical Programming.

[1]The article is an expanded version of a talk delivered at the Department of Mathematics, Theivanai Ammal College for Women, Viluppuram 605 602, Tamilnadu, on the occassion of the "The Regional Workshop on Anaysis (RWA 2004)" during the period January 30–31, 2004.

Hahn-Banach theorems in both these forms are mathematically equivalent. This means that it is possible to prove a Hahn-Banach theorem in the extension form first and then use it to prove a corresponding theorem in the separation form or prove a Hahn-Banach theorem in the separation form first and then use it to prove a corresponding theorem in the extension form. Preference among these two approaches is a matter of personal taste. Each of these approaches can be found in the text books, for example the first approach can be found in [1] whereas the second can be found in [4] .

2. Functionals and their Extensions

Hahn-Banach theorems are essentially theorems about real vector spaces. Basic theorems are first proved for real vector spaces. These are then extended to the case of complex vector spaces by means of a technical result.(See Lemma 7.1 of [4] and remarks preceding it.) In this article, we shall confine ourselves to real vector spaces. Common examples of real vector spaces are $\mathbb{R}^n$, sequence spaces l^p, $1 \leq p \leq \infty$, function spaces $C([a, b])$ with pointwise or coordinatewise operations as the case may be. A *functional* on such a vector space is a real valued function defined on it. Next we list several properties that such a functional may or may not have.

Definition 2.1. *Let V be a real vector space. A functional on V is a function*
$\phi : V \to \mathbb{R}$. *Such a ϕ is said to be*

(1) additive if

$$\phi(a + b) = \phi(a) + \phi(b)$$

for all $a, b \in V$,

(2) homogeneous if

$$\phi(\alpha a) = \alpha\phi(a)$$

for all $a \in V$ and $\alpha \in \mathbb{R}$,

(3) linear if it is additive and homogeneous,

(4) subadditive if

$$\phi(a + b) \leq \phi(a) + \phi(b)$$

for all $a, b \in V$,

(5) positive homogeneous if

$$\phi(\alpha a) = \alpha\phi(a)$$

for all $a \in V$ and $\alpha \in \mathbb{R}$ with $\alpha \geq 0$,

(6) *sublinear if it is subadditive and positive homogeneous,*
(7) *convex if*

$$\phi((1-t)a+tb) \le (1-t)\phi(a)+t\phi(b)$$

for all $a, b \in V$ *and* $t \in [0, 1]$.

Several examples of functionals having some of the above properties and not having some of the other properties can be given. These are easy to construct and can be found in most books on Functional Analysis, including [1, 4, 6]. For example, the functional ϕ defined on $\mathbb{R}^2$ by

$$\phi(x_1, x_2) := |x_1| + |x_2|$$

for $(x_1, x_2) \in \mathbb{R}^2$ is sublinear but not linear. On the other hand, the functional ψ defined on $\mathbb{R}^2$ by

$$\psi(x_1, x_2) := 2x_1 + x_2$$

for $(x_1, x_2) \in \mathbb{R}^2$ is linear. It is obvious that a linear functional satisfies all the above properties. It is a good exercise to construct examples of functionals satisfying one of the above properties but are not linear.

We are now in a position to state the first of the Hahn Banach Extension theorems. This is used in proving all the other extension theorems. It says that a linear functional defined on a subspace of a real vector space V and which is dominated by a sublinear functional defined on V has a linear extension which is also dominated by the same sublinear functional.

Theorem 2.2. *Let V be a real vector space and p be a sublinear functional defined on V. Suppose W is a subspace of V and ϕ is a linear functional defined on W such that $\phi(x) \le p(x)$ for all $x \in W$. Then there exists a linear functional ψ defined on V such that $\psi(x) = \phi(x)$ for all $x \in W$ and $\psi(y) \le p(y)$ for all $y \in V$.*

Usual proof of this theorem involves two steps.(See [6] for details.) Of course, if $W = V$, there is nothing to prove. When W is a proper subspace, some $x_0 \in V \setminus W$ is chosen and ϕ is extended to the linear span W_0 of $W \cup \{x_0\}$ in such a way that the extension ψ remains dominated by p in W_0. This is the first step. The crucial idea in this step is to choose the value of $\psi(x_0)$ in an appropriate manner. The second step makes use of Zorn's Lemma to construct a maximal subspace containing W to which ϕ can be extended satisfying the required condition. Finally it is shown using the first step that this maximal subspace must coincide with V.

Though the above formulation of the Hahn Banach Extension theorem is most basic, the most popular one is stated in the context of normed linear spaces. In order to understand this version, we need to review a few basic definitions.

Definition 2.3. *Let X be a real vector space. A norm on X is a function $\|.\| : X \to \mathbb{R}$ satisfying:*

(1) $\|x\| \ge 0$ *for all* $x \in X$ *and* $\|x\| = 0$ *if and only if* $x = 0$.
(2) $\|\alpha x\| = |\alpha|\|x\|$ *for all* $x \in X$ *and* $\alpha \in \mathbb{R}$.
(3) $\|x + y\| \le \|x\| + \|y\|$ *for all* $x, y \in X$.

A real normed linear space is a pair $(X, \|.\|)$ where X is a real vector space and $\|.\|$ is a norm on X.

Examples of real normed linear spaces include $\mathbb{R}^n$ (with several norms), sequence spaces l^p, $1 \le p \le \infty$, function spaces L^p, $1 \le p \le \infty$, $C(X)$, $C([a, b])$, $C^1([a, b])$.

Among the various norms on $\mathbb{R}^n$, the following are more frequently used. For $x = (x_1, \dots , x_n) \in \mathbb{R}^n$,

(1)
$$\|x\|_1 := \sum_{j=1}^{n} |x_j|,$$

(2)
$$\|x\|_2 := \left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2},$$

(3)
$$\|x\|_\infty := \max_{j=1\dots n} |x_j|.$$

Definition 2.4. *Let $(X, \|.\|)$ be a normed linear space. A linear functional ϕ on X is said to be bounded if*

$$\sup\{|\phi(x)| : x \in X, \|x\| \le 1\}$$

is finite. When this is the case, the above quantity is called the norm of ϕ and denoted by $\|\phi\|$.

It is easy to show (and is also proved in most textbooks) that a linear functional ϕ on X is bounded if and only if it is continuous at each $x \in X$ if and only if it is uniformly continuous on X. Also note that if ϕ is a bounded linear functional on X, then

$$|\phi(x)| \le \|\phi\|\|x\| \quad \text{for all} \quad x \in X.$$

See the references at the end, for several examples of bounded as well as unbounded linear functionals. The popular version of the Hahn Banach Extension theorem mentioned above says that every bounded linear functional defined on a subspace of a normed linear space has a norm preserving extension to the whole space.

Theorem 2.5. *Let* $(X, \|.\|)$ *be a normed linear space,* Y *a subspace of* X *and* ϕ *a bounded linear functional defined on* Y*. Then there exists a bounded linear functional* ψ *defined on* X *such that* $\phi(y) = \psi(y)$ *for all* $y \in Y$ *and* $\|\psi\| = \|\phi\|$.

This is proved by observing that the functional p defined on X by $p(x) := \|\phi\|\|x\|, \quad x \in X$, is a sublinear functional that dominates ϕ on Y and using Theorem 2.2.

This theorem is frequently used to deal with questions about approximations. Typically we want to know whether an element $a \in X$ can be approximated by elements in a subspace Y, that is whether a belongs to the closure of Y. For example one may want to know whether a continuous function f defined on the interval [0, 1] can be approximated by polynomials. This amounts to taking $X = C([0, 1])$ and Y to be the subspace of all polynomials. The following well known corollary of the Hahn Banach Extension theorem helps in answering this question.

Corollary 2.6. *Let* $(X, \|.\|)$ *be a normed linear space,* Y *be a subspace of* X *and* $a \in X$*. Then* a *belongs to the closure of* Y *if and only if there exists no bounded linear functional* ϕ *on* X *such that* $\phi(y) = 0$ *for all* $y \in Y$ *and* $\phi(a) \neq 0$*. Equivalently,* a *does not belong to the closure of* Y *if and only if there exists a bounded linear functional* ϕ *on* X *such that* $\phi(y) = 0$ *for all* $y \in Y$ *and* $\phi(a) \neq 0$.

Given X, Y and a how does one decide whether such a bounded linear functional exists or not? In general, this is difficult to answer. In order to answer this, we need to have the knowledge of all bounded linear functionals on X. This is a theme of another class of theorems in Functional Analysis known as Riesz Representation Theorems. These theorems give complete description of bounded linear functionals on various concrete normed linear spaces. See the references at the end for more information on these theorems. An interesting application of this idea to the Poisson Integral Formula can be found in [5].

3. Convex sets and their Separation

In this section we discuss Hahn-Banach theorems in the separation form. As mentioned in the introduction, these theorems deal with the question of existence of a hyperplane that separates two given disjoint convex subsets. We first need to review a few definitions and notations.

Let V be a real vector space, A, B be nonempty subsets of V and α, a real number. Then the symbols $A + B$ and αA have their natural meaning as follows:

$$A + B := \{a + b : a \in A, b \in B\}$$
$$\alpha := \{\alpha a : a \in A\}.$$

Note that with these notations,

$$A + B = B + A$$

but $A + A$ may not be the same as $2A$. A special case of the above that occurs very frequently is when B is a singleton, say $\{x\}$. Thus $A + x$ or $x + A$ is the set

$$\{x + a : a \in A\}.$$

This is called the *translate of* A *by* x for obvious reasons. A subset C of V is said to be *convex* if for each t with $0 \leq t \leq 1$, we have

$$(1 - t)C + tC \subseteq C.$$

In other words, for each $x, y \in C$ and $0 \leq t \leq 1$,

$$(1 - t)x + ty \in C.$$

This is usually expressed by saying that if C contains two points, then it should also contain the line segment joining those two points. We come across several examples of convex sets. For example, every subspace is a convex set. A translate of a subspace by a nonzero element is not a subspace, but it is a convex set. Convex sets and functionals are closely related concepts in the sense that we can associate convex sets with functionals and vice versa. For example, if ϕ is a convex functional (See Definition 2.1) and α is a real number, then the set

$$\{x \in V : \phi(x) \leq \alpha\}$$

is a convex set. If ϕ is a linear functional, then each of the following is a convex set.

(1) $\{x \in V : \phi(x) \leq \alpha\}$,
(2) $\{x \in V : \phi(x) \geq \alpha\}$,
(3) $\{x \in V : \phi(x) = \alpha\}$.

Next recall that a subspace is called *proper* if it is different from the whole space V. A *hyperspace* is a maximal proper subspace, that is a proper subspace not properly contained in any other proper subspace. If V is of finite dimension, say

n, then every subspace of dimension $n-1$ is a hyperspace. Thus in $\mathbb{R}^2$, every straight line passing through the origin is a hyperspace. Similarly, hyperspaces in $\mathbb{R}^3$ are the planes passing through the origin. It is well known and is also easy to prove that if ϕ is a nonzero linear functional on V, then the *null space of* ϕ defined by

$$N(\phi) := \{x \in V : \phi(x) = 0\}$$

is a hyperspace. Also every hyperspace is a null space of some nonzero linear functional. Next, a *hyperplane* is a translate of a hyperspace. Thus hyperplanes in $\mathbb{R}^3$ are the planes.(This may very well be the reason for the name "hyperplane".) Further, if H is a hyperplane, then there exists a nonzero linear functional ϕ and a vector $a \in V$ such that

$$H = a + N(\phi).$$

Now suppose $\phi(a) = \alpha$. Let $h \in H$. Then $h = a + y$ for some $y \in N(\phi)$. Hence

$$\phi(h) = \phi(a) + \phi(y) = \alpha.$$

On the other hand, suppose $x \in V$ is such that $\phi(x) = \alpha$. Let $y = x - a$. Then $\phi(y) = 0$, that is $y \in N(\phi)$ and

$$x = a + y \in a + N(\phi) = H.$$

Thus we have proved that with every hyperplane H, we can associate a pair (ϕ, α), where ϕ is a nonzero linear functional and α is a real number such that

$$H = \{x \in V : \phi(x) = \alpha\}.$$

For example, consider a plane given by the equation:

$$2x_1 - x_2 + 7x_3 - 5 = 0$$

in $\mathbb{R}^3$. Note that this is a hyperplane and the associated pair is $(\phi, 5)$ where the functional ϕ is given by

$$\phi(x) := 2x_1 - x_2 + 7x_3, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Every such hyperplane H divides the whole space V into two convex sets:

$$H_l = \{x \in V : \phi(x) \leq \alpha\}$$

and

$$H_r = \{x \in V : \phi(x) \geq \alpha\}$$

known as *halfspaces*. We say that two nonempty subsets A and B of V are separated by H if A and B lie on different sides of H, that is A and B are contained in different half spaces formed by H.

We are now ready to present the basic separation theorem.

Theorem 3.1. *Let A and B be nonempty disjoint convex subsets of a real vector space V. Then A and B can be separated by a hyperplane, that is, there exists a nonzero linear functional ϕ and a real number α such that $\phi(x) \leq \alpha \leq \phi(y)$ for all $x \in A$ and $y \in B$.*

A proof of this can be found in [1]. Also note that we can not think of separating nonconvex sets by hyperplanes in this way. For example, suppose A is a circle in $\mathbb{R}^2$ with the centre at the origin and radius 1 and B is the singleton set consisting of the origin. Then no straight line can separate A and B. Recall that straight lines are hyperplanes in $\mathbb{R}^2$.

Again as in the case of extension theorems, though the above theorem is very basic, the more popular ones are stated (and proved) in the context of a normed linear space. (See Definition 2.3). Here we are looking for separation of convex sets not by just arbitrary hyperplanes but by closed hyperplanes. This means that the associated linear functionals must be continuous. (See Definition 2.4 and the remarks following it.) The following is usually known as the Hahn Banach Separation Theorem in normed linear spaces. Its proof can be found in [1, 4, 6].

Theorem 3.2. *Let A and B be nonempty disjoint convex subsets of a real normed linear space X. Suppose A has an interior point. Then A and B can be separated by a closed hyperplane, that is, there exists a nonzero continuous linear functional ϕ and a real number α such that $\phi(x) < \alpha \leq \phi(y)$ for all $x \in A$ and $y \in B$.*

The following Corollary of this theorem is also very popular and is used in several applications.

Corollary 3.3. *Let A and B be nonempty disjoint convex subsets of a real normed linear space X. Suppose A is compact and B is closed. Then A and B can be strictly separated by a closed hyperplane, that is, there exists a nonzero continuous linear functional ϕ and real numbers α and β such that $\phi(x) < \alpha < \beta < \phi(y)$ for all $x \in A$ and $y \in B$.*

As mentioned in the introduction, these separation theorems have applications to the problems of optimization, in particular to a class of problems known as convex programming problems. We only mention a very interesting and famous result known as *Farkas' Lemma* which is a direct consequence of the separation theorem.

Theorem 3.4. *Let A be a matrix of order $m \times n$ with real entries and $b \in \mathbb{R}^m$, regarded as a column matrix of order $m \times 1$. Then exactly one of the following alternatives hold.*

(i) The system of equations $Ax = b$ has a nonnegative solution $x \in \mathbb{R}^n$. (Here nonnegative means $x_j \geq 0$ for each $j = 1, \ldots, n$.)

(ii) The system of inequalities $y^T A \geq 0$ and $y^T b < 0$ has a solution $y \in \mathbb{R}^m$.

It is easy to see that both the alternatives can not hold simultaneously, because if they do, then there exist x as in (i) and y as in (ii). Then

$$0 \leq (y^T A)x = y^T(Ax) = y^T b < 0,$$

a contradiction. To prove that one of the alternatives must hold, the convex set

$$C := \{Ax : x \in \mathbb{R}^n, x \geq 0\} \subseteq \mathbb{R}^m$$

is considered. If $b \in C$, the alternative (i) holds. Otherwise, C and $\{b\}$ are nonempty disjoint convex sets in $\mathbb{R}^m$ and hence can be separated by a hyperplane H. A proof consists of noting that with every such hyperplane H in $\mathbb{R}^m$, we can associate a vector $y \in \mathbb{R}^m$ and a real number α such that

$$H = \{u \in \mathbb{R}^m : y^T u = \alpha\}$$

and then showing that in this particular case α can be taken to be 0. Details of this proof as well as the application of Farkas' Lemma to the duality theory of Linear Programming can be found in [2]. A highly refreshing and readable account of Farkas' Lemma and its connection with several problems in Mathematics and Economics is given in [3].

4. Limitations

After having said so many things about the importance, relevance etc. of Hahn Banach theorems, we finally also point out some limitations of these theorems and also of methods based on these theorems. In fact, these limitations are common to many theorems and methods of Functional Analysis. Statements of all these Hahn Banach theorems are existence statements, that is, these statements assert the existence of some linear functionals. However all the proofs are nonconstructive in nature. This means that these proofs give no clue about how to find the linear functional whose existence is asserted by a theorem, even when the vector space under consideration is finite dimensional. (The situation is very similar to various proofs of the Fundamental Theorem of Algebra, none of which say anything about how to find a root of a given polynomial, though the Theorem asserts that every such polynomial must have a root.) For example, suppose we are given two finite sets in $\mathbb{R}^{10}$. One way of giving these sets would be to give two matrices each having ten rows. Let A and B denote the convex hulls of these two sets. (Recall that a convex hull of a set is the smallest convex set containing the given set.) Suppose we consider the following problem: To determine whether A and B are disjoint and in case these are disjoint, to find a hyperplane in $\mathbb{R}^{10}$ separating A and B. It may appear at first that the Hahn Banach Separation Theorem may be useful to tackle this problem. It is useful in the sense that if A and B are disjoint, the Theorem says that there exists a hyperplane in $\mathbb{R}^{10}$ separating A and B. However it says nothing about how to find such a hyperplane. Completely different methods have to be used to tackle this problem. (See [2].)

References

[1] B. Bollobás, *Linear analysis*, Cambridge Univ. Press, Cambridge, 1990. MR1087297 (92a:46001)

[2] J. Franklin, *Methods of mathematical economics*, Springer, New York, 1980. MR0602694 (82e:90002)

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[4] B. V. Limaye, *Functional analysis*, Second edition, New Age, New Delhi, 1996. MR1427262 (97k:46001)

[5] W. Rudin, *Real and complex analysis*, Third edition, McGraw-Hill, New York, 1987. MR0924157 (88k:00002)

[6] A. E. Taylor and D. C. Lay, *Introduction to functional analysis*, Second edition, John Wiley & Sons, New York, 1980. MR0564653 (81b:46001)

Summer School on Multiscale Modeling and Simulation in Science

Bosön, Stockholm
June 4–15, 2007

The purpose of the summer school is to bring together leading scientists in computational physics, computational chemistry and computational biology and in scientific computing with Ph.D. students in these fields to solve problems with multiple scales of research interest.

Lectures on computational multiscale techniques will be presented in the first week. The afternoons are devoted to the solution of computational exercises in small groups. Cross-disciplinary and challenging problems will be given by experts in the second week of the summer school. Groups of students and the experts will work together on the solution of the problems. Ph.D. students can receive credit for the summer school at their home departments.

The summer school is intended for Ph.D. students in applied mathematics, numerical analysis, scientific computing, computational physics, computational chemistry and computational biology.

Students affiliated with a university in Finland, Norway, and Sweden are encouraged to apply for a scholarship to cover the expenses at the summer school.

Organizers:

Department of Numerical Analysis (NADA) KTH
Department of Information Techonology
Centre for Dynamical Processes (CDP)
Uppsala University (UU)

Sponsors: Support is received from the National Graduate School in Scientific Computing (NGSSC), Swedish Foundation for Strategic Research (SSF), Centre for Dynamical Processes and Structure Formation (CDP), NordForsk and the Research Council of Norway.

Checkpoints: February 15, 2007, Last day for registration.

Contact Address and Further Details:

E-mail: ssmmss@it.uu.se
http://user.it.uu.se/~ngssc/ngssc_home//S2M2S2/

Extremal Problems in Complex and Real Analysis

Peoples Friendship University of Russia, Moscow, Russia
May 22–26, 2007

Minicourses: There will be three minicourses offered during the conference. The minicourses will be directed towards a broad audience starting with Ph.D. students and up to experienced researchers. They will be given by: Borislav Bojanov, Peter L. Duren, Edward Saff.

Topics of the Conference: Topics covered at this conference include but are not limited to: Approximation Theory, Theory of Spaces of Analytic and Harmonic Functions, Optimal Recovery, Geometric Function Theory in One and Several Variables, and Functions Related Operator Theory.

The goal is to bring together a large group of mathematicians working in different but related fields in order to increase awareness and enhance interaction and collaboration between individual mathematicians as well as various groups currently working in overlapping areas.

Contact Information: stessin@math.albany.edu, kosipenko@yahoo.com, amontes@us.es

For Further Details Visit: http://www.albany.edu/~pb6916/

UFA International Mathematical Conference "Function Theory, Differential Equations, Numerical Mathematics"

Alexei Leontiev Memorial Conference, UFA, RUSSIA
May 28–June 1, 2007

Registration Form:

1. Title (Dr/Prof/Mr/Mrs)
2. Full Name
3. Affiliation
4. Country
5. Mail address
6. Contact phone/fax
7. E-mail, homepage (optional)
8. Section
9. Title of the talk

Registration form should be sent to Organizing Committee not later than January 15, 2007.

- by E-mail: conf@matem.anrb.ru
- by fax +7-347-272-59-36
- by post:

Institute of Mathematics with Computer Center,
Organizing Committee
A. V. Zakharov
Chernyshevsky st., 112
RUSSIA, 450 077, Ufa

Abstracts: Abstracts should be submitted to the Organizing Committee by March 1, 2007. The length of the abstract should not exceed one typewritten page. You can use either LaTeX or MSWord.

The Conference Will Cover the Recent Developments in Fields:

- Theory of Real Variable Function
- Complex Analysis
- Non-Linear Integrable Differential Equations
- Asymptotic Methods of Solving Differential Equations
- Numerical Mathematics

The conference abstracts will be published.

Accommodation and Registration Fee: Information about accommodation and registration fee will be fixed in the second announcement (March, 2007). If you have any questions, please, contact the Organizing Committee.

Phone: +7-347-272-59-36, +7-347-273-33-42
Fax: +7-347-272-59-36
http://www.matem.anrb.ru/conf/index-en.htm

Indo-Australian Workshop on a CFD Approach on Fluid Flow, Heat and Mass Transfer

12–14 April, 2007

Symposium on CFD Applications in Multidisciplinary Areas

Organizer:

Department of Mathematics
Indian Institute of Technology
Roorkee 247 667(UA), India

Co-Sponsers:

Group of Eight: Universities of Australia (Go-8)

About the Department: The Department of Mathematics besides providing core P.G. courses: M.Sc. in Applied Mathematics and Industrial Mathematics plays a significant role in providing liberal education at both U.G. and P.G. levels to Engineering departments of the Institute in order to broaden the horizon of students. Interdisciplinary in orientation, it is running successfully the M.C.A. course which is well renowned in leading software scenario in India. The department also offers research in core Mathematics as well as in various multidisciplinary areas and has rich experience, adequate resources and facilities for organizing academic conferences.

Theme: The two day Workshop will provide a platform for enriching the Mathematical flavor and culture in the field of CFD by cross-pollination of Mathematical thoughts and numerical techniques. Topics of interest include Computing techniques of CFD, Grid generation, Multi-phase flows, Modeling to Micro, nano and Biofluids and Transport processes.

Registration Fees: Indian Delegate: - Rs. 2500/-

Organizing/Academic Secretary:

Profs. Rama Bhargava/V. K. Katiyar
Department of Mathematics
I.I.T. Roorkee, INDIA
Phone :+91 1332-285039
Mobile: 09319650553
E-mail: bhargava_iitr@rediffmail.com
E-mail: vktmafma@iitr.ernet.in

National Conference on Modern Analysis and Allied Areas

(Under UGC-DRS Programme)
February 23–24, 2007

Banaras Hindu University, Varanasi 221 005, India

Call for Papers: The Department of Mathematics, Banaras Hindu University, Varanasi is organizing a National Conference on Modern Analysis and Allied Areas, at BHU, Varanasi, under the DRS programme, during Feb. 23–24, 2007. The conference is meant for researchers and practitioners engaged in various facets of the theory and applications of Modern Analysis and Allied Areas. It will provide a forum for the exchange of ideas, sharing of experience, and discussions of related issues through invited talks, technical paper presentations and panel discussions.

Topics to be Covered (but not limited to): Functional Analysis, Operator Theory, Probabilistic Functional Analysis, Measure Theory, Complex Analysis, Sobolev Spaces and Distribution Theory, Wavelets and Applications, Fuzzy Analysis, Infinite-dimensional Optimization Theory, Fourier Analysis, Summability Theory, Ergodic Theory and Dynamical System etc.

The **Registration Fee** is Rs. 300 (Rs. 200 for students), payable by bank draft favouring "National Conference on Modern Analysis at BHU" (latest by Jan. 31, 2007). Accomodation will be arranged within the University Campus. However, on advance intimation, hotel accommodation can also be arranged on additional payment.

Funding: The funds at the disposal of organizers are limited. Hence participants are encouraged to use their own resources for TA/DA.

Deadlines:

1) Dec. 20, 2006: For receipt of TWO copies of the extended Abstract of paper/talk (of about 2 pages).
2) Jan. 15, 2007: Notification of Acceptance of the Abstract.
3) Jan. 31, 2007: For receipt of TWO hard copies of Full paper. Electronic submission, preferably as LaTeX/PDF/PS file, is strongly encouraged.

Please send the Abstracts along with the *Registration Details*, giving your (1) Name, (2) Designation, (3) Institution, (4) Address, (5) Title of the Paper/Talk, (6) Telephone/Fax/ E-mail (7) Accommodation needed/not needed, to any of the followings persons at:

Department of Mathematics
Faculty of Science
Banaras Hindu University
Varanasi 221 005

1) Professor R. S. Pathak, (Ex-coordinator, DRS Programme), E-mail: ramshankarpathak@yahoo.co.in
2) Professor A. K. Srivastava (coordinator, DRS programme): E-mail: arunksrivastava@gmail.com
3) Dr. Harish Chandra (Convener), E-mail: harish_pathak2004@yahoo.com

Note: Please do provide E-mail addresses for speedy communication.

UGC-SAP (DRS) National Conference on Mathematics, Computing and Modeling

3–4, March 2007

Organised by

Department of Mathematics
Gandhigram Rural University
Gandhigram 624 302, Tamilnadu

Theme of the Conference: National Conference on Mathematics, Computing and Modeling 2007 (NCMCM-2007) provides a National forum for faculty members, researchers working in the field of soft Computing/computational intelligence and application developers from many different area.

About the Department: The Department of Mathematics was established in the year 1965 and it offered three year under graduate degree in Mathematics, two year post graduate degree in Mathematics and Computer Applications until 2004, thereafter the two year PG course was upgraded to three year M.Sc Tech (Industrial Mathematics with Computer applications). In addition M.Phil and doctoral programme is offered in several branches of Mathematics and Computer Applications namely Optimization Theory and Algorithms, Mathematical Modeling, Control Theory, Neural Networks, Pattern Recognition, Stochastic differential equations, Cryptosystems, Graph Theory, Fractal theory and fuzzy set theory. The department has a moderately good library with over 3500 books and back volumes of several journals. This library is receiving NBHM library grant in addition to the UGC grants every year.

Contact Address: For registration details, paper submission, etc. may be obtained from

Prof. P. Balasubramaniam
Convenor, NCMCM-2007
Department of Mathematics
Gandhigram Rural University
Gandhigram 624 302.
Dindigul Dist., Tamil Nadu
Phone: 0451-2452371(O), 0451-2453091(R)
Mobile: 9443928891
E-mail: grumath@gmail.com
Web address: www.ruraluniv.ac.in

Tata Institute of Fundamental Research (A Deemed University)

Homi Bhabha Road, Colaba, Mumbai 400 005
Visiting Students' Research Program - 2007
(SNS, STCS & NCRA)

General Information: The research activities in TIFR are organized under three Schools: the School of Natural Sciences (SNS), the School of Mathematics, and the School of Technology and Computer Science (STCS) and at several centers. A summary of the areas of research relevant to VSRP-2007 are given below. Please check the website http://www.tifr.res.in/∼vsrp for additional details. For a detailed information about the VSRP programme, one may look at http://www.tifr.res.in/∼vsrp/info_brochure.pdf

TIFR offers summer programs exposing academically bright students to research in many areas of Biology, Chemistry, Computer & Systems Sciences, Mathematics and Physics (Astronomy & Astrophysics, Condensed Matter Physics & Material Sciences, High Energy Physics, Nuclear & Atomic Physics and Theoretical Physics).

The programme is held at TIFR, Mumbai for the subjects mentioned above and in addition, at the Institute's National Centre for Radio Astrophysics (Pune/Ooty) for Physics.

The VSRP programme is from May 21 to July 13, 2007. Participants are paid Sleeper class return railway fare, a stipend (Rs. 4000/- per month) and provided hostel accommodation.

Eligibility for Biology, Chemistry, Mathematics, Physics: Pre-final year students of

(1) M.Sc. in Applied Maths, Astronomy, Biosciences, Chemistry, Mathematics, Physics.
(2) B.E./B.Tech with interest in Physics.
(3) Medicine/Engineering.
(4) Exceptionally bright B.Sc. students in Mathematics may be considered.

Computer & Systems Sciences: Pre-final year students of B.E./B.Tech./M.E./M.Tech. in Computer Science/Electrical Engineering.

How to Apply: Download the VSRP-2007 application form from the Application Material section of this website or send a self-addressed, stamped (Rs. 10/-) envelope (25 cm × 17 cm) superscribed: VSRP-2007 to: "Assistant Registrar (Academic), University Cell, Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Mumbai 400 005.

Last Date for Requesting Application Forms: January 12, 2007
Last Date for Submitting Filled-in Forms: January 25, 2007.

Report of Workshop on Quasiconformal Mappings and their Applications (IWQCMA05)

December 27, 2005–January 1, 2006

The Department of Mathematics, IIT Madras, Chennai hosted International Workshop on Quasiconformal Mappings and their Applications (IWQCMA05) December 27, 2005–January 1, 2006. The workshop was co-organized by Chennai Mathematical Institute, Chennai and The Institute of Mathematical Sciences, Chennai.

This event was the first one in India on this active research area which has its roots in geometric function theory and which is closely connected with several topics of mathematical analysis.

The organizers gratefully acknowledge the financial support of

1) National Board for Higher Mathematics (DAE), India
2) National Science Foundation, USA

3) The Abdus Salam International Center for Theoretical Physics, Italy
4) Commission on Development and Exchanges of the International Mathematical Union, Italy
5) Indian National Science Academy, India
6) Council of Scientific and Industrial Research, India
7) Department of Science and Technology, India
8) Forum d'Analystes, Chennai, India.

Prior to the start of the workshop, preworkshop lectures were given by Dr. Antti Rasila and Prof. Raimo Näkki from Finland. The participants, who represented many different countries, received in most cases financial support from their national funding organizations to cover their expenses. ICTP's generous support was useful in supporting mathematicians from developing countries. Without the invaluable support from the aforementioned organizations, this conference would not have been possible in its present form.

The main goal of the conference was to bring together internationally well-known experts representing geometric function theory and some related topics. They were requested to deliver a series of lectures for postgraduate students on their respective areas. The audience consisted of mathematicians ranging from graduate students to well-known experts from all the participating countries.

Conformal invariance and conformally invariant metrics have been important research topics in geometric function theory during the past century. These topics also were discussed or mentioned in several of the lectures. The organizing committee was pleased to observe that the lectures were very well received and lead to many lively discussions afterwards. We were also pleased to receive positive response from the speakers to our request to contribute their lectures for the proceedings. It is our hope that the publication of these proceedings results presented in this Workshop and also this research area and its challenging open problems more widely known for a wide readership than what is the case presently. The editorial work was carried out at IIT Madras, and the www-pages

1) `http://mat.iitm.ac.in/~samy/`
2) `http://www.cajpn.org/madras/`
3) `http://www.math.utu.fi/projects/madras`
   contain a copy of these proceedings.

On behalf of the Organizing committee
S. Ponnusamy

Edited Volume Quasiconformal Mappings and their Applications

Edited by

S. Ponnusamy (IIT Madras, India)
T. Sugawa (Hiroshima University, Japan)
M. Vuorinen (University of Turku, Finland)

Publisher: Narosa Publishing House, India, pp. 354

Year of Publication: 2007

About the Title: Quasiconformal Mappings And Their Applications covers conformal invariance and conformally invariant metrices, hyperbolic type metrices, and hyperbolic geodesics, isometries of relative metrices, uniform spaces, and Gromov hyperbolicity, quasiregular mappings and quasiconformal mappings in higher dimensional spaces, universal Teichmueller space and related topics, quasiminimizers and potential theory, and numerical conformal mappings and circle packings.

For Further Detail Contact:
S. Ponnusamy at `samy@iitm.ac.in`

Lars Ahlfors Centennial Celebration

Organizing Institute:

University of Helsinki
Department of Mathematics and Statistics
Faculty of Science
Helsinki, Finland
August 20–24, 2007

Plenary Speakers Include:

Lennart Carleson
Alice Chang
Guy David
David Gabai
Juha Heinonen
Bruce Kleiner

Olli Lehto
Curt McMullen
Jill Pipher
Yum-Tong Siu
Dennis Sullivan
Xavier Tolsa
Gunther Uhlmanni

For Further Inquiries Contact:
Kirsi Peltonen at kirsi.peltonen@tkk.fi.

For Updated Information Visit:
http://mathstat.helsinki.fi/ahlfors100

International Conference on Industrial & Applied Mathematics

Jammu University, Jammu, India
March 31–April 3

Topics: To be emphasized are more relevant to real world problems such as wavelets, fractals, scientific computation, inverse problem.

Deadline: For participation request is January 30, 2007.

Information: Interested persons in participation of the conference may contact one of the following persons.

Prof. B. S. Komal at bskomal2@yahoo.co.in
Prof. Manchanda at pmanch2k1@yahoo.co.in
Prof. A. H. Siddiqi at siddiqi.abulhasan@gmail.com

Details can be found on http://www.siam-india.org.

Instructional Conference Solution Methods for Diophantine Equations

Lorentz Center, Leiden, The Netherlands
May 7–11, 2007

Aim: The instructional conference is meant for advanced master students, Ph.D. students and young post docs up to 35 years. The aim of the conference is to give an introduction to the techniques which are used today in solving Diophantine equations, coming from Diophantine approximation (linear forms in logarithms, hypergeometric method), p-adic analysis (Chabauty's method), and modular functions and Galois representations (Wiles' method).

Speakers: Lectures will be given by

Mike Bennett (UBC, Vancouver)
Nils Bruin (SFU, Vancouver)
Yann Bugeaud (Univ. Strasbourg) and
Samir Siksek (Univ. Warwick)

For More Information Visit:
http://www.lc.leidenuniv.nl/lc/web/2007/231/info.php3?wsid=231.

New Trends in Complex and Harmonic Analysis an International Conference on Analysis and Mathematical Physics

May 7–12, 2007

Location:
University of Bergen (UiB)
Norwegian University of Science and Technology (NTNU)
Voss, Norway

Description: The purpose of the Conference is to bring together specialists in Analysis with experts in Mathematical Physics, Mechanics and adjacent areas of applied sciences and numerical analysis. The participants will present their results and discuss further developments of the frontier research exploring the bridge between Complex, Real Analysis, Potential Theory, PDE and modern topics of Fluid Mechanics and Mathematical Physics. The Conference will feature invited expository 1 hour lectures, 45 min. talks and short 25 min. communications.

For Further Information Visit:
http://analysis2007.uib.no

Spring School on Analysis: Function Spaces, Inequalities and Interpolation

May 27–June 2, 2007

For Details:

E-mail: pasejune@karlin.mff.cuni.cz

http://www.karlin.mff.cuni.cz/katedry/kma/ss/jun07/

The International Conference "Differential Equations, Theory of Functions, and Applications"

May 28–June 2, 2007

(Dedicated to the Centennial of Academician Ilia N. Vekua, Novosibirsk, Russia)

Novosibirsk State University with the cooperation of the Siberian Branch of the Russian Academy of Sciences will convene the International Conference "Differential Equations, Theory of Functions, and Applications", May 28–June 2, 2007, Dedicated to the Centennial of Ilia Nestorovich Vekua.

Conference Topics:

Differential Equations
Theory of Functions and Complex Analysis
Theory of Elasticity
Mathematical Modeling

Scientific Program: The scientific program will consist of plenary lectures, invited 45-minute lectures, 30-minute talks, and 20-minute talks.

Registration forms and abstracts should be sent by E-mail to vekua07@math.nsc.ru

For additional information visit the site of the Conference:

http://math.nsc.ru/conference/invconf/vekua07/eng/.

Contact Address:

Alexander Kozhanov
Sobolev Institute of Mathematics
4 Acad. Koptyug ave. Novosibirsk 630 090
Russia
Phone: +7 383 3333492
E-mail: kozhanov@math.nsc.ru

The Fifth International Conference on Dynamic Systems and Applications

(Morehouse College, Atlanta, Georgia)

May 30–June 2, 2007

Topics: Dynamical Systems, Computational Mathematics, Simulation, Stochastic/Deterministic: Differential Equations, Partial Differential Equations, Integral Equations, Integro-Dif-ferential Equations, Difference Equations, and related topics.

Call for Papers: Authors are requested to submit an abstract of their research presentation in

http://atlas-conferences.com/cgi-bin/abstract/submit/catb-01

on or before March 31, 2007. Or please E-mail the abstract to icdsa5@yahoo.com.

For Further Information Visit:

http://www.dynamicpublishers.com/icdsa5.htm

Conference Coordinator:

M. Sambandham, ICDSA 05
Department of Mathematics
830 Westview Dr,
S.W., Morehouse College,
Atlanta, GA 30314, U.S.A.
Ph: (404) 215-2614
E-mail: icdsa5@yahoo.com

12th International Summer School in Global Analysis & Applications

August 20th–August 24th, 2007
Czech Republic or in Slovakia

The place of the summer school has not been determined yet. This 5-day course, organized by Department of Algebra and Geometry, Palacky University, Olomouc, Czech Republic, is aimed at mathematicians and theoretical physicists (at a doctoral or post-doctoral level) who are working in differential geometry, the calculus of variations on manifolds, differential equations and mathematical physics.

The lecturers will be Prof. Peter Olver, from University of Minnesota, USA, and Prof. Demeter Krupka, from Palacky University, Olomouc, Czech Republic.

For more information and online registration, see the official webpage of Summer School at:

http://globanal.upol.cz/summer07.html

International Conference of Numerical Analysis & Applied Mathematics 2007 (ICNAAM 2007)

16–20 September, 2007
HOTEL MARBELLA, CORFU, GREECE

Invited Speakers so far:

- Professor Dr. Carl R. de Boor, Department of Computer Sciences & Department of Mathematics, University of Wisconsin - Madison, USA
- Professor Dr. C. W. Gear (Bill), Senior Scientists, Chemical Engineering, Princeton University (zero-time appointment), Emeritus President, NEC Research Institute, Emeritus Professor, Department of Computer Science, University of Illinois at Urbana-Champaign, USA
- Professor Dr. Mariano Gasca, Depto. Matema?tica Aplicada, Fac. Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain
- Professor Dr. G. Alistair Watson, University of Dundee, Division of Mathematics, Dundee DD1 4HN, Scotland.

We are proud to announce that the Proceedings of ICNAAM 2007 will be published in the very famous AIP (American Institute of Physics) Conference Proceedings. More information can be found at:

http://www.icnaam.org/proceeding.htm

We note that the Proceedings of ICNAAM 2004 have been abstracted/indexed in: ISI Proceedings, Zentrablatt fur Mathematik, MathSciNet. The Proceedings of ICNAAM 2005 have been abstracted/indexed in: Zentrablatt fur Mathematik, MathSciNet. Selected Full Papers of ICNAAM 2007 will be published in appropriate journals.

The session and symposium organizers have free registration and a participation in the accommodation fee.

If you want leaflets and posters for ICNAAM 2007, please send your request to tsimos@mail.ariadne-t.gr (with a carbon copy to: tsimos.conf@gmail.com)

We mention that leaflet can be downloaded from the URL address of the Conference: **http://www.icnaam.org/**

Complex Analysis & Wave Processes in Mechanics

August 19—26, 2007

Will be held within the framework of Bogolyubov Readings 2007 in Zhitomir

Program of the Conference foresees the following directions

1. Hypercomplex analysis.
2. Methods of potential theory and their application in problems of mechanics.
3. Liquid flows with free surfaces.
4. Mathematical methods of investigation of wave processes in mechanics.

Organizing Committee: Co-Chairmen of the Organizing Committee P. M. Tamrazov, Corresponding member of NAS of Ukraine, O. S. Limarchenko

Members of the Organizing Committee: O. F. Gerus, S. A. Plaksa, A. V. Pokrovskii, Yu. B. Zelinskii.

We ask you to sent applications for participation in the Conference before June 1 2007 to address Institute of Mathematics NAS of Ukraine, 3 Tereshchenkivska street, Kiev-4, 01601, Ukraine or by E-mail: `olelim2000@yahoo.com` or `plaksa@imath.kiev.ua` with mark "Complex 2007"

Application must include family names and initials of authors, their affiliation, position, scientific degrees, abstract on 1 page in two samples.

International Conference Bogolyubov Readings 2007 "Mathematical Problems of Nonlinear Mechanics"

Will be held in Kiev, August 27-–September 2 (sections 1–3) and in Zhitomir, August 19—26 (section 4). The Conference is dedicated to 90th anniversary of academician of National Academy of Sciences of Ukraine and Russian Academy of Sciences Yuri Alekseevich Mitropolskiy. Program of the Conference foresees the following sections

1. Modern problems of nonlinear mechanics.
2. Perturbation methods in mathematical physics.
3. Asymptotic methods of investigation of differential equations.
4. Complex analysis and wave processes in mechanics.

Organizing Committee: Honorable Chairman of the Organizing Committee academician of NAS of Ukraine Yu. A. Mitropolskiy Chairman of the Organizing Committee academician of NAS of Ukraine A. M. Samoilenko Deputy Chairmen O. S. Limarchenko, M. A. Perestyuk, Corresponding member of NAS of Ukraine Members of the Organizing Committee I. A. Lukovskiy, academician of NAS of Ukraine S. A. Plaksa, V. G. Samoilenko, P. M. Tamrazov, Corresponding member of NAS of Ukraine, V. V. Sharko, Corresponding member of NAS of Ukraine.

We ask you to sent applications for participation in the Conference before June 1 2007 to address Institute of Mathematics NAS of Ukraine, 3 Tereshchenkivska street, Kiev-4, 01601, Ukraine or by E-mail: `olelim2000@yahoo.com` with mark "Bogolyubov Readings 2007".

Application must include family names and initials of authors, their affiliation, position, scientific degrees, abstract on 1 page in two samples.

2007 Workshop/Summer School in Saariselkä

Qualitative Properties of Solutions to Elliptic & Parabolic Equations European Science Foundation Programme

June 7–10, 2007 in Saariselkä, Finland

Place: The meeting takes place in Saariselkä, Finland, that is a small town (village) in the Finnish Lappland, 250 km north of the Arctic Circle.

Topics:

- p-Laplacian type equations
- Free boundary value problems
- Nonlinear parabolic equations

Invited speakers: Xavier Cabré (Barcelona), Emmanuele DiBenedetto (Vanderbilt) [not confirmed], Bernd Kawohl (Cologne), John L. Lewis (Kentucky), Peter Lindqvist (NTNU, Trondheim), José G. Llorente (Barcelona), Juan J. Manfredi (Pittsburgh), Giuseppe Mingione (Parma), Henrik Shahgholian (KTH, Stockholm), José Miguel Urbano (Coimbra), Changyou Wang (Kentucky).

Organizers:

Petri Juutinen (E-mail: `peanju 'at' maths.jyu.fi`)
Tero Kilpeläinen (E-mail: `terok 'at' maths.jyu.fi`)
Juha Kinnunen (E-mail: `juha.kinnunen 'at' oulu.fi`)
Xiao Zhong (E-mail: `zhong 'at' maths.jyu.fi`)

Programme: 3 mini-courses (4 hours each), 1 hour talks.

More information: http://www.math.jyu.fi/saariselka/

National Conference on Analysis and Graph Theory (NCAGT)

March 9–10, 2007

The National Conference on Analysis and Graph Theory (NCAGT) will be held during March 9–10, 2007 at the beautiful campus of Bharathidasan University, Tiruchirappalli, Tamil Nadu, India as part of the Silver Jubilee Celebrations of Bharathidasan University. The Conference focuses on Differential Equations, Computational Methods, Numerical Analysis, Functional Analysis and Graph Theory.

Registration Fee:

Students	Rs. 400/-
Other Delegates	Rs. 500/-

Dates to Remember:

Submission of Abstracts	5th January, 2007
Registration	31st January, 2007
Request for Hotel reservation	31st January, 2007
Submission of full paper	31st January, 2007

Contact:

E-mail: ncagt@yahoo.com
ncagt2007@gmail.com
Phone +91-431-2407065 (off)
Fax +91-431-2407045

For more details visit: http://www.bdu.ac.in/ncagt.htm

The 19th International Conference of the Jangjeon Mathematical Society

February 22–24, 2007

Organizers:

Department of Mathematics
Central College
Bangalore University
Bangalore 560 001, India and
The Jangjeon Mathematical Society
South Korea

Address for Correspondence:

Dr. M. S. Mahadeva Naika,
Convenor, ICJMS,
Department of Mathematics,
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National Conference on Mathematical Methods and Applications (MMA-2007)

17–17 March, 2007

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15th Mathematics Training and Talent Search Programme
(Funded by National Board for Higher Mathematics)

Aim: The aim of the programme is to expose bright young students to the excitement of doing mathematics, to promote independent mathematical thinking and to prepare them for higher aspects of mathematics.

Academic Programme: The programme will be at three levels: Level O, Level I and Level II. In Level O there will be courses in Linear Algebra, Analysis and Number Theory/ Discrete Mathematics. In Levels I and II there will be courses on Algebra, Analysis and Topology. There will be seminars by students at all Levels.

The faculty will be active mathematicians with a commitment to teaching and from various leading institutions. The aim of the instructions is not to give routine lectures and presentation of theorem-proofs but to stimulate the participants to think and discover mathematical results.

Eligibility: Level O: Second year undergraduate (B.Sc./Stat./ B.Tech.etc.) students with Mathematics as one of their subjects. Good first year students may also apply.
Level I: Final year undergraduate (B. Sc./B. Stat./B. Tech. etc.) students with Mathematics as one of their subjects.
Level II: First year postgraduate (M. Sc./M. Stat./M. Tech. etc.) students with Mathematics as one of their subjects.

Venues and Duration: There are three venues for MTTS 2007. All the three levels will be held at the Department of Applied Mathematics, University Institute of Chemical Technology (UICT), Mumbai during the period May 14–June 9, 2007.

Level O of the Programme will also be conducted at the Department of Mathematics, IIT Guwahati, during the period June 11–July 7, 2007 and at the Department of Mathematics, Panjab University, Chandigarh during the period June 4–June 30, 2007.

How to Apply?: Details and application forms can be had from the Head, Department of Mathematics, of our Institution. They can also be downloaded from MTTS Websites given below.

In case of difficulties, write to Professor S. Kumaresan by sending a self addressed and stamped (Rs. 10) envelope of size 10 cm $\times$ 22 cm to the address given below. Write "MTTS-2007 Application Form" on the cover of your letter. The completed application form should reach the programme director latest by 18th February, 2007.

Selection: The selection will be purely on merit, based on consistently good academic record and the recommendation letter from a mathematics professor closely acquainted with the candidate. Only selected candidates will be informed of their selection by the 3rd week of March 2007. The list of selected candidates will be posted on the Websites of MTTS.

Candidates selected for the programme will be paid sleeper class return train fare by the shortest route and will be provided with free board and lodging for the duration of the course.

For More Details Contact/Write/Visit:

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http://www.geocities.com/mttsprogramme
http://math.mu.ac.in/mtts/index.html

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