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Projective Modules Over Smooth Real Affine Varieties

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1. Introduction

There are many useful analogies and relations between algebra and topology. The most obvious example is that of a polynomial algebra over a field, the algebraic analogue of a Euclidean space. Let us cite another instance of one such analogue viz.

'Projective Module' over a commutative ring and 'Vector Bundle' over a topological space. In fact these two are closely related as follows.

Let X be a compact Hausdorff space and let $\mathcal{C}(X)$ denote the ring of real valued continuous functions on X. Let $E \to X$ be a (topological) real vector bundle over X. If P is the $\mathcal{C}(X)$-module of all continuous sections of E over X, then P is a finitely generated projective module over $\mathcal{C}(X)$ and in fact this leads to a bijective correspondence between the isomorphism classes of topological real vector bundles over X and the isomorphism classes of finitely generated projective $\mathcal{C}(X)$-modules (see [Sw] for details).

In view of the above correspondence, results on vector bundles on X get easily translated into corresponding (algebraic) results on projective modules over $\mathcal{C}(X)$. Unfortunately the class of rings of the type $\mathcal{C}(X)$ is very special. Hence one can ask whether there exist purely algebraic analogues of known results on topological vector bundles for projective modules over arbitrary commutative (noetherian) rings, at least in the case rings are nice say smooth affine over a field.

In this article we will highlight some results in the area of projective modules which give a partial answer to the above query.

2. Projective Modules Over Smooth Affine Varieties

Let us begin by recalling the definition of a projective module.

Let A be a commutative ring. A finitely generated A-module P is said to be projective if there exists a finitely generated A-module Q such that $P \oplus Q \simeq A^n$ where A^n denotes the free A-module of rank n.

It can be shown very easily that if the ring A is local (i.e. having only one maximal ideal) then every projective A-module is free. Using this fact, one can define the rank of a projective module (in a manner analogous to the rank of a vector bundle) over an arbitrary ring. The following result is an algebraic analogue of a well known result from topology (see [S]).

Theorem. **(Serre).** *Let A be a commutative noetherian ring of (Krull) dimension d. Let P be a projective A-module of rank $> d$. Then P splits off a free direct summand of rank 1 i.e. $P \simeq A \oplus Q$.*

The theorem of Serre immediately tells us that, over a ring A of dimension d, a projective module Q with rank $Q > d$ has a decomposition $Q = A^r \oplus P$, where rank $P = d$. In particular if $d = 1$ then every projective A-module is a direct sum of a free module and a projective module of rank 1 and it can be shown that such a decomposition is unique. So henceforth we assume that $d > 1$. The following example shows that the result of Serre is best possible in general.

Example. Let $A = \mathbb{R}[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1)$ be the coordinate ring of real two sphere. Let x, y, z denote images of X, Y, Z in A respectively.

Let P be the kernel of the surjection $\alpha : A^3 \twoheadrightarrow A$, which is defined by

$$\alpha(e_1) = x, \alpha(e_2) = y, \alpha(e_3) = z.$$

Then P is a projective A-module of rank $2 = \dim(A)$ such that $A \oplus P \simeq A^3 = A \oplus A^2$.

Since P "corresponds" to the tangent bundle of the real two sphere S^2, if P splits off a free direct summand of rank 1, then the tangent bundle would have a nowhere vanishing section, which is known to be false as its Euler class is not zero. The Euler class is a topological invariant associated to an orientable vector bundle over an orientable smooth manifold. This association is such that if the rank of the vector bundle and the dimension of the manifold are the same, then vanishing of the Euler class is a necessary and sufficient condition for the vector bundle to have a nowhere vanishing section. In view of this, it is natural to ask:

Question. Let A be a commutative noetherian ring of dimension d and P a projective A-module of rank d. Can one attach an invariant to P (analogous to the Euler class) the vanishing of which would ensure that P splits off a free summand of rank 1?

To tackle this question, we now restrict ourself to a class of rings namely, coordinate rings of smooth affine varieties over the field k which will be either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers.

Recall that an affine variety $X \subset k^n$ is defined as the set of common zeros of finitely many polynomials in n variables over k. We always assume that X is not empty. The totality of all polynomials vanishing on X forms an ideal $I(X)$ in the polynomial algebra $k[X_1, \ldots, X_n]$. The quotient ring $A = k[X_1, \ldots, X_n]/I(X)$ is called the coordinate ring of X and the (Krull) dimension of A is called the dimension of X. The affine variety X (equivalently A) is said to be smooth if every maximal ideal of A is *locally* generated by d elements where $d = \dim(A)$. If A is smooth and $k = \mathbb{R}$, then X is a smooth real manifold in the Euclidean topology with finitely many connected components.

In this setup, one associates to A an abelian group $CH_0(A)$ called the Chow group of zero cycles. In brief, $CH_0(A)$ is the free abelian group on maximal ideals of A modulo certain relations.

Let P be a projective A-module of rank $d = \dim(A)$. We associate to P an element $C_d(P)$ (called the top Chern class of P) belonging to the group $CH_0(A)$ as follows.

By a reduced generic surjection we mean a surjection $\alpha : P \twoheadrightarrow J$ where $J \subset A$ is a finite intersection of maximal ideals of A. It can be shown that P has plenty of reduced generic surjections. Let $P \twoheadrightarrow J$ be one such surjection and let $J = \bigcap_{i=1}^{r} m_i$ (m_i: maximal ideal). It can be shown that image of $\sum_{i=1}^{r}[m_i]$ in $CH_0(A)$ is independent of the choice of the reduced generic surjection.

We define $C_d(P)$ to be the image of the cycle $\sum_{i=1}^{r}[m_i]$ in $CH_0(A)$.

In the definition of $CH_0(A)$, relations are so designed that $P \simeq A \oplus Q \Rightarrow C_d(P) = 0$. So it is natural to ask: Does $C_d(P) = 0$ imply $P \simeq A \oplus Q$? In this connection we now state a result of Murthy ([Mu]).

Theorem. (Murthy). *Let A be the coordinate ring of a smooth affine variety X of dimension d over $\mathbb{C}$. Let P be a projective A-module of rank d. If $C_d(P) = 0$ then $P \simeq A \oplus Q$.*

Thus, if A is a smooth affine domain over $\mathbb{C}$ and P is a projective A-module of rank = dim (A), then the "top Chern class" of P is a nice algebraic analogue of the topological notion of "Euler class" of a vector bundle. However, if A is the coordinate ring of an *even* dimensional real sphere S^{2r} and P is the projective A-module corresponding to the tangent bundle of S^{2r} (see the above example in the case $r = 1$) then it can be proved that the top Chern class $C_{2r}(P) = 0$. It is a well known result from topology that the Euler class of the associated vector bundle (namely tangent bundle) is not zero. This underlines the fact that if the base field is $\mathbb{R}$ then "top Chern class" is not sufficiently good analogue of "Euler class". To overcome this deficiency, Nori associated a subtler invariant to P (called its *algebraic* "Euler class") and posed the question:

Does P split off a free summand of rank one if its (algebraic) Euler class vanishes?

The precise definition of the algebraic Euler class is rather technical and therefore we omit it. However we would like to mention that Bhatwadekar and Raja Sridharan have answered the question of Nori affirmatively (see [B-RS]). In view of the result of Bhatwadekar- Raja Sridharan and the example of the tangent bundle of an even dimensional real sphere, it is of interest to investigate the following question: If the base field is $\mathbb{R}$, under what further restrictions vanishing of top Chern class of P would imply that its (algebraic) Euler class is zero (equivalently $P \simeq A \oplus Q$)?

We conclude this article by quoting the following result which answers the above question (see [B-D-M]).

Theorem. (Bhatwadekar–Raja Sridharan-Das-Mandal). *Let A be the coordinate ring of a smooth real affine variety X of dimension d. Assume that the manifold X is connected. Let P be a projective A-module of rank d such that the top Chern class $C_d(P)$ of P is zero. Then $P \simeq A \oplus Q$ in the following cases:*

(I) *The manifold X is not compact.*

(II) *The manifold X is compact and d is an odd integer.*

Moreover if X is compact and d is even then there exists a family of projective A-modules P of rank d with $C_d(P) = 0$ but $P \not\simeq A \oplus Q$.

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Symmetric, Quasi-Symmetric Designs and Strongly Regular Graphs

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This article is in the general area of combinatorial configurations. Which combinatorial configurations are more interesting (than some others) and why is that so? How are they linked to each other? This is the theme that we will apply to designs, graphs, codes, groups etc. In between, we will also look at how other branches, especially linear algebra help in the study of combinatorial configurations. We also draw on other areas in particular, number theory. The author has felt that it is necessary for a serious worker in combinatorics to know about other branches.

Combinatorial Designs first appeared in the mediaveal literature mainly as puzzles. We will dwell on this theme somewhat later during the course of this talk. In the twentieth century, Combinatorial designs, mainly in the form of constructions were first looked at by statisticians. Specifically a designs or a 2-design has the following definition. By an incidence structure, we mean a triple, consisting of the set of points, the set of blocks (a block is nothing but a distinguished subset of the point-set) and the incidence (a point belongs or does not belong to a block). Since the blocks are incomplete and generally not repeated, every block may be simply viewed as a subset of the point-set. An incidence structure is called a *design* (balanced incomplete block design or a 2-design) if every block has a constant size k, which is strictly less than v, the number of points, and any two points occur together (are commonly contained in) the same number λ of blocks. If v is the number of points, b the number of blocks and r the number of blocks containing a given point, then such a configuration is called a (v, b, r, k, λ)-design or sometimes also called a (v, k, λ)-design. Let N denote the (point-block) incidence matrix of a design **D**. Then the property of constant row and column sums on N produces the following two equations:

$$\lambda(v-1) = r(k-1)$$
$$vr = bk$$

while the fact that the rows of N are linearly independent yields the *Fisher's inequality* $v \leq b$, i.e., the number of points is less than or equal to the number of blocks. Further if D is a design with $v = b$, the incidence structure in which blocks are points and points are blocks is also a design with the same parameter set (v, k, λ) (this is called the combinatorial dual of the given design). A design with $v = b$ is called a *symmetric design*.

There are two classical families of symmetric designs which we briefly mention. Let H denote a Hadamard matrix of order $4t$. From such a matrix one can construct an incidence matrix of a $(v, k, \lambda) = (4t-1, 2t-1, t-1)$ symmetric design which is called a Hadamard symmetric design. We have already seen the special example of Fano plane. In general, we can look at a projective space of dimension 2 over $GF(q)$ and declare its points as points and its lines as blocks of a symmetric design. This is called a projective plane and has parameters $(q^2+q+1, q+1, 1)$. This construction works for every prime power q. More generally, we may take an n-dimensional projective space over $GF(q)$ and declare its points as points and its hyperplanes as blocks of a symmetric design. This is called a projective design and has parameters

$$v = \frac{q^{n+1}-1}{q-1}, k = \frac{q^n-1}{q-1}, \lambda = \frac{q^{n-1}-1}{q-1}$$

For the reasons of structural symmetry and better connections with group theory, symmetric designs are objects of considerable interest. Many constructions of symmetric designs are known. The existence question for Symmetric designs is the question of constructing a (0, 1)-matrix satisfying the matrix equations given above. Algebraic number theory has been employed in order to answer this existence question and the relevant seminal result is called the Bruck–Ryser–Chowla theorem . Unfortunately, it works only in one direction. That is, it provides us with only a necessary condition which, may not be sufficient. For example, it is not known whether there is a projective plane of order twelve but it is known, thanks to the Bruck–Ryser–Chowla theorem that : If q is the order of a projective plane such that $q \equiv 1, 2 (\text{mod } 4)$ then q is a sum of two integer squares. In particular, there are no projective planes of orders q such that $q \equiv 6 (\text{mod } 8)$. An extensive search for almost 200 hours on the fastest CRAY computer available then proved in the late 1980s that there is no projective plane of order ten. There have also been a large number of new constructions of new symmetric designs in recent times. Based, in terms of ideas, on an earlier work of Dinesh Rajkundlia, Ionin in the last decade constructed new families of symmetric designs many of which were open questions for a long time. About 23 infinite families of symmetric designs are known. However, all the *known examples* of symmetric designs seem to be only of the following types:

- When $\lambda = 1$, we have a projective plane of order q with parameters $(q^2+q+1, q+1, 1)$. These exist for every prime power q. *No other examples are known.*
- When $\lambda = 2$, we have a biplane with parameters $(\frac{k}{2+1}, k, 2)$. These are known to exist for the following values of k: 3, 4, 5, 6, 9, 11, 13. *No other examples are known.*
- When $\lambda = 3$, all the *known examples have k bounded by* 15.
- When $\lambda \geq 4$, all the *known examples have k bounded by* $\lambda^2 + \lambda$.

The known situation led to the following informal conjecture attributed to M. Hall.

M. Hall's conjecture: $\forall\lambda \geq 2$, there exist only finitely many symmetric (v, k, λ).

Stronger: $\forall\lambda \geq 4$, k satisfies $k \leq \lambda^2 + \lambda$.

We now switch our attention to a new concept. Recall that a regular graph Γ of degree k is a simple graph in which every vertex has degree k. The graph Γ with n vertices and regular of degree k is called a strongly regular graph with parameters (n, k, a, c) if we have two more constants a and c such that

- If x and y are adjacent vertices then the number of vertices commonly adjacent to both of them is a.
- If x and y are non-adjacent vertices then the number of vertices commonly adjacent to both of them is c.

The Petersen graph P is a graph on 10 vertices is an example of a strongly regular graph and has parameters (10, 3, 0, 1). This graph is also a quotient of the graph of the 1-skeleton of a regular dodecahedron. For graphs, we have an adjacency matrix A which is a square matrix of order n with entry 1 at (i, j)-th place iff the i-th and j-th vertices are adjacent and 0 otherwise. Notice that A is a real symmetric matrix and hence must have real eigenvalues. For the adjacency matrix of a strongly regular graph, we have the matrix equation: $A^2 - (a-c)A - (k-c)I = cJ$. Since Γ is connected and regular of degree k, the other eigenvalues can be read off from the above matrix equation and they satisfy the quadratic equation:

$$x^2 - (a-c)x - (k-c) = 0. \qquad (*)$$

It was shown by Shrikhande and Bhagwandas that a regular and connected graph Γ is strongly regular iff it has exactly two eigenvalues besides the degree of regularity.

Some interesting questions: In a gathering of n people every two persons have exactly one common friend. The Friendship theorem then asserts that there is a person who knows all the rest (and the corresponding graph is the windmill graph). Moore graphs are characterized by $a = 0$ and $c = 1$. Here we neither have triangles (because $a = 0$) nor 4-cycles (because $c = 1$) but have a very large number of pentagons. The counting equation produces $n = k^2 + 1$ and the only possibilities are $k = 3$ when the graph is the Petersen graph, $k = 7$ and $n = 50$ and we get the Hoffman-Singleton graph. There is also one more possibility: $k = 57$ but the existence of a graph in this situation is still an open question.

Take a design D, not necessarily symmetric. An integer x is called a block intersection number of D if we have two blocks X and Y the cardinality of whose intersection is x. Which numbers occur as block intersection numbers of a design? Thanks to the proof of Fisher's inequality, we see that D has exactly one block intersection number iff it is a symmetric design. *A design with two block intersection numbers is called a Quasi-symmetric design.* Reasons for studying quasi-symmetric designs are many. A mundane and practical reason is that symmetric designs are more difficult to study (this is not completely true but sometimes believed to be so). On a more serious level quasi-symmetric design allows one to construct its block graph which in most cases of interest can be shown to be strongly regular. Finally quasi-symmetric designs are connected with combinatorial configurations arising out of finite simple groups. Here is a class of examples. Let D be a point-block incidence structure constructed from $PG(n, q)$, a projective geometry of dimension n over a field with q elements by taking as points the points of the geometry and as blocks the subspaces of codimension two (where $n \geq 3$). There are other classes of examples particularly the affine geometries (where $x = 0$) There is also a classical object called the Witt design on 23 points which is associated with the Mathieu group M_{23} on 23 letters. A lot of reseach work in the area of quasi-symmetric design stems from the linear algebra of its incidence and adjacency matrix. The later turns out to be a strongly regular graph and this paves way for further analysis of these designs.

A topic that we did not look at so far is that of coding theory. Coding theory initially began as a helper to information theory in the sense that 'good codes' made it possible to correct large number of errors. To be in the theme of this talk, a code C is a vector subspace of the vector space $\mathbf{F}^n$. Here $\mathbf{F}$ is the field with two elements. A codeword z in C has weight $w(z)$ where $z = z_1, z_2, \ldots, z_n$ and $w(z)$ equals $|\{i : z_i \neq 0\}|$. The weight enumerator of C is a polynomial in formal variables x and y that counts the number of codewords of a given weight. Obtaining weight enumerator (and hence the weight distribution) of a code C is an important question of mathematical coding theory. This study is facilitated by the use *MacWilliams identity* which gives the weight enumerator of the dual of a code C in terms of the weight enumerator of C. Note that being in the situation of finite fields, we do have the possibility that $C \subset C^{\perp}$ and even $C = C^{\perp}$. This puts severe restrictions on the dimension of C. Any binary matrix N gives rise to a code C by taking C to be the linear span of the rows of N. So the game is as follows: Beginning with a putative design D, along with its incidence matrix N we make a code of the design, derive some results on its weight enumerator and sometimes can even rule out the existence of D itself due to the inconsistencies! This has been employed to rule out the existence of a large number of putative parameters of both symmetric and quasi-symmetric designs. Specifically, looking at a putative projective plane of order ten, M. Hall was able to obtain a reasonable information on the code of such a configuration and Assmus was even able to show that the minimum weight of such a code is 11 and the codewords of minimum weight are precisely the rows of the incidence matrix N. This finally led to the proof of non-existence of a projective plane of order ten. Codes have also been used to construct some nice combinatorial configurations. For example, the Golay code $\mathbf{G}_{23}$ is a perfect code in which words of minimum weight 23 carry a quasi-symmetric design called the Witt design.

For about 100 years people have been trying to find new finite simple groups. Classical examples of finite simple groups are the alternating groups on n letters for all $n \geq 5$ as also the groups $PSL(n, q)$ for $n \geq 2$. All such classical finite groups are called the groups of 'Lie type' and the other examples of finite simple groups are called 'Sporadic Simple groups'. The first example of a sporadic simple group was constructed by Mathieu when he gave constructions of 5 sporadic simple groups. The next example took 100 years to get constructed and it was the discovery of Higman-Sims group. Over years many sporadic simple groups were discovered and by 1980, it was announced that the 'Classification' of finite simple groups is complete and we have only 26 sporadic simple groups. As such structure of finite simple groups is quite complex. Such simple groups contain a large number of copies of smaller simple groups that are, of course, not normally embedded. As an example, it is a curious fact to know that besides the six routine copies of A_5 the simple groups A_6 also contains 6 other copies of A_5 that are different in their group action, that is, have a different combinatorial action. This curious fact gives one of the constructions of the Hoffman-Singleton graph discussed earlier.

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Fine Structure of the Number Line*

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1. Introduction

The number line is foundational to present day mathematics, and in a broader way it is a major marker of the progress of human civilization. From the early beginnings of counting on fingers we have come a long way, developing a system with which one can grasp the intricate laws of the physical universe.

The path however has been quite tortuous. From natural numbers, especially their infinitude, fractions, the zero, negative numbers have each been a big conceptual stride, taken over long historical periods. The difficulty in a concept like fraction being grasped in full by the human mind may be realised if one recalls how a significant proportion of our students have difficulty in adding them. The ancient Indian civilization contributed greatly to the rise of zero; it was however not invented one fine day by some sage, as popular imagination would have it, but had a long and chequered history involving several ancient civilizations [8]. Negative numbers must have been lurking in the background for long. But a systematic approach to them appears for the first time in the work of Brahmagupta (589–668).

Notwithstanding some lacunae, various ancient civilizations may be said to have had a general understanding of fractions, or what we now call rational numbers, over two thousand years ago. Going beyond them however posed a bigger challenge. Indeed, various numbers that we now know to be irrational

*Text version of the "T. B. Hardikar Memorial Lecture" given at S.P. College, Pune, on 25 August 2007 were encountered in early times primarily in various geometric contexts. The Sulvasutras, which are compositions from the Vedangas (appendages of the Vedas) describing construction of various fire alters and various geometric principles involved in the constructions, contain the following value for the proportion of the diagonal of a square to its side, viz. the square root of 2, that we write as

$$1+\frac{1}{3}+\frac{1}{3\times 4}-\frac{1}{3\times 4\times 34};$$

(see [13] for an exposition on Sulvasutras.) The period of the Sulvasutras is difficult to ascertain, but is generally accepted to be between 800–400 BCE (Before Christian Era), with the earliest, the Baudhayana Sulvasutra near the early end of it. A cuneiform tablet (Yale Babylonian collection, No. 7289) from the old Babylonian period (around 1800–1600 BCE) shows a square with 1, 24, 51, 10 written across the diagonal; in the sexagesimal system (with base 60, in the place of 10 that we use) which the Babylonians used the number stands for

$$1+\frac{24}{60}+\frac{51}{60^2}+\frac{10}{60^3};$$

(it also stands for the multiples of the above by powers of 60, but it does not concern us here). It is quite striking that the above values for $\sqrt{2}$ adopted by the Sutrakaras and Babylonians match with the value of $\sqrt{2}$ up to five decimals, 1.4142156 . . . and 1.4142129 . . . respectively, in the place of 1.4142135 . . . . (These are however unique instances in their respective contexts, and what led to the strikingly close values in these cases is not clear.)

Sulvasutras are also concerned with transforming a square to a circle (disc) of equal area, and vice versa. For the former the construction is geometric (and the equality of areas is approximate). For the latter there is a curious formula prescribing for the ratio of the side of a square to the diameter of a circle with equal area, the number

$$1-\frac{1}{8}+\frac{1}{8\times 29}-\frac{1}{8\times 29}\left(\frac{1}{6}-\frac{1}{6\times 8}\right).$$

As in the case of $\sqrt{2}$, the actual number involved here is irrational, and the value represents an approximation, adopted for practical purposes. The ancient Egyptians also concerned themselves with the area of a circle, which they took to be $(8/9)^2$ times the square of the diameter [7].

The first systematic approach going beyond the rationals was introduced by the Greek mathematician Eudoxus of Cnidos (400-347 BCE), known as the Eudoxus's theory of proportions. In geometry, which was their main strength, the Greek mathematicians encountered magnitudes of lengths which they recognised to be "incommensurable". They had philosophical reservations about treating these magnitudes as numbers. Nevertheless they had a theory for the magnitudes, akin to the present day concept of real numbers. According to Eudoxus' theory ratio of magnitudes a and b is equal to that of magnitudes c and d if for positive integers m and n, $ma > nb$ we also have $mc > nd$ (see [3], page 89). If we choose c and d to be unit magnitudes this reduces to that a and b are equal if $ma > nb$ for all positive integers m, n with $m > n$. (Perhaps it calls for some explanation why one *defines* equality, whereas the relations $<$ and $>$ are being assumed. In this respect, without going into details let me only mention that this has to do with verifiability of the statements in terms of known, i.e. "commensurable", magnitudes.)

Crystallisation of the idea of real numbers was however to wait for over two thousand years. With the rise of calculus, and the idea of infinitesimals, in the seventeenth and eighteenth centuries the intuitive sense of continuity of the number line caught hold. Since the number line was now seen as being "continuous", a rigorous way to "fill in" between rationals was needed. An elegant definition of real numbers fulfilling this need was introduced by Dedekind in 1858. The idea involved is that for a real number as we conceive it intuitively, the collection of rational numbers less than that is a characteristic feature of the number. Taking this into account Dedekind defined real numbers as partitions (L, U) of the set of rational numbers, with the property that for any r in L and s in U, $r < s$; ((L, U) being a partition means that L and U are disjoint sets of rationals which together cover all rationals). Such a partition is called a *Dedekind cut*. Thus for example $\sqrt{2}$ is to be thought of (by definition) as the Dedekind cut (L, U) with L and U subsets of rationals consisting of $\{r \mid r \leq 0 \text{ or } r^2 < 2\}$ and $\{r \mid r > 0 \text{ and } r^2 \geq 2\}$ respectively. A rational number q may be thought of as the Dedekind cut into $\{r \mid r < q\}$ and $\{r \mid r \geq q\}$.

One can then extend the operations of addition and multiplication to the set of real numbers so defined, in a natural way (that the reader is encouraged to find for oneself). Furthermore it can be verified that every positive real number has a positive square root, cube root etc.

Dedekind's definition greatly facilitates verification of arithmetical statements about real numbers. This is dramatised in the title of [7]: Dedekind's theorem: $\sqrt{2} \times \sqrt{3} = \sqrt{6}$. While this may seem facetious (as the statement seems obvious) it may be noticed that writing a rigorous argument poses a variety of problems, including the matter of definition itself, which are readily taken care of by Dedekind's approach. The reader is referred to [7] for a discussion on how various ways thinking of real numbers through various systems of labelling, such as decimal representation or continued fractions (these we shall discuss below) do not lend themselves to the possibility of proving the simple theorem as above. For example, the decimal expansions of $\sqrt{2}$ and $\sqrt{3}$ are infinite, there is no way to write what the product of two numbers expressed in this way (since as we go down the decimal places the products of the corresponding digits would in general exceed 10, and keep calling for alteration in the digits in earlier places in a never ending way).

There are of course alternative possibilities for constructing the number line. The real numbers can also be thought of as limits of Cauchy sequences of rational numbers. However since different Cauchy sequences can have the same limit, an individual real number needs to be thought of as "equivalence class" under an equivalence relation which identifies two such Cauchy sequences $\{a_n\}$ and $\{b_n\}$ of rational numbers if the difference $a_n - b_n$ tends to 0. This can indeed be converted to a definition of real numbers and the arithmetic operations on real numbers can be introduced through them. The idea of considering equivalence classes as elements is however a late nineteenth century stratagem, that came into vogue long after Dedekind cuts. Besides even today the Dedekind cuts provide an alternative which is simpler in various ways. On the other hand the idea of using Cauchy sequence as above has other applications, such as for instance in constructing what are called p-adic numbers, for any prime number p.

2. Approximating Real Numbers by Rationals

We normally view the number line as being "uniform", looking the same everywhere, so it would seem there is no "structure" to speak of, and the title of the article may seem confusing. With some reflexion it would be clear that the intuitive sense of uniformity is the consequence of the fact that we always think of it in terms of equal subdivisions, say in terms of decimal expansion, binary expansion etc. This uniformity is of course in terms of analysis. It is however superposed, and need not be viewed entirely as constituting the structure of the number line. Recall that the number line was constructed from rational numbers, by filling in more numbers in between. In the equal subdivisions as above all rationals do not feature. To think of the structure we may think of how numbers get filled in between rational numbers, or in other words how the real numbers can be approximated rational numbers. This would involve all rational numbers rather than binary or decimal rationals. I will discuss various results on approximations. This aspect is what I have referred to in the title as the fine structure of the number line.

Before going into what I mean, let me recall what are called Farey fractions. A Farey fraction of order k is a rational number $\frac{p}{q}$ between 0 and 1, with $q \le k$. We shall consider these arranged in the increasing order. The first five rows are as shown below; (we shall write all the fractions are written in the reduced form).

$$
\begin{array}{ccccccccccc}
\frac{0}{1} & & & & & & & & & & \frac{1}{1} \\[1ex]
\frac{0}{1} & & & & & \frac{1}{2} & & & & & \frac{1}{1} \\[1ex]
\frac{0}{1} & & & \frac{1}{3} & & \frac{1}{2} & & \frac{2}{3} & & & \frac{1}{1} \\[1ex]
\frac{0}{1} & & \frac{1}{4} & \frac{1}{3} & & \frac{1}{2} & & \frac{2}{3} & \frac{3}{4} & & \frac{1}{1} \\[1ex]
\frac{0}{1} & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{2}{3} & \frac{3}{4} & \frac{4}{5} & \frac{1}{1}
\end{array}
$$

It turns out that the entries in each successive row may be obtained by inserting the fraction $\frac{m+p}{n+q}$ between consecutive entries $\frac{m}{n}$ and $\frac{p}{q}$ of the previous row, whenever $n+q$ does not exceed the number of the row being written down. This gives an alternative definition of the Farey fractions in the successive rows.

The Farey fractions have interesting properties; see [9] for instance. If $\frac{m}{n}$ and $\frac{p}{q}$ are successive entries in the k the row, then $np - mq = 1$ and $n + q \ge k + 1$. Now consider any real number α between 0 and 1 and let $\frac{m}{n}$ and $\frac{p}{q}$ be successive entries in the k th row such that $\frac{m}{n} \le \alpha \le \frac{p}{q}$. Then either $\frac{m}{n} \le \alpha \le \frac{m+p}{n+q}$ or $\frac{m+p}{n+q} \le \alpha \le \frac{p}{q}$, and together with the above this implies that either $|\alpha - \frac{m}{n}| \le \frac{1}{(k+1)n}$ or $|\alpha - \frac{p}{q}| \le \frac{1}{(k+1)q}$. Thus we have the following.

Theorem 2.1. *Let α be a real number between 0 and 1 and $\frac{p}{q}$ be the Farey fraction of order k at least possible distance from α. Then*

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{(k+1)q}.$$

As a consequence we get the following:

Corollary 2.2. *Let α be an irrational real number. Then there exist infinitely many rational numbers $\frac{p}{q}$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

This does not hold for rational α; if $\alpha = \frac{m}{n}$ the inequality as above can hold only for $\frac{p}{q}$ with $q \leq n$, which are only finitely many. Note that when there are infinitely many rationals satisfying an inequality they provide closer and closer approximation of α satisfying the restriction on the denominators.

Better approximations are possible through the study of "continued fractions". We briefly recall here the concept and some properties of continued fractions; the reader is referred to [9] and [11] for further details. Let α be a real number. If α is not an integer it can be written as $m_0 + \frac{1}{\alpha_1}$ with $\alpha_1 > 1$. Now unless α_1 is an integer it may be written as $m_1 + \frac{1}{\alpha_2}$ with $\alpha_2 > 1$, and the process may be repeated, until either we reach an integer, or indefinitely. It can be seen that the former happens if α is a rational and the latter when α is irrational. Thus any irrational number α can be written as

$$m_0 + \cfrac{1}{m_1 + \cfrac{1}{\cdot + \cfrac{\cdots}{m_k + \cfrac{1}{\cdot + \frac{1}{\cdot}}}}}$$

where m_0 is an integer and m_k, $k \geq 1$ are positive integers, and every rational number has such an expansion which stops at some k. To avoid writing the cumbersome, though illustrative, expression as above we shall denote it by $[m_0, m_1, \ldots, m_k, \ldots]$. Similarly we shall denote by $[m_0, m_1, \ldots, m_k]$ the analogous but terminating expression (omitting the part after m_k in the above expression. If $[m_0, m_1, \ldots, m_k, \ldots]$ is the representation as above for α we shall denote $[m_0, m_1, \ldots, m_k]$ by α_k, which is a rational number. A crucial thing is that the sequence α_k converges to α; this is what makes it meaningful to express α as above (otherwise the infinite expression makes no sense by itself). The expansion as above is called the continued fraction expansion of α. Conversely, given an integer m_0 and positive integers m_k, $k \geq 1$, the expression as above is the continued fraction expansion of a unique irrational number α. Note that unlike decimal or binary expansions, the continued fraction expansions of numbers are intrinsic, and do not depend on any ad hoc choice of basis.

The numbers m_k as above, associated with α, which we shall henceforth take to be irrational, are called *partial quotients* and the rationals α_k as above are called the convergents of α. The α_k, $k \geq 1$ are *best approximations* for α in the sense that if $\alpha_k = \frac{p_k}{q_k}$ then α_k is at least as close to α as any rational with denominator not exceeding q_k (namely any Farey fraction of order q_k), when $0 < \alpha < 1$. We thus have, by Theorem 2.1 $|\alpha - \frac{p_k}{q_k}| < \frac{1}{q_k^2}$. Conversely, if $\frac{p}{q}$ is a rational for which the slightly stronger approximation $|\alpha - \frac{p}{q}| < \frac{1}{2q^2}$ holds, then it is necessarily one of the convergents of α. The following theorem due to Hurwitz is proved using properties of convergents of continued fraction expansions.

Theorem 2.3. *Let α be an irrational number. Then there exist infinitely many rational numbers $\frac{p}{q}$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

Up to here the approximability properties are seen to be shared by all irrationals alike. However when we look for sharper relations this no longer holds. It turns out that for $\alpha = (\sqrt{5} - 1)/2$ analogous statement does not hold if $\frac{1}{\sqrt{5}}$ replaced by a smaller constant. Same is the case for numbers equivalent to it, namely of the form $\frac{m\alpha + n}{p\alpha + q}$, with m, n, p and q integers such that $mq - np = \pm 1$. However, but for these exceptions, for all other α the constant can be improved to $\frac{1}{\sqrt{8}}$. Then there is another number α such that the constant can not be improved any further for the numbers equivalent to it. Curiously the pattern repeats, and the constant can be improved to $\frac{5}{\sqrt{221}}$ for all others, and then again to $\frac{13}{\sqrt{1517}}$ for all but one more set of equivalent numbers, and so on; see [2] for details. The sequence $\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{8}}, \frac{5}{\sqrt{221}}, \frac{13}{\sqrt{1517}}, \ldots$ continues in analogous fashion, and tends to $\frac{1}{3}$. The numbers in the sequence are called *Markov numbers* after A. Markov who discovered the phenomenon, in 1879. It may be noted that in view of the above, in particular, given $c > \frac{1}{3}$, for all but countably many irrationals α, there are infinitely many rationals $\frac{p}{q}$ such that

$|\alpha - \frac{p}{q}| < \frac{c}{q^2}$. It turns out that $\frac{1}{3}$ is the smallest value for which this holds.

There is considerably more general theory on this topic; see for instance the recent paper [1] and the references suggested there. We will however end the general discussion here and move on to describing some extreme cases of interest.

3. Numbers with Exceptional Approximability Features

A real (or complex) number α is said to be *algebraic* if it satisfies a polynomial equation of the form $r_0\alpha^d + r_1\alpha^{d-1} + \cdots + r_{d-1}\alpha + r_d = 0$, where $r_0, \ldots, r_d$ are rational numbers, not all 0; when α is algebraic the smallest possible d for which there is such an equation satisfied by α is called the degree of α. A number which is not algebraic is called *transcendental*.

In 1844, Liouville proved the following theorem.

Theorem 3.1. *Let α be an algebraic irrational number of degree d. Then there exists $c > 0$ such that*

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^d}$$

for all rationals $\frac{p}{q}$.

As a consequence it follows that if α is a number such that for every $d \geq 1$ there exists a rational $\frac{p}{q}$ such that $|\alpha - \frac{p}{q}| < \frac{1}{q^d}$, then α is transcendental. It is quite easy to produce α with this property; consider for example the sum of the convergent series $\sum_1^\infty 2^{-k!}$. Numbers with the above property are called Liouville numbers. It can be seen that they provide an uncountable collection of transcendental numbers.

Notwithstanding the fact that there are uncountably many numbers satisfying it, the condition for Liouville numbers is viewed as very strong. A following less demanding condition has attracted much attention in literature. An irrational number is said to be *very well approximable* (VWA for short) if there exist $\epsilon > 0$ and infinitely many rationals $\frac{p}{q}$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}.$$

It turns out that even this degree of approximability is "rare" for real numbers. One way this manifests is that the Lebesgue measure of the set of VWA numbers is 0. This means that a "randomly picked" real number (e.g. a number whose binary expansion is a random binary sequence) would not be VWA.

Some questions involving higher dimensional analogue of this condition have been the subject of some recent research of D. Kleinbock and G.A. Margulis, and it may be worthwhile to recall one of their results here. A n-tuple $(\alpha_1, \ldots, \alpha_n)$ is said to be VWA if there exist $\epsilon > 0$, infinitely many n-tuples of integers $(q_1, \ldots, q_n)$, and suitable integers p such that

$$|q_1\alpha_1 + q_2\alpha_2 + \cdots + q_n\alpha_n - p| < \frac{1}{\|(q_1, \ldots, q_n)\|^{n(1+\epsilon)}}.$$

(Here $\|(q_1, \ldots, q_n)\|$ stands for "norm" which may be taken as $|q_1| + \cdots + |q_n|$.) For $n = 1$ this coincides with the notion of VWA number as above. The work of Kleinbock and Margulis [10], involving some very modern techniques, shows in particular the following: Let $n \geq 1$ and $f_1, \ldots, f_n$ be polynomials in one variable, say t, such that for any real numbers $a_0, a_1 \ldots, a_n$, not all 0, $a_0 + a_1 f_1 + \cdots + a_n f_n$ is a nonzero polynomial (i.e. 1, $f_1, \ldots, f_n$ are linearly independent polynomials). Then for "almost all" t the n-tuple $(f_1(t), \ldots, f_n(t))$ is not VWA. The work established a longstanding conjecture of Sprindzuk, which in turn was inspired by an earlier conjecture of Mahler, from 1932, in the case $f_i(t) = t^i$ as above.

We now come to a kind of behaviour which is at the other extreme. An irrational number α is said to be *badly approximable* if there exists $\delta > 0$ such that for all rational numbers $\frac{p}{q}$ we have

$$\left|\alpha - \frac{p}{q}\right| > \frac{\delta}{q^2}.$$

It may be noticed that by Liouville's theorem recalled above quadratic irrationals, namely irrational numbers which satisfy a quadratic equation, are badly approximable. Interestingly, badly approximable numbers can be completely characterised in terms of continued fractions:

Theorem 3.2. *An irrational number $\alpha = [m_0, m_1, \ldots, m_k, \ldots]$ is badly approximable if and only if m_k's are bounded, namely there exists M such that $m_k \leq M$ for all k.*

The quadratic irrationals which form a part of badly approximable numbers consist precisely of $\alpha = [m_0, m_1, \ldots, m_k, \ldots]$ which are "eventually periodic", namely there exist positive integers p and k_0 such that $m_{k+p} = m_k$ for all $k \geq k_0$.

Like the VWA numbers the badly approximable numbers also form a set of Lebesgue measure 0. It was however proved by Jarnik that they nevertheless form a large collection in terms of "Hausdorff dimension." We will not go into the definition of Hausdorff dimension here, but content myself to say that it is a nonnegative number, which need not be an integer, which captures a sense of how large a set (strictly speaking a metric space) is. Jarnik's theorem asserts that the Hausdorff dimension of the set of badly approximable numbers is 1, same as that of the number line itself, even though in general for subsets of the interval the number can be smaller than one.

Another sense in which the set is large was introduced by W. M. Schmidt. It depends on an idea of a two-person game and it would be my pleasure to recall it here, together with some results on its significance.

Consider the n-dimensional Euclidean space $\mathbb{R}^n$, for some $n \geq 1$, with the usual distance. (The game may be thought of on any complete metric space as well.) $\mathcal{A}$ and $\mathcal{B}$ are two players, and they are respectively assigned two numbers α and β (strictly) between 0 and 1. A sample procedure for the game proceeds as follows: $\mathcal{B}$ chooses a closed ball in $\mathbb{R}^n$, say B_0, with positive radius, say r_0. Then $\mathcal{A}$ chooses a closed ball, say A_1 of radius $r_1 = \alpha r_0$, contained in B_0. Then it is the turn of $\mathcal{B}$ again and he is to choose a closed ball B_1 of radius βr_1 contained in A_1. The game continues in this way with $\mathcal{A}$ and $\mathcal{B}$ taking turns in making choices: after $k \geq 1$ iterates $\mathcal{B}$ would have chosen a closed ball B_k of radius $\alpha^k \beta^k r_0$ contained in A_{k-1} and then $\mathcal{A}$ will choose a closed ball of radius $\alpha^{k+1}\beta^k r_0$ contained in B_k. As the radii of $\{A_k\}$ and $\{B_k\}$ are decreasing to 0 and the balls are contained within each other as above, it follows that there is a unique point of intersection, viz. $\cap_1^\infty A_k = \cap_1^\infty B_k = \{v\}$, with $v \in \mathbb{R}^n$. A subset S of $\mathbb{R}^n$ is preassigned, and the player $\mathcal{A}$ will be considered the winner if $v \in S$ and $\mathcal{B}$ will be considered the winner if $v \notin S$ (notice though that it involves infinitely many steps, unlike in a practical game, but the idea of winning or losing makes sense). Given the objective that the common point of intersection should be in S, $\mathcal{A}$ will try to choose the balls A_k, during his turns, so as to ensure that. On the other hand $\mathcal{B}$ will try to choose the balls B_k during his turns to avoid that happening. Now the question that concerns us is whether, given the set S, $\mathcal{A}$ has a "winning strategy", namely a way to choose the balls A_k during his turns, following the procedure as above, in such a way so that no matter what balls $\mathcal{B}$ chooses during his turns (within the rules of the game) the point of intersection will be in S (so as to be the winner). Notice that whether this is possible for the given S may also depend on the given α and β. If for S there is a winning strategy for $\mathcal{A}$ for given values α, β we say that S is an (α, β)-*winning set* (for $\mathcal{A}$ - we will consistently suppress this part in the discussion below). For certain α, β there may be no proper subset of $\mathbb{R}^n$ which is a winning set; indeed this is the case if $1-2\alpha+\alpha\beta \leq 0$, as can be readily proved. In particular if $\alpha > \frac{1}{2}$ then for sufficiently small $\beta > 0$ there is no proper subset which is an (α, β)-winning set.

It stands to reason that the winning sets (for any α, β) have to be "large sets". Firstly S has to be dense in $\mathbb{R}^n$, since otherwise $\mathcal{B}$ can choose B_0 to be outside S, in which case $\mathcal{A}$ can not win. Also it can be seen that S has to be uncountable since otherwise $\mathcal{B}$ can force the points of S out of the chosen balls one by one (in finitely many steps in each case). The "largeness" of the winning sets however goes well beyond these simple manifestations. Schmidt showed that the Hausdorff dimension of an (α, β)-winning set in $\mathbb{R}^n$ is at least $(c - n\log\beta)/|\log\alpha\beta|$, where c is a constant depending only on n. Let us call a subset S a winning set if it is a $(\frac{1}{2}, \beta)$-winning set for all $\beta > 0$ (and hence (α, β)-winning set for all $0 < \alpha \leq \beta$ and $\beta > 0$). The Hausdorff dimension of any winning set is n, the maximum possible for a subset of $\mathbb{R}^n$. There is another curious fact about the winning sets which reflects their largeness: intersection of any two winning set is also a winning set, and more strongly intersection of any sequence of winning sets is also a winning set.

Schmidt proved that the set of badly approximable numbers is a winning set in real numbers [12]. Let me now conclude with a generalisation of this that I proved in higher dimensions; see [4] and [5]; the result was inspired by certain questions in dynamics of certain flows and the geometry of certain manifolds of negative curvature, which however are beyond the scope of this article.

Theorem 3.3. *Let $\{v_k\}$ be a sequence of points in $\mathbb{R}^n$, the n-dimensional Euclidean space. Let $\{r_k\}$ be a sequence of positive numbers such that for any distinct k and l the distance between v_k and v_l is at least $\sqrt{r_k r_l}$. Let S be the set of points v such that for some $\delta > 0$, v not contained in $B(v_k, \delta r_k)$, viz. the ball of radius δr_k with centre at v_k, for any k. Then S is a winning set in $\mathbb{R}^n$.*

When $n = 1$, $\{v_k\}$ is a sequence enumerating the rationals, and $r_k = \frac{1}{q^2}$ if $v_k = \frac{p}{q}$, then the set as in the theorem consists precisely of badly approximable numbers.

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On Thin Subsets of the Real Line

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Abstract. A subset A of $\mathbb{R}^n$ is called thin if it has zero Lebesgue measure, is nowhere dense and has Hausdorff dimension equal to n. Examples of thin subsets of $\mathbb{R}^n$ are well-known when $n \geq 2$. In this note, we construct an example of a thin subset of $\mathbb{R}$.

1. Introduction

Hausdorff measures and Hausdorff dimension play a significant role in Geometric Measure Theory. For instance, it is possible to distinguish between two measure zero sets by their Hausdorff dimension; and, the critical Hausdorff measures can further distinguish between sets of same Hausdorff dimension.

In practice, the actual computation of the Hausdorff dimension and Hausdorff measure of a set can be quite involved and it would be of interest to find some general methods. In [13], W. M. Schmidt introduced the notion of (α, β)-games and of α-winning sets in a complete metric space and showed that α-winning sets in $\mathbb{R}^n$ have maximal Hausdorff dimension. He used this method to prove many interesting results and, in particular, he showed that the set of badly approximable numbers in the real line is an α-winning set implying an older result of V. Jarnik that the set of badly approximable numbers has Hausdorff dimension equal to 1. Schmidt game approach has proved to be very useful in certain geometric situations also and has been employed, for instance in [2], [5], to show the maximality of Hausdorff dimension of certain exceptional sets of ergodic systems.

However, it is not true in general that sets of maximal Hausdorff dimension in $\mathbb{R}^n$ should necessarily also be α-winning. Examples to the contrary are easy to come by. We quickly recall Schmidt's (α, β)-game for the sake of completeness.

Let X be a complete metric space and let $\alpha, \beta \in (0, 1)$. The (α, β)-game on X is played by two players $\mathcal{A}$ and $\mathcal{B}$ as follows: $\mathcal{B}$ first chooses a closed ball B_0 in X of radius r where $r \in \mathbb{R}$ is positive. Then $\mathcal{A}$ chooses a closed ball $A_1 \subset B_0$ of radius αr. Then $\mathcal{B}$ chooses a closed ball $B_1 \subset A_1$ of radius $r_1 := \beta\alpha r$. Inductively, after $\mathcal{B}$ has chosen a closed ball B_k of radius $r_k = (\alpha\beta)^k r$, $k \geq 1$, $\mathcal{A}$ chooses a closed ball $A_{k+1} \subset B_k$ of radius αr_k and $\mathcal{B}$ chooses a closed ball B_{k+1} of radius $r_{k+1} = \beta\alpha r_k$. Since $\lim_{k\to\infty} r_k = 0$, and since X is a complete metric space, $\cap_{k\geq 1} A_k$ is a singleton set $\{x\}$. A set $S \subseteq X$ is said to be an *(α, β)-winning set* (for $\mathcal{A}$) if for any choices of $\mathcal{B}$, $\mathcal{A}$ can always make choices so that $x \in S$. We say that S is *α-winning* if it is (α, β)-winning for every $\beta \in (0, 1)$.

Clearly, the whole set X is itself α-winning for any $\alpha \in (0, 1)$. Schmidt showed that if $1 - 2\alpha + \alpha\beta \leq 0$, then the only (α, β)-winning set is X itself. However, if $1 - 2\alpha + \alpha\beta > 0$, then there can be proper subsets of X which are (α, β)-winning; as mentioned above, the set of badly approximable numbers in $\mathbb{R}$ is an example of this.

It is also clear that an α-winning set must necessarily be dense in X. Hence, if U is any proper open subset of $\mathbb{R}^n$ that is not dense in $\mathbb{R}^n$, then U can not be α-winning in $\mathbb{R}^n$ though its Hausdorff dimension is equal to n. However, if the (α, β)-games are played in the complete metric space $\overline{U}$ instead of in $\mathbb{R}^n$ then it is not hard to see that U is an α-winning set in $\overline{U}$ for some $\alpha \leq 1/2$ ([2, Proposition 3.3]). Thus, one is lead to the question of whether there exist subsets of $\mathbb{R}^n$ of full Hausdorff dimension which are not α-winning on any open set in $\mathbb{R}^n$? Or, more generally, one can ask the following question:

Are there subsets of $\mathbb{R}^n$ having zero Lebesgue measure and full Hausdorff dimension but which aren't dense in any open subset of $\mathbb{R}^n$?

In [4], Besicovitch constructed closed subsets B of $\mathbb{R}^2$ of Lebesgue measure zero which contain unit length line segments in every direction. Subsets of $\mathbb{R}^2$ containing line segments in every direction are known to have full Hausdorff dimension [7, Theorem 7.9]. In particular, being closed and having zero Lebesgue measure, Besicovitch sets are nowhere dense and hence provide examples of thin sets sought after in the above question. And, when $n > 2$, subsets $B \times \mathbb{D}^{n-2} \subset \mathbb{R}^n$, where $\mathbb{D}^{n-2}$ is the unit ball in $\mathbb{R}^{n-2}$, provide examples of thin sets in these dimensions.

Since Besicovitch's construction seems special to dimension 2, it would be interesting to know if there exist such subsets of $\mathbb{R}$. Call a subset A of $\mathbb{R}^n$ *thin* if it has zero Lebesgue measure, is nowhere dense and has Hausdorff dimension equal to n. If A is a thin subset of $\mathbb{R}$ then $A \times \mathbb{D}^{n-1}$ provides a new example of a thin subset of $\mathbb{R}^n$. Newer methods of constructing Besicovitch sets have been discussed by others (for instance, see [1], [7], [7]) but, to the best of our knowledge, the problem of finding thin sets in $\mathbb{R}$ doesn't seem to have been discussed; in fact, the notion of thin sets was introduced in [3] where this problem was mentioned. The purpose of this note is to show the existence of such sets and describe one construction.

2. Thin Sets in $\mathbb{R}$

For each $k \geq 3$, consider the Cantor type set obtained as follows: First divide the interval $[0, 1]$ into k equal subintervals. Remove the second open subinterval. Each of the remaining $k - 1$ subintervals are again divided into k equal subintervals and the second of these open subintervals are removed. Repeat this procedure infinitely many times. Let C_k denote the set that remains at the end of this process. Observe that if $k = 3$ then C_k is the usual Cantor's ternary set. A similar argument as for the Cantor's ternary set shows that C_k is closed with $\mu(C_k) = 0$ where μ denotes the Lebesgue measure on $\mathbb{R}$. It can also be shown, by standard arguments, that the Hausdorff dimension of C_k is equal to $\frac{\log k}{\log(k+1)}$. However, we sketch a proof of it below for the convenience of the reader.

Let $d_k = \frac{\log k}{\log(k+1)}$. We will show that $\mathcal{H}^{d_k}(C_k) = 1$. Since the set C_k can be covered by $(k-1)^j$ intervals of length k^{-j} that occur at the jth stage in the construction of C_k, we can conclude that $\mathcal{H}^{d_k}_{k^{-j}}(C_k) \leq (k-1)^j k^{-d_k j} = 1$. Letting $j \to \infty$ we get $\mathcal{H}^{d_k}(C_k) \leq 1$.

On the other hand, we will show that for any cover $\{I_m\}$ of C_k by intervals with $|I_m| < \delta < \frac{1}{2k}$, $\sum |I_m|^{d_k} \geq 1$ (here, $|I_m|$ denotes the length of the interval I_m). By adding a length of $\frac{\epsilon}{2^m}$ to I_m for an arbitrarily small ϵ, if necessary, we can assume all the I_m's are open. Since C_k is compact, there is a finite subcover, say, $I_j = (a_j, b_j)$, $j = 1, 2, \ldots, n$. Without loss of generality we can assume $a_1 < a_2 < \cdots < a_n$ and $b_1 < b_2 < \cdots < b_n$. Choose the smallest integer i_1 such that $a_{i_1} < \frac{1}{k} < b_{i_1}$. Then $\sum_{j=1}^{i_1}(b_j - a_j)^{d_k} \geq (\frac{1}{k})^{d_k}$. Next choose the largest integer i_2 such that $a_{i_2} < \frac{2}{k} < b_{i_2}$ and smallest integer i_3 such that $a_{i_3} < \frac{3}{k} < b_{i_3}$. Then $\sum_{j=i_2}^{i_3}(b_j - a_j)^{d_k} \geq (\frac{1}{k})^{d_k}$. Note that $b_{i_3} - \frac{3}{k} < \delta$. Also since $\delta < \frac{1}{2k}$, we have $i_1 < i_2 < i_3$. Next choose the greatest integer i_4 such that $a_{i_4} < b_{i_3} < b_{i_4}$ and the least integer i_5 such that $a_{i_5} < \frac{4}{k} < b_{i_5}$. Then $\sum_{j=i_4}^{i_5}(b_j - a_j)^{d_k} \geq (\frac{1}{k} - \delta)^{d_k}$. Continuing this till we cover 1 we get $\sum_{j=1}^{n}(b_j - a_j)^{d_k} \geq (\frac{1}{k})^{d_k} + (\frac{1}{k})^{d_k} + (\frac{1}{k} - \delta)^{d_k} + \cdots + (\frac{1}{k} - \delta)^{d_k}$ where there are $(k-3)$ terms of $(\frac{1}{k} - \delta)^{d_k}$. Letting $\delta \longrightarrow 0$ we get $H^{d_k}(C_k) \geq \frac{k-1}{k^{d_k}} = 1$.

Now, set $D_k = \frac{1}{2^{k-2}} C_k + \frac{1}{2^{k-2}}$ for each $k \geq 3$ and let $D = \cup_{k=3}^{\infty} D_k \cup \{0\}$. Observe that $D_k \subset \left[\frac{1}{2^{k-2}}, \frac{1}{2^{k-3}}\right]$. We will show that D is thin. Firstly, $\mu(D) = \sum_{k=3}^{\infty} \mu(D_k) = 0$ since $\mu(C_k) = 0$ for each $k \geq 3$. And, since D contains D_k for $k \geq 3$, the Hausdorff dimension of D must be greater than or equal to $\frac{\log k}{\log(k+1)}$ for every k and, therefore, is equal to 1. We are left with showing that D is a closed set. To show this, let $\{x_j\}$ be a sequence in D which converges to $x \in [0, 1]$. If $x = \frac{1}{2^k}$ for some k or $x = 0$, then clearly $x \in D$. Otherwise $x \in \left(\frac{1}{2^k}, \frac{1}{2^{k-1}}\right)$ for some $k \geq 1$. Hence $x_j \in D_{k+2}$ for all large j. But since each D_k is closed we must have $x \in D_{k+2} \subset D$. This completes the proof that D is thin.

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Random Walks

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Abstract. We discuss the classical theorem of Pólya on random walks on the integer lattice in Euclidean space. This is the starting point for much work that has been done on random walks in other settings. We mention a tiny fraction of this work, and discuss in detail "random walks" which can be created using trigonometric functions.

1. Pólya's Theorem

Consider the integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$. Suppose you are standing at 0. Flip a coin. If the coin comes up heads, move to the right by one step. If it comes up tails, move to the left by one step. Flip the coin again. If it comes up heads, move a step to the right, if it comes up tails, move a step to the left. Repeat and continue this process. Must you come back to zero? Obviously, there is a very good chance of returning to zero: the first two flips could have been head then tail, or tail then head, and these both result in a return to zero. So of the four possible outcomes in the first two flips, half of these take you back to zero. So the probability of returning to zero is at least $\frac{1}{2}$. If you further consider that even if you fail to return to zero in two steps, that after some subsequent flips you might make it back to zero, then clearly the probability of returning to zero is greater than $\frac{1}{2}$. A natural question arises: What is the probability of returning to zero? The answer was given by Georg Pólya [9] in 1921:

Figure 1. Georg Pólya

Theorem 1. *With probability one, the random walker will return to zero in a finite number of steps.*

Let us show how to prove Pólya's theorem. For $n = 0, 1, 2, 3, \ldots$ let $u_n =$ the probability that the random walker is at 0 after n steps. We make two simple observations: $u_0 = 1$ (because the walker starts at 0) and $u_n = 0$ if $n = 1, 3, 5, 7, \ldots$, that is, the walker cannot be at 0 after an odd number of steps.

For $n = 1, 2, 3, \ldots$, set f_n= the probability that the walker returns to 0 for *the first time* at step n. Set $f_0 = 0$ (being at 0 at step 0 isn't really a *return*) and $f = \sum_{n=0}^{\infty} f_n$. The probability that the walker returns to 0 is f, the probability of not returning is $1 - f$.

We now decompose the event that the walker is at 0 at time n, $n \geq 1$, into first returns, that is, if the walker is at 0 at time n, then the walker may be back there for the first time or may have already visited there previously. This gives the relations:

$$
\begin{aligned}
u_1 &= f_0u_1 + f_1u_0 \\
u_2 &= f_0u_2 + f_1u_1 + f_2u_0 \\
&\vdots \\
u_n &= f_0u_n + f_1u_{n-1} + f_2u_{n-2} + \cdots + f_ku_{n-k} + \cdots + f_nu_0.
\end{aligned}
$$

For example, in the expression for u_n, the term f_ku_{n-k} represents the probability that the walker returned to 0 for the first time after k steps and then returned to 0 in $n-k$ more steps. Set

$$
\begin{aligned}
U(s) &= u_0 + u_1s + u_2s^2 + \ldots \\
\text{and} \qquad F(s) &= f_0 + f_1s + f_2s^2 + \ldots
\end{aligned}
$$

Then

$$
\begin{aligned}
U(s)F(s) = u_0f_0 + (u_0f_1 + u_1f_0)s \\
+ (f_0u_2 + f_1u_1 + f_2u_0)s^2 + \cdots = U(s) - 1.
\end{aligned}
$$

Lemma 1. *If* $\sum_{n=0}^{\infty} u_n = \infty$ *then the probability of returning to 0 is one.*

If $\sum_{n=0}^{\infty} u_n < \infty$ *then there is a positive probability of not returning to 0.*

Proof. Notice that for any N, $\sum_{n=0}^{N} u_n \leq \lim_{s\nearrow 1} U(s)$ so if $\sum_{n=0}^{\infty} u_n = \infty$, then $\lim_{s\nearrow 1} U(s) = \infty$ and given the relation $U(s)(1-F(s)) = 1$ this forces $f = \sum_{n=0}^{\infty} f_n = \lim_{s\nearrow 1} F(s) = 1$.

On the other hand, if $\sum_{n=0}^{\infty} u_n < \infty$, then $\lim_{s\nearrow 1} U(s) = \sum_{n=0}^{\infty} u_n < \infty$, which, due to the relation $U(s)(1 - F(s)) = 1$ forces $\sum_{n=0}^{\infty} f_n = \lim_{s\nearrow 1} F(s) < 1$.

Thus, to determine if our walker returns, we need to compute $\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} u_{2n}$. (Recall $u_n = 0$ if n is odd.) If this sum is infinite, the walker returns with probability one, if the sum is finite, there is a positive probability of the walker not returning.

Let's compute u_{2n}. In the first $2n$ steps, there are a total of 2^{2n} different routes which could be taken, each equally likely. How many of these come back to 0? This is a basic combinatorics question. Think of two urns, one labeled left and one labeled right. We want to distribute $2n$ balls, labeled $1, 2, 3, \ldots, 2n$ (corresponding to the step number) into these two urns. There are $\binom{2n}{k}$ ways to distribute k into one urn and $2n - k$ into the other; of interest here is the number of ways to put n balls into the urn labeled "left" and n into the urn labeled "right". Any such distribution of balls corresponds to a route of length $2n$ which comes back to 0. Thus, there are $\binom{2n}{n}$ routes which lead back to 0. Consequently,

$$u_{2n} = \frac{\binom{2n}{n}}{2^{2n}}$$

To get an estimate of u_{2n} we need

Lemma 2. (Stirling's formula; see e.g. Feller [7]). *As $n \to \infty$, $n! \approx \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}$.*

With this estimate, when n is large,

$$u_{2n} = \frac{(2n)!}{(n!)^2} \frac{1}{2^{2n}} \approx \frac{\sqrt{2\pi}(2n)^{2n+\frac{1}{2}} e^{-2n}}{(\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n})^2} \frac{1}{2^{2n}} = \frac{1}{\sqrt{\pi n}}$$

Therefore, $\sum_{n=0}^{\infty} u_{2n} = \infty$ so by Lemma 1, the walker returns with probability one.

2. Many Possible Directions

Pólya [9] also considered random walks in higher dimensions. Consider the integer lattice in two dimensional Euclidean space, that is, consider those points with integer coordinates. Start at $(0, 0)$, and toss two coins to determine whether to make a step to either $(1, 0)$, $(-1, 0)$, $(0, 1)$ or $(0, -1)$. Repeat, going one more step, either up, down, left or right, with equal probability. The same question can now be asked: What is the probability this walker returns to $(0, 0)$?

An examination of the proof given in the previous section reveals that the analysis up through Lemma 1 was in no way dependent on the fact that walker was walking in one dimension; this fact only entered into the proof with the computation of u_{2n}. With only slightly more complicated combinatorics, and again using Stirling's formula, one can compute that in dimension two, $u_{2n} \approx \frac{c}{n}$, as $n \to \infty$, for some constant c. Since then $\sum_{n=0}^{\infty} u_{2n} = \infty$, Lemma 1 implies that the walker returns with probability one.

The situation changes in dimension three. Here, after some combinatorics and computation, it turns out that $u_{2n} \approx Cn^{-\frac{3}{2}}$ so that by Lemma 1, there is a positive probability of the walker not returning to the origin. Pólya also showed that this is the case in any dimension greater than or equal to three.

A very drunken man staggers so as to simulate a random walk. Thus, Pólya's theorem implies that if he staggers out of a bar into a very narrow long alley, he will certainly return to the door of the bar. Similarly, if he finds himself in the middle of a large plaza, he will eventually stagger and return, with probability one, to the center of the plaza. Birds, however, should be cautioned against drunken behavior, as a drunken bird has a positive probability of not returning to the nest.

The walks we've discussed above are called symmetric random walks because at each step, each of the possible directions to go was equally likely. Going back to dimension one, let's consider the situation where we use a weighted coin to determine the direction of travel. This coin has probability p of turning up heads (so that the walker goes right) and probability $1 - p$ of turning up tails. To determine if a walker using this coin returns to 0, we reason as before and compute u_{2n}. As before, a random walk returns to zero if there are as many steps to the left as to the right, so that $u_{2n} = p^n(1-p)^n \binom{2n}{n}$. Here again we may use Stirling's formula to obtain:

$$u_{2n} = p^n(1-p)^n \frac{(2n)!}{(n!)^2} \approx p^n(1-p)^n \frac{\sqrt{2\pi}(2n)^{2n+\frac{1}{2}} e^{-2n}}{(\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n})^2}$$

$$= (4p(1-p))^n \frac{1}{\sqrt{\pi n}}.$$

If $p \neq \frac{1}{2}$ then $4p(1-p) < 1$, hence $\sum_{n=0}^{\infty} u_{2n} < \infty$ and by Lemma 1, there is a positive probability of the walker not returning to 0. In fact, $4p(1-p) < 1$ exactly when $p \neq \frac{1}{2}$ and so $\sum_{n=0}^{\infty} u_{2n}$ converges if and only if $p \neq \frac{1}{2}$. Thus, the walker returns to the origin with probability one if and only if $p = \frac{1}{2}$. This is not surprising– a gambler who continually plays a game with the odds against him can expect to eventually lose his initial stake.

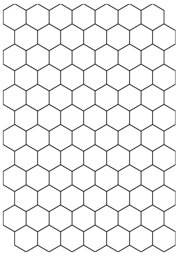

Figure 2. Hexagonal lattice

Other lattices may be considered. Our walker could start at the vertex of a hexagonal lattice in $\mathbb{R}^2$. At each step there are three possible directions to go; we could consider a walk where each of these directions is equally likely (it can be shown the walker will return) or a walk where these are not. In fact, we could consider a walk on any lattice, regular or not, in Euclidean space, and ask the same questions. See P. G. Doyle and J. L. Snell [5] for a detailed study of random walks on lattices in $\mathbb{R}^n$ as well as an excellent exposition of the connection between random walks and the theory of electrical networks.

Another possibility is to consider a sequence of identically distributed independent random variables $X_1, X_2, X_3, \ldots$. Setting $S_1 = X_1$, $S_2 = X_1 + X_2$, $S_3 = X_1 + X_2 + X_3, \ldots$ we may ask about $\{S_n(\omega)\}$ for points ω in the underlying probability space. Here the X_i may take on discrete values or may take values in $\mathbb{R}$. In the first case, this allows, say, for walks on an integer lattice in which the walker may jump several steps in a single move. In the case where the X_i take values in $\mathbb{R}$ the walker may never get back *exactly* to 0, but we can ask if he gets arbitrarily close. See K. L. Chung [4] for a treatment of random walks in this generality.

Consider a group G with a subset S which generates G. Assume S has the property that $s \in S$ implies $s^{-1} \in S$. The Cayley graph of G is constructed with vertices the elements of G and an edge from x to y if and only if there exists an element $s \in S$ such that $x = sy$. Thus, the Cayley graph depends on G and the generating set S (which is not unique). A walker starts from a vertex of this graph, and at each stage, takes a step along an adjacent edge, chosen uniformly from among the adjacent edges with equal probability. What is the probability the walker returns to the starting point?

3. Return to Pólya's Theorem; Trigonometric Walks

Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}) \\ -1 & \text{if } x \in [\frac{1}{2}, 1) \end{cases}$$

Extend this to a 1-periodic function on the real line, and continue to call this extension f. Now consider the functions $f(x)$, $f(2x)$, $f(4x), \ldots, f(2^n x), \ldots$, restrict these to the interval $[0, 1)$ and call the resulting functions $r_1(x), r_2(x), r_3(x), \ldots, r_n(x), \ldots$. These are called the Rademacher functions, after Hans Rademacher who introduced these in a paper in 1922. With the possible exception of x which are dyadic points, that is, points of the form $x = \frac{j}{2^n}$, where $n = 1, 2, 3, \ldots$, $j \in \{0, 1, \ldots, 2^n\}$ we may write $r_n(x) = \text{sgn}(\sin(2^n \pi x))$ where sgn is defined as $\text{sgn}(x) = 1$ if $x \geq 0$ and $\text{sgn}(x) = -1$ if $x < 0$. (The set of dyadic points is countable, hence of measure zero, and therefore negligible for our purposes. So we could define the r_n either way and obtain the same results.)

Set $s_0(x) = 0$, $s_1(x) = r_1(x)$, $s_2(x) = r_1(x) + r_2(x)$, $s_3(x) = r_1(x) + r_2(x) + r_3(x)$, etc. Take $x \in [0, 1)$. If $x \in [0, \frac{1}{2})$ then $s_1(x) = 1$, if $x \in [\frac{1}{2}, 1)$, then $s_1(x) = -1$. So for half the x's in $[0, 1)$, $s_1(x)$ takes the value 1, and for half it takes the value -1. Now if $x \in [0, \frac{1}{4})$, then $s_2(x) = r_1(x) + r_2(x) = 1 + 1 = 2$; if $x \in [\frac{1}{4}, \frac{1}{2})$ then $s_2(x) = r_1(x) + r_2(x) = 1 + -1 = 0$; if $x \in [\frac{1}{2}, \frac{3}{4})$, then $s_2(x) = r_1(x) + r_2(x) = -1 + 1 = 0$; if $x \in [\frac{3}{4}, 1)$ then $s_2(x) = r_1(x) + r_2(x) = -1 + -1 = -2$. Each of these four cases, occurring on exactly one-fourth of the unit interval, corresponds to the first two steps of a random walk. Continuing in this way, we have created a model of Pólya's random walk on the integers: There is a 1-1 correspondence between $\{x \in [0, 1) | x \text{ is not of the form } \frac{j}{2^n}\}$ and the set of random walks, where x corresponds to the random walk $\{s_n(x)\}_{n=1}^{\infty}$. In fact, if you express such an $x \in [0, 1)$ in binary and change all

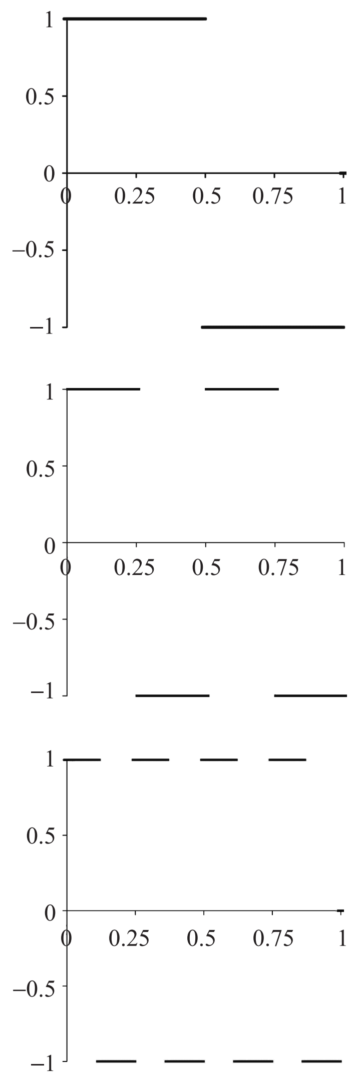

Figure 3. The functions $r_1(x)$, $r_2(x)$ and $r_3(x)$.

the $1's$ in the binary representation to $-1's$ and all the $0's$ to $1's$, then $s_n(x)$ is just the sum of the first n digits of this string of $-1's$ and $1's$.

Thinking in this way, we can rephrase Pólya's theorem:

Theorem 2. *For a.e.* $x \in [0, 1)$, *the sequence* $s_n(x), n = 1, 2, 3, \ldots$ *returns to* 0 *in a finite number of steps.*

Actually, a little more can be said. Given any integer m, it can be shown that with probability one, the walker will land on m in a finite number of steps. Thinking now of m as a starting point, then with probability one, the walker will return to this position in a finite number of steps. Once the walker has returned, there is no memory of the past, so now the situation is as if the walker is starting for the first time. So with probability one, the walker returns again to m. This continues and consequently, we may conclude:

Theorem 3. *For a.e.* $x \in [0, 1)$, *the sequence* $s_n(x), n = 1, 2, 3, \ldots$ *visits every integer an infinite number of times.*

Now consider the interval $[-\pi, \pi]$ and the functions $\sin(x)$, $\sin(2x)$, $\sin(4x)$, $\ldots$, $\sin(2^n x)$, $\ldots$ and the sums $s_1(x) = \sin(x)$, $s_2(x) = \sin(x) + \sin(2x)$, $s_3(x) = \sin(x) + \sin(2x) + \sin(4x)$, $\ldots$, $s_n(x) = \sum_{j=1}^{n} \sin(2^{j-1}x)$, $\ldots$.

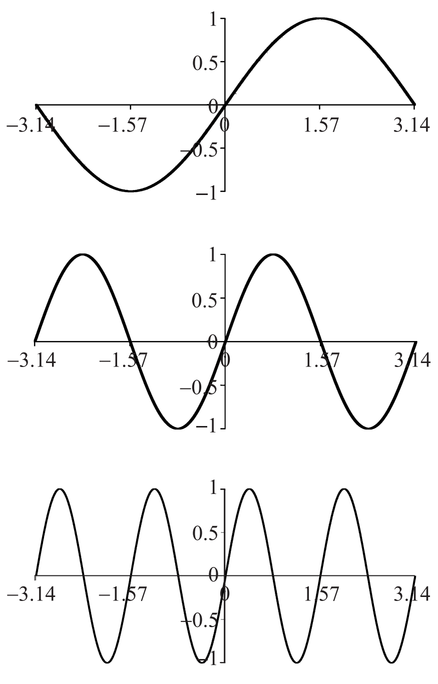

Figure 4. The functions $\sin(x)$, $\sin(2x)$, and $\sin(4x)$ on $[-\pi, \pi]$.

Given the resemblance of each of the functions $\sin(2^n x)$ to $r_n(x)$, it seems natural to investigate the "random walks" $\{s_n(x)\}_{n=1}^{\infty}$ for $x \in [-\pi, \pi]$. Here of course, sums of the $r_n(x)$ will always be integer valued, whereas sums of the $\sin(2^n x)$ certainly are not. And certainly, for any fixed x, the sequence $\{s_n(x)\}$ is countable, so it could not visit every element in an uncountable set such as the real numbers. The correct analogue to Theorem 3 is that it visits every neighborhood of every point an infinite number of times, or what is equivalent, the sequence $\{s_n(x)\}$ is dense in $\mathbb{R}$. In fact, we can be a little more general:

Theorem 4. (Grubb and Moore [8].) *Let*

$$s_n(x) = \sum_{j=1}^{n} a_j \sin(n_j x + \theta_j)$$

where the a_j, θ_j, *are real,* $|a_j| \leq 1$ *for every* j *and the* n_j *are positive integers which satisfy* $\frac{n_{j+1}}{n_j} \geq \lambda > 1$. *Suppose also that* $\sum |a_j|^2 = \infty$. *Then for a.e.* $x \in [-\pi, \pi]$, $\{s_n(x)\}$ *is dense in* $\mathbb{R}$.

It is easy to verify that the sequence $s_1(x) = \sin(x)$, $s_2(x) = \sin(x) + \sin(2x)$, $\ldots$ discussed above satisfies the hypotheses of this theorem. Note that we must have $\sum |a_j|^2 = \infty$; otherwise, the sequence of functions $\{s_n(x)\}$ converges in L^2 and couldn't satisfy the conclusion. Previously, D. Ullrich [14] had shown this result under the assumption that $|a_j| = 1$ for every j and the proof of Theorem 4 (which we describe below) borrows many ideas from his work.

Series of the form $\sum_{j=0}^{\infty} a_j z^{n_j}$ where $n_{j+1}/n_j \geq \lambda > 1$, or their real counterparts such as in Theorem 4, are called lacunary series, gap series, or often Hadamard series. They arise in the study of analytic continuation: for example, the series $\sum_{j=0}^{\infty} z^{2^j}$ represents an analytic function on the unit disk which cannot be extended to an analytic function on any larger domain. They are also of interest due to the fact, evidenced by this theorem among many others, that the partial sums $\sum_{j=0}^{n} a_j z^{n_j}$ behave like sums of independent random variables (which they are not). In particular, we mention here the central limit theorems for lacunary trigonometric series of R. Salem and A. Zygmund [11], the central limit theorem of P. Erdös and I. S. Gál [6] and the laws of the iterated logarithm of Salem and Zygmund [12] and M. Weiss [15].

The proof of Theorem 4 is not difficult. A theorem due to Zygmund [16], vol. 1, p. 205 shows that the set of x where the sequence $\{s_n(x)\}$ is bounded above or below is of measure zero. Thus, fixing an $\alpha \in \mathbb{R}$ and an x at which $\{s_n(x)\}$

bounded neither above nor below, then for an infinite number of n (which depend on x) we must have $s_n(x)$ on one side of the real number α and $s_{n+1}(x)$ on the other; that is, in the real line the sequence $s_n(x)$ crosses α an infinite number of times. With a little more work it can then be shown that $s_n(x)$ must visit any neighborhood of α an infinite number of times. See [8] for details.

What about such "trigonometric random walks" in higher dimensions? In analogy to Pólya's theorem in two dimensions, we could consider

$$s_n(\theta) = \sum_{k=1}^{n} a_k e^{in_k\theta}$$

where, as above, $\frac{n_{k+1}}{n_k} \geq \lambda > 1$, $a_k \in \mathbb{C}$, and $\theta \in \mathbb{R}$. (If each n_k is an integer, each $s_n(\theta)$ is 2π-periodic, so it suffices to consider $\theta \in [-\pi, \pi]$.) Fix a θ. Each term, $a_k e^{in_k\theta}$ can be thought of as a vector, so that $s_1(\theta) = a_1 e^{in_1\theta}$ represents one step of a random walk in $\mathbb{C}$, $s_2(x) = a_1 e^{in_1\theta} + a_2 e^{in_2\theta}$ represents two steps of a random walk, etc. Two important results concerning such random walks are due to J. M. Anderson and L. Pitt [1], [2].

To discuss these we make some definitions. Let $\varepsilon > 0$ be fixed, $\theta \in \mathbb{R}$ and suppose that for each $z \in \mathbb{C}$

$$\lim_{n\to\infty} \inf |s_n(\theta) - z| \leq \varepsilon.$$

In this case we say $\{s_n(\theta)\}$ is ε-recurrent. If $\{s_n(\theta)\}$ is ε-recurrent for almost all θ then we say $\{s_n\}$ is ε-recurrent. If $\{s_n\}$ is ε-recurrent for each $\varepsilon > 0$ we say that $\{s_n\}$ is recurrent.

Theorem 5. (Anderson and Pitt [1].) *Suppose $\{\lambda_k\}_1^\infty$ is a sequence of positive numbers such that $\frac{\lambda_{k+1}}{\lambda_k} \geq q > 1$ for all k. Suppose $\{a_k\}_{k=1}^\infty$ is a sequence of complex numbers satisfying $\|\{a_k\}\|_\infty \equiv \sup_k |a_k| < \infty$ and $\sum_{k=1}^{\infty} |a_k|^2 = \infty$. Set $s_n(\theta) = \sum_{k=1}^{n} a_k e^{i\lambda_k\theta}$. Then for $\varepsilon \geq \frac{2q}{q-1}\|\{a_k\}\|_\infty$ the sequence $\{s_n\}$ is ε-recurrent.*

The proof uses tools from complex analysis. For certain types of sums, more can be said:

Theorem 6. (Anderson and Pitt [2].) *Let $s_n(\theta) = \sum_{k=1}^{n} e^{ia^k\theta}$ where $a \geq 2$ is an integer. Then $\{s_n\}$ is recurrent.*

The proof of this is difficult and requires complex analysis, probability and number theory.

None of the theorems 4, 5, 6 can be shown to be best possible, most likely because they probably are not the best possible. In [3] J. Bretagnolle and D. Dacunha–Castelle present an extensive study of random walks created from sums of independent random variables. Consider partial sums $s_n(\omega) = \sum_{k=1}^{n} a_k X_k(\omega)$ where the X_k are real-valued independent identically distributed mean zero random variables. The hypotheses of their results are too numerous to mention explicitly, but they make several assumptions on the distribution of X_k and several technical assumptions of the a_k. Then, under these assumptions, and if $\sigma_n = \left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}} \to \infty$, they show recurrence occurs precisely when $\sum_{n=1}^{\infty} \frac{1}{\sigma_n} = \infty$. (Of course, one needs to be precise about what recurrence means here: if the X_k are integer valued and the a_k are integers, then recurrence could only occur in the integers; if the X_k are real valued then recurrence could be in the reals or in some other subgroup of the reals.) Here's a very rough sketch of the proof: By using the central limit theorem, they obtain the estimate $\text{Prob}(s_n \in I) = \frac{c_I}{\sigma_n} + o(\frac{1}{\sigma_n})$, where $I \subset \mathbb{R}$ is any interval, and c_I is a constant depending only on I. The proof is completed by the Borel-Cantelli lemma (actually variations of this) and other fairly standard techniques.

Reasoning in this way gives an idea as to what the correct version of Theorems 4, 5 and 6 should be in the trigonometric case, but it won't give us a proof. Sequences of functions such as $\{\sin(2^k x)\}$ or $\{e^{i\lambda_k\theta}\}$ are not independent random variables, yet as amply illustrated by many theorems, they do behave much like sequences of random variables. Consider, for example, series of the form $s_n(\theta) = \sum_{k=1}^{n} a_k \sin(2^{k-1}\theta)$, where the a_k are real. Set $\sigma_n = \left(\frac{1}{2}\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}}$ and suppose that $a_n = o(\frac{\sigma_n}{\log\log\sigma_n})$ as $n \to \infty$. Under these hypotheses the central limit theorem for lacunary trigonometric series of Salem and Zygmund [11] states that the distribution function of $\frac{s_n}{\sigma_n}$ tends to that of a Gaussian with mean zero and variance one. That is, as $n \to \infty$,

$$m\left(\left\{\theta \in [-\pi, \pi] : \frac{s_n(\theta)}{\sigma_n} < \lambda\right\}\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{\frac{-t^2}{2}} dt,$$

where m denotes the probability measure $d\theta/2\pi$ on $[-\pi, \pi]$.

So by this central limit theorem, if $\varepsilon > 0$, and n is large, then $\text{Prob}(|\frac{s_n}{\sigma_n}| < \varepsilon) \approx \frac{1}{\sqrt{2\pi}} \int_{\varepsilon}^{\varepsilon} e^{\frac{-t^2}{2}} dt \approx \frac{2\varepsilon}{\sqrt{2\pi}}$. Replace ε in this last equation by $\frac{\varepsilon}{\sigma_n}$ - this is, of course, quite incorrect, as the choice of n large depends on ε. Assuming such a step were correct, we would obtain $\text{Prob}(|s_n| < \varepsilon) \approx \frac{2}{\sqrt{2\pi}} \frac{\varepsilon}{\sigma_n}$, which is analogous to the estimate of Bretagnolle and Dacunha-Castelle above. Then,

$$\sum_{n=1}^{\infty} \text{Prob}\,(|s_n| < \varepsilon) \approx \frac{2\varepsilon}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{1}{\sigma_n},$$

so that if $\sum\limits_{n=1}^{\infty} \frac{1}{\sigma_n} < \infty$, then the so-called easy half of the Borel–Cantelli lemma (see e.g. Feller [7], Volume 1, Chapter VIII, section 3) implies that for a.e. x eventually $|s_n| > \varepsilon$, so that recurrence doesn't occur in any neighborhood of 0. Similar reasoning could then be used to show that recurrence doesn't occur at any point of $\mathbb{R}$. If, in addition, the functions $\sin(2^k x)$ were independent (which is another false assumption) then the Borel–Cantelli lemma (the so-called harder half) would imply that if $\sum\limits_{n=1}^{\infty} \frac{1}{\sigma_n} = \infty$ then a.e. x is in an infinite number of the sets $\{|s_n| < \varepsilon\}$, that is, recurrence occurs at 0. Again, similar reasoning could be applied to show that there is recurrence at any point of $\mathbb{R}$. Of course, this is all specious reasoning, but yet it seems to lead to some central issues, and gives an indication what conjectures to pose.

So in the case of trigonometric random walks, there is still much work to be done. Thus, despite the fact there are already thousands of papers and dozens of books written on random walks, there are still many directions left to go.

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Planar Harmonic and Quasiconformal Mappings

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Abstract. In Section 1 we recall some basic facts about planar harmonic mappings. In Section 2, we present harmonic analog of the classical Schwarz' lemma and its applications. In Section 3, we include the definition of quasiconfomal mappings along with some basic results and conformal modulus. In Section 4, our main aim is to discuss a lower bound for the module of the image annulus of a univalent harmonic mapping of an annulus. The result obtained by Lyzzaik, Weitsman and Kalaj each by a different method of proof is the first substantial solution for a 39-year open problem that was originally raised in l962 by J. C. C. Nitsche. The problem is actually originated from the study of minimal surfaces.

Keywords. Harmonic, Analytic and Univalent Functions; Conformal and Quasiconformal Mappings; Schwarz' Lemma, Bloch Constants and Ring Domains

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CONTENTS

1. Preliminaries on Harmonic Mappings

Let Ω be a domain in the extended complex plane $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. A complex-valued function $f = u + iv$ is harmonic in a domain Ω of $\mathbb{C}$ if and only if u and v are real-valued harmonic functions in Ω, i.e. f satisfies Laplace's equation

$$\Delta f = 4 f_{z\bar{z}} = 0 \text{ on } \Omega. \tag{1.1}$$

Here f_z and $f_{\bar{z}}$ denote the formal derivatives of f:

$$f_z = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \text{ and } f_{\bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

We do not require f to be univalent in Ω. Equation (1.1) implies that f_z is analytic and $f_{\bar{z}}$ is anti-analytic in Ω. This clearly shows that each f admits the canonical representation

$$f(z) = h(z) + \overline{g(z)}, \tag{1.2}$$

where h and g are analytic functions in Ω. For instance,

$$f(z) = z - \frac{1}{\bar{z}} + 2\ln|z|$$

is a univalent harmonic function from the exterior $|z| > 1$ of unit disk onto the punctured plane $\mathbb{C}\setminus\{0\}$, where

$$h(z) = z + \log z \text{ and } g(z) = -\frac{1}{z} + \log z.$$

We remark that, in the representation (1.2), h and g are single-valued if Ω is a simply connected domain, and possibly have multiple values which can be determined uniquely up to additive constants otherwise. On the other hand, in either case, $h'(z)$ and $g'(z)$ are single-valued analytic functions in Ω. If Ω is simply connected domain, we call h and g the *analytic* and the *co-analytic* parts of f, respectively. If f is (locally) injective, then f is called (locally) univalent. A *harmonic mapping* of Ω onto a domain Ω' is a complex-valued harmonic function f which maps Ω univalently onto Ω'. Thus, the class of complex-valued sense-preserving harmonic functions defined on a domain Ω of $\mathbb{C}$ can be regarded as a natural generalization of the class of analytic functions defined on Ω. Thus, the class of the *univalent analytic functions* (called conformal mappings) in Ω is a generalization of the class of all *sense-preserving univalent harmonic functions* (called harmonic mappings) in Ω. In particular, every conformal or anti-conformal mapping is harmonic. We note that the relation (1.2) can easily be used to find harmonic functions that are neither analytic nor anti-analytic.

In particular, we are interested in the case where Ω is the unit disk $\mathbb{D}$. Thus, one says that a harmonic mapping f of the unit disk $\mathbb{D}$ is sense-preserving if it has a positive *Jacobian*:

$$J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2 = |h'|^2 - |g'|^2 > 0 \quad \text{in } \mathbb{D}.$$

Recall that the *second complex dilatation* for f is defined as

$$\omega(z) := \frac{\overline{f_{\bar{z}}(z)}}{f_z(z)} = \frac{g'(z)}{h'(z)}.$$

Clearly, this function is analytic in $\mathbb{D}$ with $|\omega(z)| < 1$ in $\mathbb{D}$, and it has a special significance. At this place, it is important to recall a result of Hans Lewy [19] which shows that *a harmonic function f is locally univalent in a neighbourhood of a point z_0 if and only if its Jacobian $J_f(z)$ never vanishes at z_0*; for $z \in \mathbb{D}$, either $J_f(z) > 0$ or $J_f(z) < 0$ holds if f is locally univalent in $\mathbb{D}$. In the first case, $|\omega(z)| < 1$ and f is sense-preserving, and in the second case $|\omega(z)| > 1$ and f is sense-reversing. Of course, the local univalence of f does not imply the global univalence on the unit disk $\mathbb{D}$. Moreover, on every compact subset of $\mathbb{D}$, ω is bounded away from one (namely, by a constant k with $k < 1$) and so f is locally quasiconformal. A harmonic mapping in general need not be quasiconformal since its distortion may be unbounded along the boundary. However, if $|\omega(z)| < k$ in $\mathbb{D}$ for some $k \in (0, 1)$, then f is quasiconformal with maximal dilatation $K \leq (1 + k)/(1 - k)$, or simply K-quasiconformal [18]. Clearly, $\omega \equiv 0$ when f is a conformal map, and in general the second complex dilatation ω measures how far f is from being conformal. Also, we note that Lewy's result fails for harmonic functions in higher dimensions.

Now, we introduce the notation

$$\Lambda_f := |f_z(z)| + |f_{\bar{z}}(z)| \text{ and } \lambda_f := |f_z(z)| - |f_{\bar{z}}(z)|.$$

A best known effective device for producing harmonic mappings with prescribed dilatations is the "*method of shear*" introduced by Clunie and Sheil-Small [8] (see also [24] for some details).

In this article, we shall mainly cover the following topics in our preliminary discussion and also in our later articles:

(1) Harmonic analog of classical Schwarz' lemma
(2) Basic quasiconformal mappings in the plane
(3) Conjecture of J. C. C. Nitsche
(4) Bloch constants for planar harmonic mappings

2. Schwarz' Lemma and Area Distortion

The classical Schwarz' lemma is one of the cornerstones of the geometric function theory because of its base for many important metrics both in planar and higher dimensional theory. It also has a counterpart for quasiconformal maps [1,18,25,26]. Both for analytic functions and for quasiconformal mappings it has a form that is conformally invariant under conformal automorphisms of the unit disk $\mathbb{D}$. The invariance property is no longer valid for harmonic functions. Indeed if $\phi\colon \mathbb{D} \to \mathbb{D}$ is a conformal automorphism of the unit disk $\mathbb{D}$ and $f\colon \mathbb{D} \to \mathbb{D}$ is a harmonic mapping, then $\phi \circ f$ is harmonic in $\mathbb{D}$ only in rare exceptional cases!

Next we prove a harmonic counterpart of the classical Schwarz' lemma. This result is due to Heinz [13] (see also [9]).

Lemma 2.1. *Let $\Omega = \{w \in \mathbb{C} : |\mathrm{Re}\, w| < 1\}$ and $f\colon \mathbb{D} \to \Omega$ be analytic with $f(0) = 0$. Then the inequalities*

$$|\mathrm{Re}\, f(z)| \le \frac{4}{\pi} \arctan |z|, \qquad |\mathrm{Im}\, f(z)| \le \frac{2}{\pi} \log \frac{1+|z|}{1-|z|}$$

hold and are sharp for all $z \in \mathbb{D}$.

Proof. We define a conformal mapping g of $\mathbb{D}$ onto Ω by

$$g(z) = \frac{2i}{\pi} \log \frac{1+z}{1-z}.$$

Then $f(\mathbb{D}) \subset g(\mathbb{D})$, and $f(0) = g(0) = 0$. Hence, there exists an analytic function $\omega\colon \mathbb{D} \to \mathbb{D}$ with $\omega(0) = 0$ such that $f = g \circ \omega$. We obtain

$$\mathrm{Re}\, f(z) = -\frac{2}{\pi} \arg\left(\frac{1+\omega(z)}{1-\omega(z)}\right).$$

We note that the Möbius transformation defined by

$$w(z) = \frac{1+z}{1-z}$$

maps the circle $\partial\mathbb{D}_r$, $r < 1$ onto the circle $\partial\mathbb{D}(w_0; \delta)$ where

$$w_0 = \frac{1+r^2}{1-r^2} \ \text{ and } \ \delta = \frac{2r}{1-r^2}.$$

Thus,

$$\left|\arg\left(\frac{1+z}{1-z}\right)\right| \le \arctan\left(\frac{2r}{1-r^2}\right) = 2\arctan r$$

and

$$\left|\log \frac{1+z}{1-z}\right| \le \log \frac{1+r}{1-r}.$$

By Schwarz' lemma

$$|\mathrm{Re}\, f(z)| \le \frac{4}{\pi} \arctan |z| \ \text{ and } \ |\mathrm{Im}\, f(z)| \le \frac{2}{\pi} \log \frac{1+|z|}{1-|z|}$$

where equalities hold if and only if $f(z) = g(\alpha z)$ for some constant α with $|\alpha| = 1$. ■

The following version of Schwarz' lemma for harmonic functions was proved in [7].

Lemma 2.2. *Let f be a harmonic function of $\mathbb{D}$ such that $f(0) = 0$ and $f(\mathbb{D}) \subset \mathbb{D}$. Then*

$$\Lambda_f(0) \le \frac{4}{\pi} \tag{2.3}$$

$$\Lambda_f(z) \le \frac{8}{\pi(1-|z|^2)} \quad \text{for} \quad z \in \mathbb{D} \tag{2.4}$$

and

$$|f(z)| \le \frac{4}{\pi} \arctan |z| \le \frac{4}{\pi}|z| \quad \text{for} \quad z \in \mathbb{D}. \tag{2.5}$$

Remark 2.6. Recently in [28] this lemma has been improved by replacing (2.4) by the inequality

$$\Lambda_f(z) \le \frac{4(1+|f(z)|)}{\pi(1-|z|^2)} \quad \text{for} \quad z \in \mathbb{D}. \tag{2.7}$$

Note that this inequality includes (2.3) whereas (2.4) does not. The equality in (2.7) holds for the harmonic function

$$f(z) = \frac{2}{\pi} \arctan\left(\frac{2y}{1-x^2-y^2}\right). \qquad \bullet$$

Proof. For the proof of (2.5) we fix α, $0 \le \alpha \le 2\pi$ and consider

$$u_\alpha(z) = \mathrm{Re}\,(e^{-i\alpha} f(z)).$$

Then, because $f(0) = 0$, $u_\alpha(0) = 0$ and, by hypothesis, u_α is harmonic on $\mathbb{D}$. Because $\mathbb{D}$ is simply connected, there exists an analytic function $g_\alpha(z)$ in $\mathbb{D}$ such that $g_\alpha(0) = 0$ and $\mathrm{Re}\, g_\alpha(z) = u_\alpha(z)$. Clearly $g_\alpha(\mathbb{D})$ is contained in the strip $|\mathrm{Re}\, w| < 1$, as $|e^{-i\alpha} f(z)| < 1$ in $\mathbb{D}$. By Lemma 2.1,

$$|\mathrm{Re}\,(e^{-i\alpha} f(z))| \le \frac{4}{\pi} \arctan |z| \quad \text{for } z \in \mathbb{D}.$$

Because this inequality holds for an arbitrary α, (2.5) holds in $\mathbb{D}$. The inequality (2.3) is a direct consequence of (2.5).

For the proof of (2.7), we fix $a \in \mathbb{D}$ and consider the transformation

$$F(z) = \frac{f(\phi_a(z)) - f(0)}{1+|f(a)|}, \quad \zeta = \phi_a(z) = \frac{z+a}{1+\bar{a}z}.$$

Then F is harmonic on $\mathbb{D}$, $F(0)$, $|F(z)| < 1$ in $\mathbb{D}$. Applying the Chain rule by treating $F = F(z, \bar{z})$ as $F(\zeta, \bar{\zeta})$, we see that

$$F_z(z) = \frac{1}{1+|f(a)|} f_\zeta(\phi_a(z))\phi_a'(z),$$

and

$$F_{\bar{z}}(z) = \frac{1}{1+|f(a)|} f_{\bar{\zeta}}(\phi_a(z))\overline{\phi_a'(z)}.$$

Setting $z = 0$, this gives (note that $\phi_a(0) = a$ and $\phi_a'(0) = 1 - |a|^2$),

$$f_z(a) = \frac{1+|f(a)|}{1-|a|^2} F_z(0)$$

and

$$f_{\bar{z}}(a) = \frac{1+|f(a)|}{1-|a|^2} F_{\bar{z}}(0)$$

so that

$$\Lambda_f(a) = \frac{1+|f(a)|}{1-|a|^2} \Lambda_F(0)$$

which in turn by (2.3) implies that

$$\Lambda_f(a) \le \frac{4}{\pi}\left(\frac{1+|f(a)|}{1-|a|^2}\right).$$

Thus, (2.7) follows as a is arbitrary. For the proof of the equality case, it suffices to observe that for $x \in (-1, 1)$, $h(x, 0) = 0$ and

$$|h_z(x, 0)| + |h_{\bar{z}}(x, 0)| = \frac{4}{\pi(1-x^2)} \quad \text{for } x \in (-1, 1)$$

showing that (2.7) is sharp. We also see that the function h also gives the equality in (2.5). ■

We recall the following notation from [24]

Definition 2.8. *Let $\mathcal{S}_H$ denote the class of all complex-valued (sense-preserving) harmonic mappings that are normalized on the unit disk $\mathbb{D}$. That is,*

$$\mathcal{S}_H = \{f : \mathbb{D} \to \mathbb{C} : f \text{ is harmonic and univalent with } f(0) = 0 = f_z(0) - 1\}.$$

We note that $\mathcal{S}_H$ reduces to $\mathcal{S}$, the class of normalized univalent analytic functions in $\mathbb{D}$ whenever the co-analytic part of f is zero, i.e. $g(z) \equiv 0$ in $\mathbb{D}$. As remarked in [24], we may sometimes restrict our attention to the subclass of functions f in $\mathcal{S}_H$ for which $f_{\bar{z}}(0) = 0$. This led to the definition

$$\mathcal{S}_H^0 = \{f \in \mathcal{S}_H : f_{\bar{z}}(0) = 0\}.$$

As an example of an extremal problem for the functions of the class $\mathcal{S}_H^0$, we study the question, if there exists a function in $\mathcal{S}_H^0$ which maps the unit disk to a domain of the smallest area. It turns out that this problem can be completely solved. Namely, we obtain the following result (see [9]):

Theorem 2.9. *Let f be a function in $\mathcal{S}_H$. Then the area of $f(\mathbb{D})$ is greater than or equal to $\pi/2$. Furthermore, the minimum is attained only for the function*

$$f(z) = z + \frac{1}{2}\bar{z}^2$$

and its rotations.

Proof. As noted above, each function in S_H^0 can be written as $f = h + \overline{g}$, where h, g are analytic functions

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} b_n z^n.$$

As in the introduction, we may write $g' = \omega h'$, where $\omega\colon \mathbb{D} \to \mathbb{D}$ is analytic with $\omega(0) = 0$. Thus, by the classical Schwarz' lemma we have $|\omega(z)| \le |z|$ and by the definition of the Jacobian, we have

$$\begin{aligned} J_f(z) &= |h'(z)|^2 - |g'(z)|^2 = (1 - |\omega(z)|^2)|h'(z)|^2 \\ &\ge (1 - |z|^2)|h'(z)|^2 > 0. \end{aligned}$$

Then the area A of $f(\mathbb{D})$ has the expression

$$\begin{aligned} A &= \int_{\mathbb{D}} J_f(z)\, dx\, dy \\ &\ge \int_{\mathbb{D}} \left(1 - |z|^2\right)|h'(z)|^2\, dx\, dy \\ &= \pi \sum_{n=1}^{\infty} n|a_n|^2 - \int_{\mathbb{D}} \Big|\sum_{n=1}^{\infty} n a_n z^n\Big|^2\, dx\, dy. \end{aligned}$$

We note that

$$\begin{aligned} \int_{\mathbb{D}} |z|^{2n}\, dx\, dy &= \int_{r=0}^{1}\int_{\theta=0}^{2\pi} r^{2n} r\, d\theta\, dr \\ &= 2\pi \int_0^1 r^{2n+1}\, dr = \frac{\pi}{n+1}, \end{aligned}$$

and hence,

$$\begin{aligned} &\pi \sum_{n=1}^{\infty} n|a_n|^2 - \int_{\mathbb{D}} \Big|\sum_{n=1}^{\infty} n a_n z^n\Big|^2\, dx\, dy \\ &= \pi \sum_{n=1}^{\infty} n\left(1 - \frac{n}{n+1}\right)|a_n|^2 \end{aligned}$$

$$= \frac{\pi}{2} + \pi \sum_{n=2}^{\infty} n\Big(1 - \frac{n}{n+1}\Big)|a_n|^2$$
$$\geq \frac{\pi}{2}.$$

The minimum is attained if we choose $a_n = 0$ for all $n \geq 2$.

Then the function ω is of the form $\omega(z) = e^{i\alpha} z$ for some α, and so,

$$g'(z) = e^{i\alpha} z h'(z) = e^{i\alpha} z.$$

Thus, $|b_2| = 1/2$ and $b_n = 0$ for $n \geq 3$. The function $f(z) = z + \frac{1}{2}\overline{z}^2$ is known to be univalent in $\mathbb{D}$, and the claim follows. ∎

2.10. BLOCH CONSTANTS FOR PLANAR HARMONIC MAPPINGS. The classical Schwarz' lemma is instrumental for the development of many modern areas of research. As an application to this, we begin our discussion with Landau's theorem which provides the largest schlicht disk for the properly normalized class of bounded analytic functions in the unit disk $\mathbb{D}$. Here by a schlicht disk, we mean a disk which is univalent image of some subregion of $\mathbb{D}$.

Theorem 2.11 (Landau). *Suppose that f is analytic in $\mathbb{D}$, $|f(z)| \leq M$ in $\mathbb{D}$, $f(0) = f'(0) - 1 = 0$. Then*

(i) *f is univalent for $|z| < \rho_0$; i.e. $\frac{f(\rho_0 z)}{\rho_0} \in \mathcal{S}$,*
(ii) *$f(\mathbb{D}_{\rho_0}) \supseteq \mathbb{D}_{R_0}$,*

where

$$\rho_0 = \frac{1}{M + \sqrt{M^2 - 1}} \text{ and } R_0 = M\rho_0^2.$$

The numbers ρ_0 and R_0 are the best possible. Here $\mathcal{S}$ denotes the class of all analytic functions f that are univalent in the unit disk $\mathbb{D}$, with the normalization $f(0) = f'(0) - 1 = 0$.

Remark 2.12.

(i) If $M = 1$, then the Schwarz lemma gives that $f(z) = z$.
(ii) For the class $\mathcal{H}$ of normalized analytic functions in $\mathbb{D}$ without the boundedness condition on f, there is still the Bloch theorem which asserts the existence of a positive constant b such that for any $f \in \mathcal{H}$ the image $f(\mathbb{D})$ contains a schlicht disk of radius b. The Bloch constant is then defined to be the "largest" such constant, i.e. the supremum of all such b. One of the outstanding open problems in the classical complex analysis is perhaps that of determining the precise value of the Bloch constant for $\mathcal{H}$. We refer to the book of Ponnusamy [23] for a general and basic discussion on Bloch constant. The best known lower estimate is in [7]. ●

Although the Bloch theorem does not hold for normalized harmonic functions, the authors in [7] obtained Bloch's theorem for normalized quasiregular harmonic mappings and open harmonic mappings. In this subsection, we consider the analogous problem of estimating the Bloch constant for certain class of harmonic mappings. It turns out that one requires suitable additional assumption other than the usual normalization in order to obtain Landau and Bloch theorems. We refer to the paper of Bohner [5] for the existence of Bloch constant for K-quasiregular harmonic mappings (even in higher dimensional case).

Theorem 2.13. *Let $f : \mathbb{D} \to \mathbb{D}(M) = \{w : |w| < M\}$ be harmonic such that $f(0) = 0,\ J_f(0) = 1$. Then*

- *f is univalent in $\mathbb{D}_{\rho_0}$,*
- *$f(\mathbb{D}_{\rho_0}) \supseteq \mathbb{D}_{R_0}$,*

where

$$\rho_0 = \frac{\pi^3}{64mM^2} \text{ and } R_0 = \frac{\pi}{8M}\rho_0 := \frac{\pi}{512mM^3}.$$

Here $m \approx 6.85$ is given by

$$m = \min_{r \in (0,1)} \frac{3 - r^2}{r(1 - r^2)}.$$

Recently improved estimates for Theorem 2.13 are obtained by Grigoryan [10] and Xinzhong [28]. Indeed, Grigoryan has shown that ρ_0 and R_0 in Theorem 2.13 are

$$\rho_0 = 1 - \frac{2\sqrt{2}M}{\sqrt{\pi + 8M^2}}$$

and

$$R_0 = \frac{\pi}{4M} + 4M - 4M\sqrt{1 + \frac{\pi}{8M^2}}.$$

On the other hand, Xinzhong improved this result further by showing that

$$\rho_0 = \max_{0<\rho<1} \frac{\rho\pi}{4\Big[1 + \frac{16M^2}{\pi^2}\Big(\frac{1 + \frac{4}{\pi}\arctan\rho}{1-\rho^2}\Big)\Big]}$$

and

$$R_0 = \max_{0<\rho<1} \frac{\rho\pi^2}{32M\Big[1 + \frac{16M^2}{\pi^2}\Big(\frac{1 + \frac{4}{\pi}\arctan\rho}{1-\rho^2}\Big)\Big]}.$$

Moreover, estimates for these maximum values are also given in [28].

3. Quasiconformal Mappings in the plane

We shall outline basic definitions and properties of quasiconformal mappings in the plane. Quasiconformal mappings are a natural generalization of conformal mappings. Standard references in this topic are [1] and [18]. Quasiconformal mappings are useful tools in the theory of planar harmonic mappings. This arises from the fact that planar harmonic mappings are locally quasiconformal, as remarked in the beginning. Some familiarity with real analysis and measure theory is required for the study of [1,18].

3.1. Definitions of Quasiconformality and the Class $\mathrm{ACL}(\Omega)$.

Definition 3.2 (K-quasiconformal). *Let Ω_1, Ω_2 be domains in the extended complex plane $\mathbb{C}_\infty$. Suppose that $f\colon \Omega_1 \to \Omega_2$ is a sense-preserving homeomorphism. For each $z \in \Omega \setminus \{\infty, f^{-1}(\infty)\}$, we define the* linear dilatation *of f at z by*

$$H_f(z) = \limsup_{r\to 0} \frac{L(z,r)}{\ell_f(z,r)},$$

where

$$L_f(z,r) = \max_{|z-w|} |f(z) - f(w)|,$$

and

$$\ell_f(z,r) = \min_{|z-w|} |f(z) - f(w)|.$$

*We say that f is K-*quasiconformal*, if H_f is bounded in $\Omega \setminus \{\infty, f^{-1}(\infty)\}$ and*

$$H_f(z) \le K \text{ a.e. in } \Omega,$$

where $1 \le K < \infty$ is a uniform constant.

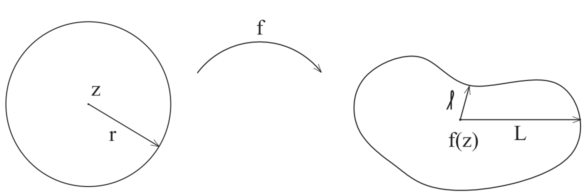

Figure 1. Description for quasiconformality

Examples 3.3.

(1) A homeomorphism $f\colon \Omega \to f\Omega$ satisfying

$$|z-w|/L \le |f(z) - f(w)| \le L|z-w|$$

for all $z, w \in \Omega$ is called L-bilipschitz. It is easy to see that L-bilipschitz mappings are L^2-quasiconformal.

(2) Not all quasiconformal mappings are bilipschitz. The standard counterexample is the quasiconformal radial stretching

$$z \mapsto |z|^{\alpha-1} z, \qquad z \in \mathbb{D},$$

where $\alpha \in (0,1)$.

Definition 3.4 (Class $\mathrm{ACL}(\Omega)$). *Let f be a complex valued function defined on subinterval I of $\mathbb{R}$. Suppose that for all $\varepsilon > 0$, there is a $\delta > 0$ such that*

$$\sum_{k=1}^{n} |f(a_k) - f(b_k)| < \varepsilon$$

for every finite, pairwise disjoint family of open subintervals $[a_k, b_k]$, $k = 1, 2, \ldots, n$ of I such that

$$\sum_{k=1}^{n} |a_k - b_k| < \delta.$$

Then f is said to be absolutely continuous in I. Obviously, an absolutely continuous function is continuous: simply take $n = 1$. See e.g. [14, 18.10].

Let Ω be a domain in $\mathbb{C}$. We say that a function $u\colon \Omega \to \mathbb{R}$ is absolutely continuous on lines *(ACL) in Ω if, for each rectangle*

$$R = [a,b] \times [c,d] \subset \Omega,$$

the function $u(x+iy)$ is absolutely continuous with respect to the variable x a.e. in $[a,b]$ and with respect to y a.e. in $[c,d]$.

The next important result [18, IV.2.3] gives an analytic characterization of quasiconformality.

A sense preserving homeomorphism $f\colon \Omega_1 \to \Omega_2$ is K-quasiconformal if and only if f is in $\mathrm{ACL}(\Omega_1)$ and

$$\max_\alpha |\partial_\alpha f(z)|^2 \le K\, J_f(z) \text{ a.e. in } \Omega_1, \tag{3.5}$$

where ∂_α denotes the derivative of f to the direction α and $J_f(z)$ is the Jacobian of f at z. This characterization is said to be the analytic definition of quasiconformality.

Remark 3.6. If the inequality (3.5) holds for an ACL(Ω_1) map f a.e. in Ω_1, but f is not necessarily homeomorphic, then f is called *K-quasiregular*. By a result of S. Stoilow a quasiregular mapping f of $\mathbb{D}$ onto a domain Ω can be represented as $f = g \circ h$, where h is a quasiconformal mapping of $\mathbb{D}$ onto itself and g is an analytic function [18]. ●

3.7. Beltrami Equation. Another approach to the theory of planar quasiconformal mappings is the following. Immediately from the analytic definition of quasiconformality, for a quasiconformal mapping $f\colon \Omega \to \mathbb{C}$, we see that there is a Lebesgue measurable function μ defined in Ω such that

$$f_{\bar{z}}(z) = \mu(z) f_z(z). \tag{3.8}$$

We also have

$$\|\mu\|_\infty = \frac{K-1}{K+1} < \infty.$$

The equation (3.8) is known as the (complex) *Beltrami equation*. The function $\mu = \mu_f = f_z/f_{\bar{z}}$ is the Beltrami coefficient of f or the complex dilatation of f. See also discussion in [24, 2.10].

3.9. Conformal Modulus. Suppose that Γ is a path family in $\mathbb{C}_\infty$. We will assign a conformally invariant quantity, modulus, to Γ, which measures the size of Γ.

Let $\rho\colon \mathbb{C} \to [0,\infty]$ be a Borel measurable function. We call ρ *admissible* for Γ (denoted $\rho \in \mathcal{F}(\Gamma)$) if

$$\int_\gamma \rho(z)\,|dz| \ge 1,$$

for each locally rectifiable $\gamma \in \Gamma$. We define the modulus of Γ by

$$\mathsf{M}(\Gamma) = \inf_\rho \int_{\mathbb{C}} \rho^2(z)\,dx\,dy, \tag{3.10}$$

where the infimum is taken over all $\rho \in \mathcal{F}(\Gamma)$.

The following basic properties of the conformal modulus are well-known:

Lemma 3.11.

(1) *The modulus is an outer measure in the space of all path families in $\mathbb{C}$, i.e.*

(i) $\mathsf{M}(\emptyset) = 0$,

(ii) *If* $\Gamma_1 \subset \Gamma_2$ *then* $\mathsf{M}(\Gamma_1) \le \mathsf{M}(\Gamma_2)$, *and*

(iii) $\mathsf{M}\big(\bigcup_j \Gamma_j\big) \le \sum_j \mathsf{M}(\Gamma_j)$.

(2) *We say that Γ_2 is* minorized *by Γ_1 and write $\Gamma_1 < \Gamma_2$ if every $\gamma \in \Gamma_2$ has a subpath in Γ_1. If $\Gamma_1 < \Gamma_2$ then* $\mathsf{M}(\Gamma_1) \ge \mathsf{M}(\Gamma_2)$.

(3) *If Γ is the family of paths in Ω such that $\ell(\gamma) \ge r$, then $\mathsf{M}(\Gamma) \le m(\Omega) r^{-2}$, where $m(\Omega)$ is the Lebesgue measure of Ω.*

Proof.

(1) (i) Since the zero function is admissible for $\emptyset$, $\mathsf{M}(\emptyset) = 0$.

(ii) If $\Gamma_1 \subset \Gamma_2$ then $\mathcal{F}(\Gamma_2) \subset \mathcal{F}(\Gamma_1)$ and hence $\mathsf{M}(\Gamma_1) \le \mathsf{M}(\Gamma_2)$.

(iii) We may assume that $\mathsf{M}(\Gamma_j) < \infty$ for all j. Let $\varepsilon > 0$. Then we may choose for each j a function ρ_j admissible for Γ_j such that

$$\int_{\mathbb{C}} \rho_j^2\,dx\,dy \le \mathsf{M}(\Gamma_j) + 2^{-j}\varepsilon.$$

Now let

$$\rho = \sup_j \rho_j \quad \text{and} \quad \Gamma = \bigcup_j \Gamma_j.$$

Then $\rho\colon \mathbb{C} \to [0,\infty]$ is a Borel function. Moreover, if $\gamma \in \Gamma$ is locally rectifiable, then $\gamma \in \Gamma_j$ for some j,

$$\int_\gamma \rho\,|dz| \ge \int_\gamma \rho_j\,|dz| \ge 1,$$

and hence, ρ is admissible for Γ. Now

$$\begin{aligned}\mathsf{M}(\Gamma) &\le \int_{\mathbb{C}} \rho^2\,dx\,dy \\ &\le \int_{\mathbb{C}} \sum_j \rho_j^2\,dx\,dy \\ &\le \sum_j \mathsf{M}(\Gamma_j) + \varepsilon,\end{aligned}$$

and the claim follows by letting $\varepsilon \to 0$.

(2) If $\Gamma_1 < \Gamma_2$ then obviously $\mathcal{F}(\Gamma_1) \subset \mathcal{F}(\Gamma_2)$. Hence $\mathsf{M}(\Gamma_1) \ge \mathsf{M}(\Gamma_2)$.

(3) The claim follows immediately from (3.10) and the fact that the function $\rho = \chi_\Omega/r$ is admissible for Γ. ■

Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ and $f\colon \Omega_1 \to \Omega_2$ be a continuous function. Suppose that Γ is a family of paths in Ω_1. Then $\Gamma' = \{f \circ \gamma : \gamma \in \Gamma\}$, the image of Γ under f, is a family of paths in Ω_2.

Theorem 3.12. *If* $f\colon \Omega_1 \to \Omega_2$ *is conformal, then* $\mathsf{M}(f(\Gamma)) = \mathsf{M}(\Gamma)$ *for all path families* Γ *in* Ω_1.

Proof. Let $\rho_1 \in \mathcal{F}(f(\Gamma))$, and define

$$\rho(z) = \rho_1\big(f(z)\big)|f'(z)|$$

for $z \in \Omega_1$ and $\rho(z) = 0$ otherwise. Because f is conformal,

$$\begin{aligned}\int_\gamma \rho\,|dz| &= \int_\gamma \rho_1\big(f(z)\big)|f'(z)|\,|dz| \\ &= \int_{f\circ\gamma} \rho_1\,|dz| \geq 1\end{aligned}$$

for every locally rectifiable $\gamma \in \Gamma$. Hence, $\rho \in \mathcal{F}(\Gamma)$, and

$$\begin{aligned}\mathsf{M}(\Gamma) &\leq \int_{\mathbb{C}} \rho^2\,dx\,dy \\ &= \int_{\Omega_1} \rho_1^2\big(f(x)\big)|J_f(x)|\,dx\,dy \\ &= \int_{\Omega_2} \rho_1^2\,dx\,dy = \int_{\mathbb{C}} \rho_1^2\,dx\,dy\end{aligned}$$

for all $\rho_1 \in \mathcal{F}(f(\Gamma))$, and thus $\mathsf{M}(\Gamma) \leq \mathsf{M}(f(\Gamma))$. The inverse inequality follows from the fact that f^{-1} is conformal. ■

We now recall the geometric definition of quasiconformality. A sense preserving homeomorphism $f\colon \Omega_1 \to \Omega_2$ is K-quasiconformal if and only if it satisfies the inequalities

$$\mathsf{M}(\Gamma)/K \leq \mathsf{M}(f\Gamma) \leq K\mathsf{M}(\Gamma) \qquad (3.13)$$

for every family of paths Γ in Ω_1 (see [18, IV.3.3]).

3.14. BASIC PROPERTIES OF QUASICONFORMAL MAPPINGS. The following properties of quasiconformal mappings are well-known.

(1) If $f\colon \Omega_1 \to \Omega_2$ is K_1-quasiconformal and $g\colon \Omega_2 \to \Omega_3$ is K_2-quasiconformal, then $f \circ g\colon \Omega_1 \to \Omega_3$ is K_1K_2-quasiconformal. The inverse mapping $f^{-1}\colon \Omega_2 \to \Omega_1$ of a K-quasiconformal mapping is K-quasiconformal. These properties follow immediately from the geometric definition of quasiconformality.
(2) A mapping f of Ω is 1-quasiconformal if and only if it is a conformal mapping in $\Omega \setminus \{\infty, f^{-1}(\infty)\}$ [18, I.5.1].
(3) If $f\colon \Omega_1 \to \Omega_2$ is K-quasiconformal and Ω_1, Ω_2 are Jordan domains, then f can be extended to a homeomorphism from $\overline{\Omega}_1$ onto $\overline{\Omega}_2$ [18, I.8.2].
(4) Let $E \subset \Omega_1$ is a closed set, and suppose that E can be expressed as an enumerable union of rectifiable curves. If $f\colon \Omega_1 \to \Omega_2$ is a homeomorphism, and f is K-quasiconformal in each component of $\Omega_1 \setminus E$, then f is K-quasiconformal in Ω_1 [18, V.3.4].

3.15. ESTIMATES WITH MODULUS. In general, it is difficult to calculate the modulus of a given path family. However, it is possible to obtain effective upper and lower bounds for the modulus in many situations. Estimates of this type can be used in the geometric theory of quasiconformal mappings. In particular, this technique is used for obtaining distortion results for quasiconformal mappings.

At first we introduce the concept of conformal modulus or simply modulus of a quadrilateral. A Jordan domain Ω in $\mathbb{C}$ with marked (positively ordered) points $z_1, z_2, z_3, z_4 \in \partial\Omega$ is a quadrilateral and denoted by $(\Omega; z_1, z_2, z_3, z_4)$. We use the canonical conformal mapping of quadrilateral onto a rectangle $(\Omega'; 1 + ih, ih, 0, 1)$, with the vertices corresponding, to define the modulus $\mathrm{mod}(Q) = h$ of a quadrilateral $Q = (\Omega; z_1, z_2, z_3, z_4)$. The modulus of $(\Omega; z_2, z_3, z_4, z_1)$ is $1/h$. By this we mean that for $h = \mathrm{mod}(Q)$ and for this value only there exists a unique conformal mapping from Q onto the rectangle Ω', which takes the four points $z_1, z_2, z_3, z_4 \in \partial\Omega$ onto the four points $1 + ih, ih, 0, 1$ of the rectangle Ω'. Moduli of quadilaterals and path families connected are connected to each other as follows.

Lemma 3.16. *Let* $Q = (\Omega; z_1, z_2, z_3, z_4)$ *be a quadrilateral and denote by* γ_j *the boundary arc of* Ω *connecting* z_j *and* z_{j+1} *for* $j = 1, 2, 3$ *and* z_4, z_0 *for* $j = 4$. *Then* $\mathrm{mod}(Q) = 1/\mathsf{M}(\Gamma)$, *where* Γ *is the family of paths connecting* γ_2 *and* γ_4 *in* Ω.

Proof. By conformal invariance, we may assume that Q is the rectangle with boundary points $1 + ih, ih, 0$ and 1. We show that $\mathsf{M}(\Gamma) = 1/h$.

Choose $\rho \in \mathcal{F}(\Gamma)$, and let γ_y be the vertical segment $[y, y+ih]$, where $y \in (0, 1)$. Then $\gamma_y \in \Gamma$. We note that by Hölder's inequality

$$\begin{aligned}1 &\leq \left(\int_{\gamma_y} \rho\,|dz|\right)^2 \\ &\leq \left(\int_{\gamma_y} |dz|\right)\left(\int_{\gamma_y} \rho^2\,|dz|\right) \\ &= h\int_{\gamma_y} \rho^2\,|dz|.\end{aligned}$$

This holds for all $y \in (0, 1)$, and hence by Fubini's theorem

$$\int_{\mathbb{C}} \rho^2 \, dx \, dy \ge \int_0^1 \left(\int_{\gamma_y} \rho^2 \, |dz| \right) dt \ge \frac{1}{h}.$$

Since the above holds for any $\rho \in \mathcal{F}(\Gamma)$,

$$\mathsf{M}(\Gamma) \ge \frac{1}{h}.$$

Next we choose $\rho = 1/h$ inside Ω and $\rho = 0$ otherwise. Then ρ is admissible for Γ and

$$\mathsf{M}(\Gamma) \le \int_{\mathbb{C}} \rho^2 \, dx \, dy = \frac{1}{h}. \qquad \blacksquare$$

Remark 3.17. Suppose that Ω is a region in the complex plane whose boundary $\partial\Omega$ consists of a finite number of regular Jordan curves, so that at every point, except possibly at finitely many points, of the boundary a normal ∂n is defined. Then the following problem is known as the *Dirichlet-Neumann problem*.

Let ψ be a real-valued continuous function defined on $\partial\Omega$. Let $\partial\Omega = A \cup B$ where A, B both are unions of Jordan arcs. Find a function u satisfying the following conditions:

(1) u is continuous and differentiable in $\overline{\Omega}$.

(2) $u(t) = \psi(t), \qquad t \in A.$

(3) If $\partial/\partial n$ denotes differentiation in the direction of the exterior normal, then

$$\frac{\partial}{\partial n} u(t) = \psi(t), \qquad t \in B.$$

It is possible to express the modulus of a quadrilateral $(\Omega; z_1, z_2, z_3, z_4)$ in terms of the solution of the Dirichlet-Neumann problem as follows.

Let $\gamma_j, \ j = 1, 2, 3, 4$, be the arcs of $\partial\Omega$ between (z_1, z_2), (z_2, z_3), (z_3, z_4), (z_4, z_1), respectively. If u is the (unique) harmonic solution of the Dirichlet-Neumann problem with boundary values of u equal to 0 on γ_2, equal to 1 on γ_4 and with $\partial u/\partial n = 0$ on $\gamma_1 \cup \gamma_3$, then by [2, p. 65/Thm 4.5]:

$$\operatorname{mod}(\Omega; z_1, z_2, z_3, z_4) = \int_{\Omega} |\nabla u|^2 \, dx \, dy. \tag{3.18}$$

The solutions of the Dirichlet-Neumann problems can be approximated by the method of finite elements, see [11, pp. 305–314], [22]. Hence, this method can be used to obtain numerical approximations of the conformal modulus, see e.g. [4]. ●

3.19. Ring Domains. A domain Ω in $\mathbb{C}_\infty$ is called a ring, if $\mathbb{C}\setminus\Omega$ has exactly two components. If the *boundary* components are E and F, we denote the ring by $R(E, F)$. For $E, F, \Omega \subset \mathbb{C}_\infty$ we denote by $\Delta(E, F; \Omega)$ the family of all nonconstant paths joining E and F in Ω.

The conformal modulus of a ring domain $R(E, F)$ is defined by

$$\operatorname{mod} R(E, F) = \frac{2\pi}{\mathsf{M}\big(\Delta(E, F; \Omega)\big)},$$

provided that $\mathsf{M}(\Delta(E, F; \Omega)) \ne 0$. Otherwise, we define $\operatorname{mod} R(E, F) = \infty$. Note that this happens only if either E or F is a singleton.

Lemma 3.20. *Let $0 < a < b < \infty$, $A = \mathbb{D}(b) \setminus \mathbb{D}(a)$ and*

$$\Gamma_A = \Delta\big(S(a), S(b); A\big),$$

where $\mathbb{D}(r)$ and $S(r)$ denote the disk $\{z : |z| < r\}$ and the circle $\{z : |z| = r\}$, respectively. Then

$$\mathsf{M}(\Gamma_A) = \frac{2\pi}{\log \frac{b}{a}}.$$

Proof. Let $\rho \in \mathcal{F}(\Gamma_A)$. We may assume that $\rho(z) = 0$ for $z \notin A$. For each w with $|w| = 1$, let $\gamma_w : [a, b] \to \mathbb{C}$ be the radial line segment defined by $\gamma_w(z) = |z|w$. By Hölder's inequality we obtain

$$\begin{aligned} 1 &\le \left(\int_{\gamma_w} \rho \, |dz| \right)^2 \\ &\le \left(\int_a^b \rho(sw)^2 s \, ds \right) \left(\int_a^b \frac{1}{s} \, ds \right) \\ &= \log \frac{b}{a} \int_a^b \rho(sw)^2 s \, ds. \end{aligned}$$

By integrating over the unit circle, we have

$$2\pi \le \log \frac{b}{a} \int_{\mathbb{C}} \rho^2 \, dx \, dy.$$

Taking the infimum over all admissible ρ yields

$$2\pi \le \left(\log \frac{b}{a} \right) \mathsf{M}(\Gamma_A).$$

Next we define $\rho(z) = 1/\big(|z| \log(b/a)\big)$ for $z \in A$, and $\rho(z) = 0$ otherwise. Clearly ρ is admissible for Γ_A, and hence

$$\begin{aligned} \mathsf{M}(\Gamma_A) &\le \int_{\mathbb{C}} \rho^2 \, dx \, dy \\ &= 2\pi \left(\log \frac{b}{a} \right)^{-2} \int_a^b \frac{1}{s} \, ds \\ &= \frac{2\pi}{\log \frac{b}{a}}. \end{aligned} \qquad \blacksquare$$

Remark 3.21. Every ring domain R can be mapped conformally onto the annulus $\{z : 1 < |z| < e^M\}$, where $M = \operatorname{mod} R$ is the conformal modulus of the ring domain R. Hence the conformal modulus and Lemma 3.20 can be used to characterize conformally equivalent ring domains (see also Theorem 4.3).

Remark 3.22. The connection between ring and quadrilateral moduli is given in [17, p.102] or [18, p. 36]. If we map the annulus

$$A_r = \{z : 1 < |z| < r\},$$

with the segment $[1, r]$ on the real axis removed, by

$$z \mapsto \log z = \log|z| + i \arg z, \qquad 0 < \arg z < 2\pi,$$

the image is the rectangle R with vertices $(\log r + 2\pi i, 2\pi i, 0, \log r)$ and we see that

$$\operatorname{mod}(R; \log r + 2\pi i, 2\pi i, 0, \log r) = \frac{2\pi}{\log r}.$$

Thus $2\pi/\log r$ has this interpretation as the modulus of a quadrilateral.

Lemma 3.23. *Let Q be either a quadrilateral or ring domain Q such that $\overline{Q} \subseteq \Omega$. Then, for a K-quasiconformal mapping f of a domain Ω onto $f(\Omega)$,*

$$\frac{\operatorname{mod}(Q)}{K} \le \operatorname{mod}(f(Q)) \le K \operatorname{mod}(Q).$$

Proof. Since every quadrilateral is conformally equivalent to a rectangle, we may assume that f maps a rectangle onto a rectangle. If f is K-quasiconformal, then the claim follows from (3.13) and Lemma 3.16.

Similarly, the result for ring domains follows from (3.13) and Lemma 3.20. ■

Remark 3.24. In fact, a sense-preserving homeomorphism $f : \Omega_1 \to \Omega_2$ is quasiconformal if and only if the modules of quadrilaterals are K-quasi-invariant (see [1, Chapter II]) or [18, I.5.3]).

3.25. Canonical Ring Domains. The complementary components of the *Grötzsch ring* $B_G(s)$ in $\mathbb{C}_\infty$ are the unit circle $S = \{z : |z| = 1\}$ and $[s, \infty]$, $s > 1$, and those of the *Teichmüller ring* $B_T(s)$ are $[-e_1, 0]$ and $[s, \infty]$, $s > 0$. We define two special functions $\gamma(s)$, $s > 1$ and $\tau(s)$, $s > 0$ by

$$\begin{cases} \gamma(s) = \mathsf{M}(\Delta(S, [s, \infty]; \mathbb{C})), \\ \tau(s) = \mathsf{M}(\Delta([-1, 0], [s, \infty]; \mathbb{C})), \end{cases}$$

respectively. We shall refer to these functions as the *Grötzsch modulus* and the *Teichmüller modulus*. We also use the bounded version of the Grötzsch ring $B(s) = \mathbb{D} \setminus [0, s]$, where $s \in (0, 1)$. By conformal invariance, the modulus of the path family connecting boundary components of $B(s)$ is $\gamma(1/s)$.

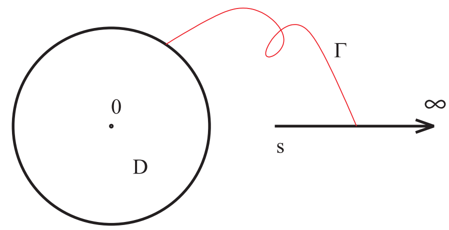

Figure 2. Grötzsch ring domain $B_G(s)$

Grötzsch and Teichmüller moduli functions are strictly decreasing and continuous with range $(0, \infty)$. They are connected by the identity [3, Theorem 8.37]:

$$\gamma(s) = 2\tau(s^2 - 1), \qquad s > 1. \tag{3.26}$$

For $s > 1$, we also have the identity [3, 8.57]:

$$\gamma(s) = \frac{4}{\pi}\mu\Big(\frac{s-1}{s+1}\Big),$$

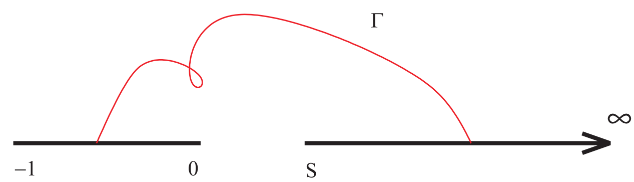

Figure 3. Teichmüller ring domain $B_T(s)$

where, for $r \in (0, 1)$, $\mu(r)$ is the quantity

$$\mu(r) = \frac{\pi}{2}\frac{\mathcal{K}(r')}{\mathcal{K}(r)}, \quad \mathcal{K}(r) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}},$$

and $r' = \sqrt{1 - r^2}$. The function $\mu(r)$ is the conformal modulus of the bounded Grötzsch ring domain $B(r) = \mathbb{D} \setminus [0, r]$ and hence for $r > 1$,

$$\gamma(r) = \frac{2\pi}{\mu(1/r)}. \tag{3.27}$$

These relations allow us to obtain useful estimates for the functions γ and τ (see e.g. [3, Chapters 5 and 8]).

The following functional identities for $\mu(r)$ are well known [3, 5.2]:

$$\mu(r)\mu(r') = \frac{\pi^2}{4},$$

$$\mu(r)\mu\Big(\frac{1-r}{1+r}\Big) = \frac{\pi^2}{2},$$

$$\mu(r) = 2\mu\Big(\frac{2\sqrt{r}}{1+r}\Big). \tag{3.28}$$

Canonical ring domains are useful in the theory of quasiconformal mappings because they can be used for obtaining a lower bound for the conformal modulus of any given ring domain.

Lemma 3.29. [3, Theorem 8.44] *Let $R = R(E, F)$ be a ring in $\mathbb{C}_\infty$ and let $a, b \in E$ and $c, \infty \in F$ be distinct points. Then*

$$\frac{2\pi}{\mathrm{mod} R} \geq \tau(s), \qquad s = \frac{|a-c|}{|a-b|}.$$

Obviously, equality holds for the Teichmüller ring $B_T(s)$.

3.30. Schwarz' Lemma for Quasiconformal Mappings. From the estimates above, it is easy to obtain a version of Schwarz' lemma for quasiconformal mappings. This result also useful in studying harmonic mappings.

Lemma 3.31. *Let $f : \mathbb{D} \to f(\mathbb{D}) \subset \mathbb{D}$ be a K-quasiconformal mapping with $f(0) = 0$. Then*

$$|f(z)| \leq \varphi_K(|z|),$$

where φ_K is the distortion function

$$\varphi_K(r) = \mu^{-1}\left(\frac{\mu(r)}{K}\right).$$

For a fixed $K > 1$, the bound $\varphi_K(r)$ increases with r from 0 to 1.

Proof. Fix $r \in (0, 1)$, and let $s = \max_{z\in S(r)} |f(z)|$. We may assume that $f(r) = s$. Then by Lemma 3.11 (2) and Lemma 3.23,

$$\mu(r) = \mathrm{mod} B(r) \leq K \mathrm{mod} R\big(f([0, r]), S(1)\big).$$

Let

$$g_1(z) = \left(\frac{1+z}{1-z}\right)^2 \quad \text{and} \quad g_2(z) = \frac{1-g(z)}{|1-g(s)|}.$$

We note that $g = g_2 \circ g_1$ maps $\mathbb{D}$ conformally onto the domain $\mathbb{C}\backslash\{x \in \mathbb{R} : x \geq (1-s)^2/(4s)\}$ with $g(0) = 0$ and $g(s) = -1$. Then by Lemma 3.29, (3.26), (3.27) and (3.28) we have

$$\begin{aligned}\mathrm{mod} R\big(f([0, r]), S(1)\big) &\leq \frac{2\pi}{\tau\big(4s/(1-s)^2\big)} \\ &= \frac{4\pi}{\gamma\big((1+s)/(2\sqrt{s})\big)} \\ &= 2\mu\left(\frac{2\sqrt{s}}{1+s}\right) = \mu(s).\end{aligned}$$

Because $\mu(s)$ is a strictly decreasing function, the claim follows. ■

4. Conjecture of J. C. C. Nitsche

Before we address the problem of Nitsche, it will be useful to provide two other related cases.

4.1. Conformal Case. By Riemann mapping theorem, any two simply connected domains Ω_1 and Ω_2 ($\neq \mathbb{C}$) are conformally equivalent. This means that if $f_j : \Omega_j \to \mathbb{D}$ ($j = 1, 2$) is a conformal mapping of Ω_j onto $\mathbb{D}$ ($j = 1, 2$), then $f = f_2^{-1} \circ f_1$ is the required conformal equivalence (see Figure 4) which is an important property of simply connected domains.

Problem 4.2. *Can any two doubly connected domains, for instance annuli centered at the origin, be conformally equivalent to each other?*

The answer is no. But then we ask: when can two doubly connected domains be conformally equivalent? Equivalently, are there good necessary and sufficient conditions, for example for any two annulus regions to be conformally equivalent to each other.

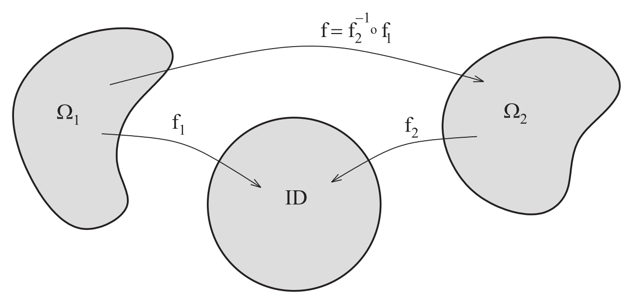

Figure 4. The composed map $f = f_2^{-1} \circ f_1$ from Ω_1 to Ω_2

	Table-I
(1)	$0 < \lvert z\rvert < R$ ($R < \infty$)
(2)	$0 < \lvert z\rvert < \infty$ ($R = \infty$)
(3)	$r < \lvert z\rvert < R$ ($0 < r < R < \infty$)
(4)	$r < \lvert z\rvert < \infty$ ($0 < r,\ R = \infty$)
	Table-II
(A)	$0 < \lvert w\rvert < S$ ($S < \infty$)
(B)	$0 < \lvert w\rvert < \infty$ ($S = \infty$)
(C)	$s < \lvert w\rvert < S$ ($0 < s < S < \infty$)
(D)	$s < \lvert w\rvert < \infty$ ($0 < s,\ S = \infty$)

Clearly, any one in Table-I can correspond to any one in Table-II. We have the following:

- The domains $0 < |z| < R$ and $\frac{1}{R} < |w| < \infty$ are conformally equivalent and is given by the mapping
$$z \mapsto \frac{e^{i\theta}}{z}$$
so that there is a conformal map from (1) $\to$ (D), and (4) $\to$ (A).
- The domains $0 < |z| < R$ and $0 < |w| < S$ are conformally equivalent and is given by the mapping
$$z \mapsto \frac{S}{R}e^{i\theta}z$$
so that there is a conformal map from (1) $\to$ (A).
- Neither the domains $0 < |z| < R$ and $0 < |w| < \infty$ nor the domains $0 < |z| < R$ and $s < |w| < S$ are conformally equivalent. Suppose on the contrary that $0 < |z| < R$ and $0 < |w| < \infty$ are conformally equivalent. Then there exists an analytic function (inverse)
$$g : \mathbb{C} \setminus \{0\} \to A(0, R)$$
for which either $|g(w)| \to 0$ as $w \to 0$, or $|g(w)| \to R$ as $w \to 0$. In either case g has a removable singularity at $w = 0$ and hence g can be extended to be analytic in $\mathbb{C}$ such that $|g(w)| \le M$ in $\mathbb{C}$ for some $M > 0$. Consequently, by Liouville's theorem, g must be a constant which is clearly a contradiction.
- As in the last case, we see that $A(0, R)$ and $A(s, S)$ cannot be conformally equivalent. Indeed suppose not. Then there exists a conformal mapping f between them such that either $|f(0)| \to s$ as $z \to 0$, or $|f(0)| \to S$ as $z \to 0$. Consequently, f has a removable singularity at $z = 0$ and so it can be extended to be continuous at $z = 0$, and so on $|z| < R$. But $f(0)$ cannot be an interior point to $\{w : s < |w| < S\} \cup \{f(0)\}$, which is a contradiction. Thus, we have
$$(1) \not\to (B) \qquad (1) \not\to (C)$$
$$(2) \not\to (A) \qquad (3) \not\to (A)$$
- Similarly, $A(0, \infty)$ and $A(s, S)$ cannot be conformally equivalent; because the extension of $f : A(0, \infty) \to A(s, S)$ to $f : \mathbb{C} \to \{w : s \le |w| < S\}$ or $\{w : s \le |w| \le S\}$ is entire and bounded and hence a constant, by Liouville's theorem. This contradiction shows that
$$(2) \not\to (C)$$
$$(3) \not\to (B)$$
- A conformal map from $\{z : r < |z| < \infty\}$ onto itself is possible, because
$$z \mapsto \frac{se^{i\theta}}{r}z$$
does the job. Thus, the conformal correspondence between (4) and (D) exists.
- Also, there is a conformal map from $0 < |z| < \infty$ onto itself, because
$$z \mapsto e^{i\theta}z \quad \text{or} \quad z \mapsto \frac{e^{i\theta}}{z}$$
does the job. This gives the implication (2) $\to$ (B).

Finally, we are left with only the case of examining the conformal equivalence of the annuli
$$A(r, R) \text{ and } A(s, S), \quad 0 < r, s, S, R < \infty.$$

Theorem 4.3. *Let $0 < r_i < R_i < \infty$ $(i = 1, 2)$. The annular domains $A(r_1, R_1)$ and $A(r_2, R_2)$ are conformally equivalent if and only if they are similar, i.e. $\frac{R_1}{r_1} = \frac{R_2}{r_2}$; Equivalently, if and only if the annuli have the same moduli:*
$$\frac{1}{2}\log\left(\frac{R_1}{r_1}\right) = \frac{1}{2}\log\left(\frac{R_2}{r_2}\right).$$

Proof. Let f be a conformal mapping of $A(r_1, R_1)$ onto $A(r_2, R_2)$. Without loss of generality, we may suppose that f maps the inner boundary circle $|z| = r_1$ onto the inner boundary $|w| = r_2$ (otherwise consider an inversion map, if necessary). Then we apply Schwarz' reflection principle (see [1,23]). The formula shows that the reflected mapping carries the reflected annulus onto an annulus. By successive reflections over the inner and outer boundaries, we then obtain a conformal map g, which is the extension of f, mapping $\mathbb{C} \setminus \{0\}$ onto itself. But then the extended map is bounded near the origin. Thus, g has a removable singularity at $z = 0$ which may be removed by setting $g(0) = 0$. Finally, the resulting map g is a conformal self mapping of the complex plane $\mathbb{C}$ and therefore, $g(z) = az$ (because $g(0) = 0$) for some complex constant a. For a proof of this fact, we refer to [23]. Hence, f must be of the form $f(z) = cz$. Proof of the theorem then follows.

Moreover, it is clear that $|z| = r_1$ corresponds to either $|w| = r_2$ or $|w| = R_2$. In the first case, $f(z) = cz$ with $r_2 = |c|r_1$. This gives the map
$$f(z) = \frac{r_2}{r_1}e^{i\theta}z$$
with $r_1R_2 = R_1r_2$, as we need to map $|z| = R_1$ onto $|w| = R_2$.

In the second case, $f(z) = c/z$ with $R_2 = |c|/r_1$ so that

$$f(z) = \frac{R_2 r_1 e^{i\theta}}{z}$$

with $R_2 r_1 = r_2 R_1$, because we need to map $|z| = R_1$ onto $|w| = r_2$. Thus, the map will depend on the choice of the boundary correspondence. ■

Corollary 4.4. *If f is a conformal mapping of $A(r, 1)$ onto $A(R, 1)$, then we must have $R = r$. More precisely, all conformal mappings of $A(r, 1)$ onto $A(R, 1)$ are given by*

- $f(z) = e^{i\theta} z$ *if* $f(\partial\mathbb{D}) = \partial\mathbb{D}$
- $f(z) = \frac{r}{z} e^{i\theta}$ *if* $f(|z| = r) = \partial\mathbb{D}$,

where in both the cases $R = r$.

Corollary 4.5. *Conformal self maps of the annulus $A(r, R)$ are given by*

- $z \mapsto e^{i\theta} z$ *if* $|z| = r \mapsto |w| = r$
- $z \mapsto \dfrac{Rre^{i\theta}}{z}$ *if* $|z| = r \mapsto |w| = R$.

It is well-known that every ring domain is conformally equivalent to a unique annulus $A(1, r)$. The counterpart of Riemann mapping theorem may be phrased as follows "*Every ring domain is conformally equivalent to an annulus $r < |z| < R$ $(0 \le r < R \le \infty)$.*" As a consequence, every ring domain is conformally equivalent to one of the following:

(i) $0 < |z| < \infty$,
(ii) $1 < |z| < \infty$,
(iii) $1 < |z| < R \quad (R < \infty)$.

Thus, unlike the case of simply connected domains, which fall into three equivalence classes, ring domains posses infinitely many conformal equivalence classes. In the case of (iii) above, R determines the equivalence class, through the module of ring domains.

4.6. Quasiconformal Case. To state the quasiconformal analog of Theorem 4.3, we require Lemma 3.23. Indeed, as a consequence of Lemma 3.23, we suppose that $Q = A(r, 1)$, $f(Q) = A(R, 1)$ and f is a K-quasiconformal mapping of Q onto $f(Q)$. Then

$$M(Q) = \frac{1}{2\pi} \log(1/r) \ \text{ and } \ M(f(Q)) = \frac{1}{2\pi} \log(1/R)$$

so that, by Lemma 3.23, one has

$$\frac{1}{K} \log(1/r) \le \log(1/R) \le K \log(1/r).$$

This gives

$$\frac{1}{r^{1/K}} \le \frac{1}{R} \le \frac{1}{r^K}, \quad \text{or } r^K \le R \le r^{1/K}$$

and we conclude the quasiconformal analog of Theorem 4.3 in the following form.

Theorem 4.7. *If f is a K-quasiconformal mappings of $A(r, 1)$ onto $A(R, 1)$, then*

$$r^K \le R \le r^{1/K}.$$

In particular, for conformal mappings $R = r$.

We are interested in finding harmonic analog of this result. More precisely, we demonstrate that for univalent harmonic mappings we also have some restriction for the possible values of R.

4.8. Harmonic Case. We begin the discussion with the following problem.

Problem 4.9. *For a fixed $r \in (0, 1)$, are there many (orientation preserving) univalent harmonic mappings f of $A(r, 1)$ onto $A(0, 1)$?*

The answer is indeed yes as we present in the example below.

Example 4.10. Fix $r \in (0, 1)$ and consider

$$f(z) = \lambda z + \frac{1-\lambda}{\overline{z}}, \quad \text{where } \lambda = \frac{1}{1-r^2} > 1. \tag{4.11}$$

This is one such map and there are many other ones. This disproves the conjecture of Nitsche that f in Problem 4.9 must be of the above form.

In order to show that f defined by (4.11) meets the required properties, we set $\zeta = \rho e^{i\theta}$ $(r < \rho < 1)$. Then

$$f(\rho e^{i\theta}) = \frac{1}{1-r^2} \left(\rho - \frac{r^2}{\rho}\right) e^{i\theta} := R(\rho) e^{i\theta}.$$

First observe that f is a (sense preserving) univalent harmonic mapping, because f has the form

$$f = h + \overline{g}$$

where

- $h(z) = \lambda z$ and $g(z) = (1-\lambda)/z$ are analytic in $\mathbb{C} \setminus \{0\}$
- $f_z(z) = \lambda, \ f_{\overline{z}}(z) = -\dfrac{1-\lambda}{(\overline{z})^2}$
- $J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2 = \lambda^2 - \dfrac{(1-\lambda)^2}{|z|^4} > \lambda^2 - \dfrac{(1-\lambda)^2}{r^4} = 0$
- $w(z) = \dfrac{g'(z)}{h'(z)} = \dfrac{r^2}{z^2}$ so that $|w(z)| < 1$ on $A(r, 1)$.

Next we see that f maps concentric circles centered at the origin onto concentric circles at centered at the origin and, in particular, f maps $A(r, 1)$ onto $A(0, 1)$. Indeed,

- $\rho = r \mapsto f(\rho e^{i\theta}) = f(re^{i\theta}) = 0$
- $\rho = \rho_1 \mapsto f(\rho_1 e^{i\theta}) = R(\rho_1)e^{i\theta}$, where

$$\rho < \rho_1 \Longleftrightarrow R(\rho) < R(\rho_1)$$

- $\rho = 1 \mapsto f(1.e^{i\theta}) = f(e^{i\theta}) = e^{i\theta}$, i.e. $f(\partial\mathbb{D}) = \partial\mathbb{D}$.

Finally, we note that

$$r < \rho < 1 \Longleftrightarrow 0 = R(r) < R(\rho) < R(1) = 1$$

and the image of the inner circle $|z| = r$ is a single point, namely the origin. Thus, we see that if we consider the class of all univalent harmonic mappings of the annulus $A(r, 1)$ onto the annulus $A(R, 1)$ for some $R \in [0, 1)$, R can possibly be zero. This example shows that if f is neither conformal nor quasiconformal, then R is possibly zero. ●

Thus, in the class of univalent harmonic mappings of $A(r, 1)$ onto $A(R, 1)$, the problem of finding the minimum value of R makes no sense. On the other hand, R admits a universal upper bound (less than 1), as shown in 1962 by Nitsche that $R(r)$ cannot be arbitrarily close to one. To state this result, let $K(r)$ denote the class of univalent harmonic mappings of $A(r, 1)$ onto $A(R, 1)$, and $k(r)$ be the supremum of $R = R(r)$ as f ranges over all $f \in K(r)$. In [21], Nitsche considered all possible values of $R = R(r)$ for fixed r and proved the following interesting result.

Theorem 4.12. *The value of $k(r)$ is less than* 1.

Proof. Set $f = u + iv$. Choose any compact subset of $A(r, 1)$, e.g., the circle

$$\gamma = \left\{ z : |z| = \frac{1+r}{2} \right\}.$$

By the well-known Harnack's inequality, there exists a constant $M = M(\gamma, \phi) > 1$ such that

$$\phi(z) \le M\phi(z')$$

for all harmonic functions ϕ on $A(r, 1)$ and all $z, z' \in \gamma$.

Topological reasoning implies that $f(\gamma)$ surrounds the origin contained in the annulus $A(R, 1)$ in the w-plane. Thus, there exist points z_1 and z_2 in γ such that $u(z_1)$ is to the right

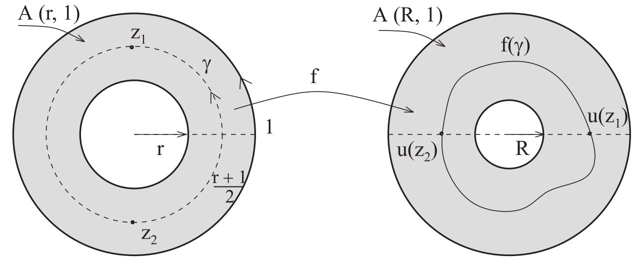

Figure 5. Description for Harnack inequality

of the point $z = R$ whereas $u(z_2)$ is to the left of the point $z = -R$. That is

$$R < u(z_1), \text{ and } u(z_2) < -R$$

so that (see Figure 5)

$$1 + R < 1 + u(z_1) \le M(1 + u(z_2)) < M(1 - R),$$

because $1 + u = \text{Re}\,(f(z) + 1)$ is a positive harmonic function on $A(r, 1)$. The last inequality implies that

$$1 + R < M(1 - R), \text{ or } R < \frac{M-1}{M+1} \; (< 1)$$

and we conclude the proof. ■

In Example 4.10, we have shown that R can be as small as possible. The above theorem shows that the image domain cannot be arbitrarily too thin, but can be degenerated to a punctured disk $0 < |w| < 1$.

Example 4.13. Consider

$$f_t(z) = tz + \frac{1-t}{\overline{z}} := \left(t\rho + \frac{1-t}{\rho}\right)e^{i\theta}, \quad z = \rho e^{i\theta}.$$

It is a simple exercise to see that each f_t maps concentric circles onto concentric circles and maps $A(r, 1)$ onto $A(R(t), 1)$ if and only if

$$\frac{1}{1+r^2} \le t \le \frac{1}{1-r^2}, \quad R(t) = tr + \frac{(1-t)}{r}.$$

Indeed it is easy to observe that

- $R(t) \ge 0 \Longleftrightarrow t \le \dfrac{1}{1-r^2}$
- $R(t)$ $(= \phi(r))$ is increasing function of r if and only if $t \ge \frac{1}{1+r^2}$. Also $R'(t) < 0$, $R(t)$ attains its maximum value at $t_0 = \frac{1}{1+r^2}$. We compute that

$$R(t_0) = \frac{2r}{1+r^2} \text{ and } f_{t_0}(z) = \frac{1}{1+r^2}\left(z + \frac{r^2}{\overline{z}}\right). \qquad ●$$

Thus we have the following theorem due to Nitsche.

Theorem 4.14. *For each $r \in (0,1)$, there is an $R_0(r) \in (0,1)$ such that if f is a univalent harmonic mapping from $A(r,1)$ onto $A(R,1)$, then $R \le R_0(r)$. Moreover, $R_0(r) \le \frac{2r}{1+r^2}$ and equality is achieved by the harmonic function*

$$f(z) = \frac{1}{1+r^2}\left(z + \frac{r^2}{\bar{z}}\right).$$

By this theorem it follows that, for a given $r \in (0,1)$, there exists no univalent harmonic mapping from $A(r,1)$ onto $A(R,1)$ if $R > 2r/(1+r^2)$. Example 4.13 led Nitsche to suggest the following conjecture.

Conjecture 4.15. $k(r) \le R_0(r) = 2r/(1+r^2)$.

Remark 4.16.

(1) Looking closely at Nitsche's proof in Theorem 4.12, Bshouty and Hengartner (see eg. [6]) observed that the proof does not use the univalency of f but rather the fact that $f(\gamma)$ contains a point in each of the vertical strips

$$\{w\colon R < \operatorname{Re} w < 1\} \text{ and } \{w\colon -1 < \operatorname{Re} w < -R\}.$$

As a consequence the proof also applies to a wider class of mappings that are not necessarily univalent and that admit a point in each of these vertical strips.

(2) The proof can also be applied to other image domains

$$\Omega' = \mathbb{D} \setminus [-R, R] \quad \text{or}$$
$$\Omega' = \{w\colon |w| > R,\ |\operatorname{Re} w| < 1\}, \quad R < 1.$$

Example 4.17. Consider

$$f(z) = z - \frac{1}{\bar{z}}, \quad z \in A(1,\infty).$$

Clearly, this function is obtained by dividing the function $f_t(z)$ in Example 4.13 by $t > 0$ and then allowing $t \to \infty$. We observe that

- f is harmonic for $|z| > 1$
- $J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2 = 1 - \frac{1}{|z|^4} > 0$ for $|z| > 1$
- $f(re^{i\theta}) = (r - 1/r)e^{i\theta}$

so that f maps concentric circles in $A(1,\infty)$ onto concentric circles in $\mathbb{C} \setminus \{0\}$. Thus, f is a univalent harmonic mapping of $A(1,\infty)$ onto $\mathbb{C} \setminus \{0\}$. Note also that

$$|\omega_f(z)| = \left|\frac{\overline{f_{\bar{z}}(z)}}{f_z(z)}\right| = \frac{1}{|z|^2} < 1 \text{ for } |z| > 1$$

and $|\omega_f(z)| = 1$ on $|z| = 1$. Therefore, f is not a quasiconformal mapping of $A(1,\infty)$ onto $\mathbb{C} \setminus \{0\}$ and hence is an example of sense preserving univalent harmonic mapping which is not quasiconformal. Note that the unit circle $|z| = 1$ is mapped onto the origin. ●

Using the extremal module property of the Grötzsch ring domain, Lyzzaik [20] presented a quantitative upper bound for $k(r)$, in terms for the Grötzsch ring domain $B(s)$, $0 < s < 1$, in the unit disk $\mathbb{D}$, which is the doubly-connected open subset of $\mathbb{D}$ whose boundary components are the unit circle $|z| = 1$ and the segment $\{x : 0 \le x \le s\}$.

Theorem 4.18. **[20]** *Let f be a univalent harmonic mapping from $A(r,1)$ onto $A(R,1)$, and let $B(s)$ be the Grötzsch ring domain that is conformally equivalent to $A(r,1)$. Then $R \le s$.*

It follows from this theorem that $k(r) \le s$. Then the idea of Lyzzaik is based on a number of observations. For example, we have

- $\operatorname{mod}(B(s)) = \mu(s)$.
- With r and s as Theorem 4.18, we have

$$\mu(s) = \operatorname{mod}(A(r,1)) = \log(1/r),$$

  by the hypothesis.
- Every ring domain Q is conformally equivalent to a unique annulus $A(1,r)$. This implies every annulus is conformally equivalent to a unique Grötzsch ring domain. This implies

$$\operatorname{mod}(Q) = \begin{cases} \log\left(\frac{1}{r}\right) & \text{if } 0 < r < 1 \\ \infty & \text{if } r = 0. \end{cases}$$

- mod is known to be conformally invariant and is strictly increasing with respect to the set inclusion. More precisely, if $Q \subset Q'$, then

$$\operatorname{mod}(Q) \le \operatorname{mod}(Q'),$$

  where Q' is another ring domain.

Define

$$w = \phi_r(z) = \frac{z + r}{1 + rz}.$$

Then we see that $|z| < 1 \iff |w| < 1$ and

$$|z| > r \iff \left|w - \frac{r}{1+r^2}\right| > \frac{r}{1+r^2}.$$

Set $B = \phi_r(A(r,1))$ which is clearly a ring domain. This gives

$$B \subsetneq B(s_0) \text{ with } s_0 = 2r/(1+r^2).$$

Also, we have

$$\text{mod}\,(B) = \text{mod}\,(A(r,1)) < \text{mod}\,(B(s_0))$$

so that $\mu(s) < \mu(s_0)$ which yields that

$$\frac{2r}{1+r^2} < s,$$

because μ is known to be a strictly decreasing function. Finally, a rough idea of the proof of Lyzzaik is to associate f with an analytic function of $\text{mod}\,(A(r,1))$ whose image surface embeds properly in a smooth doubly connected covering surface X of the plane $\mathbb{C}$. Such an association provides the inequality $\mu(s) < \text{mod}\,(X)$. Also, Lyzzaik has shown that $\text{mod}\,(X) \le \mu(R)$. A comparison of the last two inequalities implies that $s > R$ and the proof is complete. We advise the reader to refer the original paper of Lyzzaik [20] for technical details of the proof as well as for a number of interesting observations.

On the other hand, Weitsman [27] uses real analytic methods using the Green function of the annulus to get

Theorem 4.19. *If f is a univalent harmonic mapping from $A(r,1)$ onto $A(R,1)$, then*

$$R \le R_0(r) = \frac{1}{1+(r^2/2)(\ln r)^2}.$$

In [16], Kalaj improved the result of Weitsman to

$$R \le R_0(r) = \frac{1}{1+(1/2)(\ln r)^2}.$$

Remark 4.20.

(1) The article by Lyzzaik [20] also relates univalent harmonic mappings between annuli and doubly connected coverings

(2) Neither the result of Weitsman nor that of Kalaj is of any value for small values of r as

$$\lim_{r\to 0} r^2(\ln r)^2 = 0$$

so that $R_0(r) \to 1$ as $r \to 0$.

(3) Lyzzaik's estimate is asymptotically good for r close to 0.

(4) Weitsman's estimate is asymptotically exact for r close to 1. ●

In conclusion, Nitsche's conjecture remains an unsolved open problem.

5. Exercises

(1) Discuss the equality case of Theorem 2.2.

(2) Is it possible to map the domain $A(1,\infty)$ conformally or even quasiconformally onto $\mathbb{C}\setminus\{0\}$?

Hint: More general exercise is to show that

$$f_c(z) = z - \frac{1}{z} + c\ln|z|$$

are sense preserving univalent harmonic mapping of $A(1,\infty)$ onto $\mathbb{C}\setminus\{0\}$, whenever $|c| \le 2$, see Section 4 and a number of papers due to D. Bshouty and W. Hengartner, and many others [9].

(3) Define $f_t(z) = tz + \dfrac{1-t}{\overline{z}}$, $z \in \mathbb{D}\setminus\{0\}$. Show directly that f_t is univalent for $0 < |z| < 1$, whenever $t \ge \frac{1}{2}$.

Hint: For $z_1, z_2 \in \mathbb{D}\setminus\{0\}$ $(z_1 \ne z_2)$, observe that

$$\frac{f_t(z_1)-f_t(z_2)}{z_1-z_2} = t - (1-t)\left(\frac{\overline{z}_1-\overline{z}_2}{z_1-z_2}\right)\frac{1}{\overline{z_1 z_2}}.$$

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Conference on Vector Bundles in Honour of S. Ramanan (On the Occasion of his 70th Birthday)

June 16–20, 2008

Venue: Miraflores de la Sierra (Madrid, Spain).

This conference is devoted to the broad area of research interests of Professor S. Ramanan (Chennai Mathematical Institute, India) in celebration of his 70th birthday.

This is a joint activity of the Spanish Semester on Moduli Spaces (January–June 2008) and the international research group Vector Bundles on Algebraic Curves (VBAC).

The conference will take place at the Residencia La Cristalera in Miraflores de la Sierra (Madrid, Spain). Most participants will be accommodated in this residence. Talks will start on the morning of Monday 16th of June and will end by lunch time on Friday 20th of June.

Venue and Accommodation: The meeting will take place at a Residence and Conference Hall called La Cristalera. This is about 2 Km from Miraflores de la Sierra, a small village located 50 Km North of Madrid. Most participants will be accommodated in this residence.

For Further Details Visit:

`http://www.mat.csic.es/webpages/moduli2008/ramanan/index.html#contact`

Conference on Algebra and its Applications (In Honor of 70th Birthday of S. K. Jain)

June 18–21, 2008

Venue: Ohio University, Athens.

The last date for submission of abstracts is May 15, 2008.

OU Inn is designated as the conference hotel. The conference rate is $ 65 plus taxes per night. Reservation must be made by May 30, 2008 to get the conference rate.

For Help Regarding the Accommodation, You May Contact the Organizers:

Dinh Van Huynh
(`huynh@math.ohiou.edu`)

S. R. Lopez–Permouth
(`lopez@math.ohiou.edu`)

More Information Can Be Obtained At:

`http://www.math.ohiou.edu/~algebra/conference/ou.html`

46th International Symposium on Functional Equations

June 22–29, 2008

Venue: Opava–Hradec and Moravici, Czech Republic.

Topics: Functional equations and inequalities, mean values, functional equations on algebraic structures, Hyers–Ulam stability, regularity properties of solutions, conditional functional equations, functional-differential equations, iteration theory; applications of the above, in particular to the natural, social, and behavioral sciences.

Information: Participation at these annual meetings is by invitation only. Those wishing to be invited to this or one of the following meetings send details of their interest and, preferably, publications (paper copies) and/or manuscripts with their postal and E-mail address to:

R. Ger
Institute of Mathematics, Silesian University, Bankowa 14
PL-40-007 Katowice, Poland
(`romanger@us.edu.pl`) before February 15, 2008.

Local Organizer
Jaroslav Smital, Mathematical Institute,
Silesian University,
Na Rybníèku 1, CZ-74601 Opava
Czech Republic
E-mail: `isfe46@math.slu.cz`.

Homotopical Group Theory and Topological Algebraic Geometry

June 23–27, 2008

Venue: Max Planck Institute for Mathematics Bonn (Germany)

Conference Topics: The conference focuses on the new interactions of Algebraic Topology with Group Theory, Algebraic Geometry and Mathematical Physics which come from looking at these fields through the eye of a homotopy theorist. It celebrates one of the contributors to the subject by honoring the 60th birthday of Haynes Miller (MIT). One week before the conference there will be a workshop at the University of Copenhagen with lecture series by Bill Dwyer (Notre Dame) and by Paul Goerss (Northwestern).

Registration: All participants are required to complete the registration page. This applies to the conference and the workshop. The registration deadline is 01/31/08.

Schedule: The talks start Monday morning June 23rd and end Friday afternoon June 27th. A more detailed schedule will be posted here in due course.

Financial Support: The conference is partially supported by the DFG Graduiertenkolleg 1150 "Homotopy and Cohomology" and the Max Planck Institute Bonn. The workshop is funded by the University of Copenhagen. Support from the National Science Foundation for beginning researchers is under application.

For Further Details Visit:

`http://www.ruhr-uni-bochum.de/topologie/conf08/`

Analysis, PDES and Applications. (On the Occasion of the 70th Birthday of Vladimir Maz'ya)

Venue: INDAM
(Istituto Nazionale di Alta Matematcia "Francesco Severi")
— Università degli Studi di Roma "La Sapienza"
Roma, Italy

Topics: Linear and non-linear PDEs, Asymptotic and numerical methods for PDEs including homgenization and boundary elements, Spectral theory, Harmonic Analysis, Approximation theory, Wavelets, Elasticity Theory, Function Spaces, Ill-posed problems, Non-linear potential theory, Fluid Mechanics, History of Mathematics.

Deadlines:

April 30, 2008 – deadline for submission of abstracts
May 15, 2008 – deadline for lower registration rates
June 30, 2008 – start of the conference

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For Further Details Visit:

`http://www.mat.uniroma1.it/ mazya08/index.html`

VIII International Colloquium on Differential Geometry

July 7–11, 2008

Venue:

Faculty of Mathematics,
University of Santiago de Compostela (Spain)

The Department of Geometry and Topology (Univ. of Santiago de Compostela, Spain) announces the celebration of the "VIII International Colloquium on Differential Geometry (E. Vidal Abascal Centennial Congress)".

This new edition of the series of International Colloquia on Differential Geometry of Santiago de Compostela (Spain) is thought as a celebration of the centenary of the birth of Prof. Enrique Vidal Abascal (1908–1994), organizer of the three first Colloquia in the series, held in 1963, 1967 and 1972, and who made, with this innovator initiative, a fundamental contribution to the beginning of the change of mathematical research in Spain in the second half of the XX century.

This new Colloquium will offer the participants an opportunity for the presentation of their more recent results in the framework of the two sections in which the Colloquium will be scheduled:

Section 1: Foliation Theory
Section 2: Riemannian Geometry

At the same time, plenary lectures will be scheduled in the charge of prestigious mathematicians who will provide a view of the "state of the art" in the most specific areas of their personal interest, always in the framework of both sections.

Deadlines For:

Reduced registration	:	May 15th, 2008
Ordinary registration	:	June 30th, 2008
Student registration	:	June 30th, 2008
Accommodation grant	:	March 31st, 2008
Contribution	:	April 15th, 2008

Contact Address:

`http://xtsunxet.usc.es/icdg2008/benvida.html`
E-mail at `icdg2008@usc.es`

Conference on Modules and Representation Theory (Babes–Bolyai University Cluj–Napoca)

July 7–12, 2008

Location: Babes–Bolyai University, Cluj–Napoca, Romania

Description: As described in the title, this is an Algebra Conference with special emphasis to Abelian Groups, Modules and Representation Theory (including the usual AGRAM on 2008).

Important Dates:

Deadline for registration and submission of abstracts	:	May 15, 2008
Deadline for submission of manuscripts	:	October 1, 2008

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Web Site:

`http://math.ubbcluj.ro/~aga_team/AlgebraConferenceCluj2008.html`

First Announcement for the International Conference on Ring and Module Theory

August 18–22, 2008

Venue: Hacettepe University, Ankara, Turkey

Dead Line:

Submission of title and abstracts	:	May 1, 2008
Preregistration	:	May 1, 2008

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http://www.algebra2008.hacettepe.edu.tr/index.htm

International Conference on Complex Analysis and Related Topics

The XI$^{\text{th}}$ Romanian–Finnish Seminar Alba Iulia, Romania

August 14–19, 2008

First Announcement

Within the tradition of the collaboration between the Romanian and Finnish mathematicians initiated by Rolf Nevanlinna (1895–1980) and Simion Stoilow (1887–1961), a conference on Complex Analysis and Related Topics, the XI$^{\text{th}}$ Romanian–Finnish Seminar, will take place in Alba Iulia (Romania), August 14–19, 2008.

The organizers are the Institute of Mathematics "Simion Stoilow" of the Romanian Academy, "1 Decembrie 1918" University of Alba Iulia, the University of Bucharest and the Faculty of Mathematics and Informatics of the "Babeş–Bolyai" University of Cluj–Napoca, in cooperation with the Universities of Helsinki, Joensuu and Jyväskylä from Finland.

The following sections will be included:

1. Analytic functions of one complex variable
   1A. Univalence,
   1B. Other subjects;
2. Quasiconformal mappings and Teichmüller spaces;
3. Several complex variables;
4. Potential theory;
5. Functional analytical methods in complex analysis.

There will be plenary lectures, lectures in sections and short communications.

Alba Iulia (Apulum) is located 350 km north of Bucharest in Transylvania near the Western Carpathians Mountains. The two millennium old city of Alba Iulia was the capital of Dacia–Apulensis (first century) and Transylvania Principality (sixteenth century).

The nearest airports are those from Sibiu and Cluj–Napoca.

The **preliminary registration** may be done online before **January 31$^{\text{st}}$, 2008**, following the link in the web page of the conference:

http://www.imar.ro/~purice/conferences/

The participation fee is 50 Euro (covering the conference materials, refreshment, opening reception), to be payed at the registration desk.

More details will be given in the Second Announcement and in the web page of the conference.

Address:

Complex Analysis and Related Topics
c/o Institute of Mathematics
"Simion Stoilow" of the Romanian Academy
P. O. Box 1-764, 014700, Bucharest, Romania
Fax: +40 21 319 65 05
E-mail: rofinsem@gmail.com

M. T. & T. S. 2008
16th Mathematics Training and Talent Search Programme

Funded by

National Board for Higher Mathematics

Aim: The aim of the programme is to expose bright young students to the excitement of doing mathematics, to promote independent mathematical thinking and to prepare them for higher aspects of mathematics.

Academic Programme: The programme will be at three levels: Level O, Level I and Level II. In Level O there will be courses in Linear Algebra, Analysis and Number Theory/ Discrete Mathematics. In Levels I and II there will be courses on Algebra, Analysis and Topology. There will be seminars by students at all Levels.

The faculty will be active mathematicians with a commitment to teaching and from various leading institutions. The aim of the instructions is not to give routine lectures and presentation of theorem-proofs but to stimulate the participants to think and discover mathematical results.

Eligibility

Level O: First and second year undergraduate (B. Sc./B. Stat./ B. Tech. etc.) students with Mathematics as one of their subjects.

Level I: Final year undergraduate (B. Sc./B. Stat./B. Tech. etc.) students with Mathematics as one of their subjects. Bright second year students may also apply.

Level II: First year postgraduate (M. Sc./M. Stat./M. Tech. etc.) students with Mathematics as one of their subjects. Bright final year undergraduate students may also apply.

Venues and Duration: There are two venues for MTTS 2008. All the three levels will be held at the Regional Institute of Education (RIE), Mysore during the period May 19–June 14, 2008.

Level O of the Programme will also be conducted at the Department of Mathematics, Indian Institute of Technology, Guwahati, during the period May 26–June 21, 2008.

How to Apply?

Details and application forms can be had from the Head, Department of Mathematics, of your Institution. They can also be downloaded from MTTS Websites given below. In case of difficulties, write to Professor S. Kumaresan by sending a self addressed and stamped (Rs. 10) envelope of size 10 cm × 22 cm. Write "*MTTS-2008 Application Form*" on the cover of your letter. The completed application form should reach the programme director latest by 23rd February, 2008.

Selection: The selection will be purely on merit, based on consistently good academic record and the recommendation letter from a mathematics professor closely acquainted with the candidate. Only selected candidates will be informed of their selection by the 3rd week of March 2008. The list of selected candidates will be posted on the Websites of MTTS.

Candidates selected for the programme will be paid sleeper class return train fare by the shortest route and will be provided with free board and lodging for the duration of the course.

Home Page

http://www.geocities.com/mttsprogramme
http://mathstat.uohyd.ernet.in/~mtts/

Programme Director

Professor S. Kumaresan
Programme Director, MTTS
Depat. of Maths & Stats
University of Hyderabad
P. O. Central University
Hyderabad 500 046
Email: kumaresa@gmail.com
mttsprogramme@gmail.com

Last Date for Applying 23rd February, 2008.

Venues for MTTS 2008

RIE Mysore (All levels)
IIT Guwahati (Level O)

Duration of MTTS 2008

Mysore: May 19–June 14, 08
Guwahati: May 26–June 21, 08

Ramanujan Mathematical Society Lecture Notes Series in Mathematics

Ramanujan Mathematical Society (RMS) is publishing the DST-sponsored RMS Lecture Notes Series in Mathematics. This series publishes monographs and proceedings of conferences which report on important developments in Mathematics. This series provides an outlet for publication of vast amount of important material that is generated from conferences, seminars and courses held in different parts of the world.

Proceedings of Conferences which focus on a topic of current research, tutorials and instructional workshops will be considered suitable for publication in this series.

Monographs which consist of notes from seminars and courses, exposition of important topics for which there is no readily available text-books will also be considered for publication in this series.

Volumes 1–4 in this series have already been published and made widely available all across the world and at an affordable price within India. The details are as follows:

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Vol 2.: The Riemann Zeta Function and Related Themes: Papers in honor of Professor K. Ramachandra (Proc. of International Conference held at National Institute of Advanced Studies, Bangalore during 13–15 December, 2003)

Volume Editors: R. Balasubramanian and K. Srinivas.
ISBN No.: 978-81-902545-2-6
Price: In India: Rs. 350/- + Rs. 25/- (Postage)
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Volume Editors: Kamala Krithivasan and R. Rama.
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Price: In India: Rs. 350/- + Rs. 25/-(Postage)
Outside India: U. S. $ 35

Vol 4.: "Commutative Algebra and Combinatorics" Proceedings of the International Conferences organized by Bhaskaracharya Pratishthana, Pune and Harish–Chandra Research Institute, Allahabad, during 8–13 December, 2003.

Part 1 – Authored by Diane Maclagan and Rekha R. Thomas
Part 2 – Edited by Winfried Bruns

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For Details See the Ramanujan Mathematical Society Website (www.ramanujanmathsociety.org)

First Announcement
Asian Mathematical Conference

June 22–June 26, 2009

Introduction: The Asian Mathematical Conference (AMC) series is a major South East Asian Mathematical Society (SEAMS) conference, held every 4–5 years and hosted by countries in Asia. The first AMC was held in Hong Kong (1990), the second in Thailand (1995), the third in the Phillipines (2000) and the latest was in Singapore (2005). Malaysia will be hosting the conference in 2009.

The School of Mathematical Sciences, Universiti Sains Malaysia, together with the Malaysian Mathematical Sciences Society and Mathematics Departments of Malaysian Public Universities will be organizing the 5th Asian Mathematical Conference from June 22–June 26, 2009 at Putra World Trade Center in Kuala Lumpur.

Objective: The objective of the conference is to provide a forum for mathematics researchers from Asia to foster links and collaborations among themselves and with mathematicians from other parts of the world through discussion of issues, exchange of ideas and the presentation of research findings.

Focus Areas: All major areas of mathematics including Mathematics Education.

Activities:

- Keynote address by international renowned mathematicians
- Invited talks by prominent regional mathematicians
- Contributed Papers
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Organizer:

- Malaysian Mathematical Sciences Society
- Universiti Sains Malaysia
- Mathematics Departments of Malaysian Public Universities

Collaboration With:

- Ministry of Higher Education, Malaysia

Conference Website: http://math.usm.my/amc2009/

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