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Developing Knowledge in Mathematics by Generalising and Abstracting

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Why Teach Mathematics?

Everybody who is involved in teaching mathematics has probably at some point asked themselves this question. Why should the students I am teaching learn mathematics? What kind of mathematics should they learn, and how much? Maybe one will also ask, *how* should I teach mathematics, meaning what aspects of the nature of mathematics should I emphasise when I teach? What kind of understanding do I want my students to develop? Depending on the type of students one is teaching these questions may have different answers. If the students are studying for a particular professional career some of the questions may have rather obvious answers but if the questions concern the teaching of pupils and students within the compulsory schooling system the answers may not be so obvious. In this case the central question will be *what kind of mathematics, and how much, do all people in a modern society need to know?*

It can be argued that mathematics is becoming more and more important in the society. More and more decisions are made and actions are being taken on the basis of mathematical models. This is due to the fact that the existence of computer technology has led to larger and larger parts of the society being influenced by mathematics. Computer technology is made possible because of mathematical knowledge, and increased use of technology requires more and more mathematical knowledge. Therefore one can say that mathematics and technology reinforce one another, and the society is mathematised (Gellert, Jablonka, & Keitel, 2001). On the other hand the existence of computers and calculators reduces the need for the individual member of the society to perform mathematical tasks in his/her daily life. Routine calculations are taken over by the machines, and I can use my computer and the equipment connected to it without understanding the mathematics that lies behind the construction of this equipment. I can also understand and make use of the weather forecast without understanding the mathematical models that have been used to develop it. Therefore one can say that although the society as such is mathematised, the individual citizen is demathematised. This has been expressed as the relevance paradox in mathematics (Niss, 1994). Mathematics has an objective relevance in the society in the sense that the importance of mathematics is growing but a subjective irrelevance in the sense that the need for mathematical knowledge in daily life seems to be decreasing.

Clearly only a small proportion of the citizens in a society will need mathematical knowledge on the level of a research mathematician or for developing advanced technology. Based on this it could be argued that the majority of people do not need to learn much mathematics because one can manage one's daily tasks with only a basic knowledge. Still mathematics is a large subject in compulsory schooling in most countries and this fact shows that most societies are of the opinion that it is important to spend time and money so that the children can learn mathematics. Also governments and politicians as well as the general public are concerned when international tests like PISA (OECD, 2006) and TIMSS (IEA, 2007) show bad records for their country. So why is mathematics for all still important? Paul Ernest (2000) has listed four possible purposes of mathematics in school:

1. To reproduce mathematical skill and knowledge-based capability
2. To develop creative capabilities in mathematics
3. To develop empowering mathematical capabilities and a critical appreciation of the social applications and uses of mathematics
4. To develop an inner appreciation of mathematics – its big ideas and nature (Ernest, 2000, pp. 10–14)

Looking at old school curricula one can get an impression of the changing role of mathematics in basic schooling. In Norway a monumental school reform was carried out in 1939. This resulted in a national curriculum based on the leading pedagogical ideas of the time. This curriculum also had a political side to it, based on social democratic ideas, reflecting the fact that Norway for the first time got a stable social democratic government in 1935 (Telhaug & Mediås, 2003). The national curriculum of 1939 contained many pedagogical ideas that even today would be considered as modern. According to this curriculum the teachers should not work with the subjects strictly separate but instead they are encouraged to work with broader topics incorporating several subjects at the same time. Further it is required that the pupils should be active and independent in the sense that they should learn how to obtain information and material that they need for their work themselves. It is also emphasised that the goal of the education is to teach the pupils in ways that in the best possible manner suit each pupil's abilities (Normalplannemnda, 1964).

I will now turn to the role of mathematics, a word which by the way was not used in the curriculum from 1939. At that time compulsory schooling in Norway ended at age 14. After age 14 the children could either leave school or they could choose between various types of vocational training and more theoretical schools where essentially the ultimate goal was to prepare for university studies. Only in these theoretical schools was the word mathematics used, and to a large extent one can say that the main difference from earlier schooling was the introduction of algebra. In the compulsory school the subject was called *rekning*, corresponding to the English term *arithmetic*. The word 'rekning' literally means computing or calculating. In the national curriculum of 1939 the following goals are set for the subject rekning:

1. To help the pupils to understand the common numbers (whole numbers, decimal numbers and common fractions) and to use the numbers in a sensible way in simple computations, so that they in swift, practical and secure ways can solve simple computing tasks required in daily life, and to explain the computations by displaying them in a clear way.
2. To give the pupils knowledge of the shape and size of the most common surfaces and objects, for example by letting the pupils practice measuring simple surfaces and objects and compute their area and volume. (Normalplannemnda, 1964, p. 137, my translation)

Central to these goals is the ability to do computations quickly and securely. In daily life, and also in many professions, it was important to perform computations and to be able to trust that the answer was correct. It was also important to display the computations so as to make it easier, both for oneself and for others, to check for and detect possible errors.

Going back to Ernest's (2000) four points listed earlier, the purpose of mathematics (rekning) in the Norwegian curriculum of 1939 is essentially based on reproducing. One could even argue that the main purpose expressed in the curriculum, managing daily tasks, is missing from Ernest's list. Today mathematics in school is often justified by expressing the need for *mathematical literacy*. This term is for example used in the PISA framework where it is expressed in the following way.

> *Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen.* (OECD, 2006)

Mathematical literacy is also sometimes taken to include mathematics as a way of thinking, including looking for patterns and structures, classifying and formalising, conjecturing, arguing and proving, and also including the ability to design and apply mathematical models (Gellert, Jablonka, & Keitel, 2001).

Taken together, the content of the concept mathematical literacy described above could be said to address points 2 and 3 on Ernest's list and, depending on the actual content and ways of presenting it, it could also address point 4 on the list. In short, the creative and critical aspects of mathematics tend to be recognised as important. The critical aspect of mathematics has been written about at length by for example Skovsmose (1994) and I will not go further into this here. I will however discuss the creative aspect, including generalising, conjecturing, proving and refuting. The ability to generalise is central to developing knowledge in many fields but it could be argued that it is in particular important for developing knowledge in mathematics. At least it is fair to say that it is definitely very important in mathematics, without taking into consideration whether it is more or less important in other subjects. I will include two quotes from John Mason that express this very strongly. He writes (Mason, 1996) that "[g]eneralization is the heartbeat of mathematics, and appears in many forms" (p. 65) and further: "Lessons that are not imbued with generalization and conjecturing are not mathematics lessons, whatever the title claims them to be" (p. 84). I will return to the aspect of generalisation later in the paper.

In concluding this section I will say that it seems that mathematics is recognised by a large number of countries in the world as being important, and as a consequence of that lots of resources are spent on teaching mathematics on all levels. Still there is a great concern in many countries that pupils and students do not learn mathematics well enough. Some countries, like Norway, experience that they end up further down the ranking list for each round of tests in for example PISA. In addition it could be added that there is a lot of research going on in mathematics education. The amount of literature about educational aspects of mathematics (didactics of mathematics) is probably larger than in any other school subject. Despite all these efforts the situation for the subject does not seem to be satisfactory. So if one thinks that one has found a reasonable answer to the question why learn mathematics, it is relevant to ask why is it so difficult to learn mathematics?

Mathematical Objects

It is commonly recognised that many learners, in all age groups, struggle with learning mathematics. What are the obstacles to learning mathematics? One might argue that one obstacle is the complexity of the mathematical language. However, a complex professional language is not special for mathematics. Every subject has its own particular concepts, but unlike most other subjects, mathematics is characterised by the importance of semiotic representations, and indeed the large variety of semiotic representations. If one thinks about a mathematical object, be it the most basic number concept or a more advanced concept like for example *function*, one will realise that the object itself is an abstract entity which cannot be accessed apart from through some representation. Duval (2006) claims that the fact that mathematical objects are only accessible through their semiotic representations (signs and symbols), and that there is a need for transformations between representations, are the most distinctive features of mathematical knowledge. This is also emphasised by David Pimm when he writes that "[t]he symbolic aspect of written mathematics is one of the subject's most apparent and distinctive features" (Pimm, 1991, p. 19). Signs and symbols have two functions. Firstly, they have a semiotic function: A sign is something (a signifier) that stands for something else (the signified). Secondly, they have an epistemologic function: The sign contains knowledge about the object it represents – the signified. The signs and symbols make it possible to talk about the mathematical objects but they *are* not the mathematical objects, they are only *representations* of them. This is well illustrated by Steinbring (2006) in his epistemological triangle.

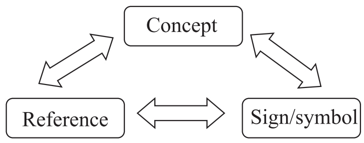

Figure 1.

Here the mathematical concept or object is placed in one corner and the sign or symbol (representation) in another corner. In the third corner are specific references to which the concept is applied. As an example, take the concept of cardinal number, for example the number five. This is the abstract object occupying the top corner of the triangle. In the lower right hand corner can be put the sign 5, the sign ||||| (tally marks), or the sign V (roman numeral). In the lower left hand corner can be put a group of five children or a heap of five apples on the table. A child in the process of developing the number concept needs to move between the various corners of the triangle, and she also needs to move between different references and different signs and symbols, all representing the same mathematical object. In a process of concept development, the content of one corner of the triangle may move to another corner. A hand with all the fingers stretched out could at one point be a sign representing a group of five children (reference). Later the numeral 5 could be a sign representing the fingers on the hand. Then the fingers move from being a sign to being a reference – the signifier becomes the signified. I will return to this in the next section.

To develop understanding of a concept there is often a need to be able to work with a variety of representations and to do transformations from one semiotic representation to another. Because different representations bring different features of the mathematical object to the fore it is not a question of finding the "correct" representation. This is again one of the obstacles to learning mathematics, as expressed by Duval: "Changing representation register is the threshold of mathematical comprehension for learners at each stage of the curriculum" (Duval, 2006, p. 128). The need for different semiotic representations is very well described in the case of functions where the transformations between situation, formula, graph and table were discussed by Claude Janvier about 20 years ago (Janvier, 1987). It is easy to realise that if work with functions were to be restricted to only one of these representations, many important aspects of functions would be lost.

Generalisation and Abstraction

The process of developing understanding for a concept is characterised by going towards higher abstraction. In this process the role of signifier and signified will change, as I also pointed out in the previous section. What is a signifier at one level will become the signified at the next level. I will illustrate this with an anecdote. In a social setting I was talking to a little girl and I asked her how old she was. She replied by holding up all fingers on one hand. Her fingers were in this case the signifier for her age (the signified). I followed up by asking "how much is this?", and she replied by pointing to each finger on the hand she held up and simultaneously uttering "one – two – three – four – five". I interpret this as the counting procedure (pointing and saying out number words) being the signifier and the fingers being the signified. We did not pursue this further so I have no evidence to say whether or not she made any connection between the counting procedure and her age. If she did this is an example of how the previous signifier (the fingers) becomes the new signified, and the new signifier (the counting) becomes a signifier both for the fingers and her age. This is an example of a semiotic chain, or chain of signifiers, where "[t]he new signifier stands for all that went before" (Presmeg, 2002, p. 302). An abstraction process is characterised by building semiotic chains where the new signifiers cover more and more aspects of the concept than the previous signifiers. If the girl in the anecdote did not make the connection between the counting and her age the episode is an example of a case where the semiotic chain is not yet established. At some point it will certainly be established and the further development that will take place is that the last uttered word – five – will become a signifier for the result of the counting process (the concept of cardinality), and later the sign 5 will become a signifier for the word 'five'. A similar example is discussed in (Walkerdine, 1988, pp. 129–130).

If emphasis is placed on the purpose of mathematics expressed by the second point on Paul Ernest's list, to develop creative capabilities in mathematics, the ideas of generalising and justifying are very important. Generalising often has to do with seeing a pattern, being able to extract certain properties from one or more examples while ignoring others. Here is a simple example.

$$\begin{aligned} 3+4+5 &= 3\cdot 4 \\ 4+5+6 &= 3\cdot 5 \\ &\vdots \end{aligned}$$

What is the next line? Going from one line to the next, what is the same and what is different? What is invariant, what is changing?

Starting from a simple example like this one can generalise by asking what could be changed without obstructing the essential features of the pattern? Try for example by changing the number of terms on the left hand side. Another variation could be to change the difference between the terms. What if the terms are not consecutive? Try 3 + 5 + 7 or 3 + 6 + 9. What is the same and what is different now? The ideas that lie behind this way of thinking about generalising are much inspired by the writings of John Mason. See for example (Mason, 2005, 2006a) for a more detailed exposition.

The Role of Algebra

I have said that many pupils and students have problems learning mathematics. I will argue that to a large extent the problems can be narrowed down to problems having to do with learning algebra. Many pupils have little or no problems with mathematics in the early school years. They might even enjoy working with mathematics but at a certain point they start to feel that they do not master it and they also often start to find it meaningless. The point when this happens often coincides with the introduction of algebra into the curriculum. My introductory question "Why teach mathematics?" could then be replaced by "Why teach algebra?" Or maybe it is better to ask the question "What aspects of algebra should be taught?" Kaput (1999) has formulated the following four aspects of algebraic thinking.

1. Algebra as the generalization and formalization of patterns and constraints
2. Algebra as syntactically guided manipulation of (opaque) formalisms
3. Algebra as the study of structures abstracted from computations and relations
4. Algebra as the study of functions, relations, and joint variation
5. Algebra as a cluster of modeling and phenomena-controlling languages

Looking back at the concept of mathematical literacy it can be said that the idea of generalising, working with patterns and structures, classifying and formalising, conjecturing, arguing and proving fit very well with the aspects of algebra as expressed by Kaput. Also designing and applying mathematical models can be found in the list. Mason links the problems learning algebra very strongly to the lack of working with generalisation when he conjectures that "when awareness of generality permeates the classroom, algebra will cease to be a watershed for most people" (Mason, 1996, p. 65).

In order to work with mathematical models it is necessary to have a good understanding of functions, and working with functions beyond an intuitive level requires a good basis of algebra. Therefore, if the students have not developed a good understanding of algebra, further work will be difficult because, to use the language developed earlier in the paper, the semiotic chain will be broken.

Expressing generalities is part of conjecturing, from conjecturing follows proving or refuting, and algebra is a tool to express and manipulate generalities (Mason, 1996). It is not obvious what it means for a learner to prove a conjecture. At an early stage a child may be convinced that a property is true in general by trying out a few examples, a level of proving referred to by Balacheff as naïve empiricism (Balacheff, 1988). From the teacher's point of view this will not be sufficient, and therefore the learner has to go through a development process in order to obtain a different view on what it means that a property is proved to hold in general. To be able to express a proof on what Balacheff refers to as the generic example level, using algebra is sometimes necessary, or at least it will make the way of expressing the proof more efficient. Therefore it can be said that algebra is fundamental to mathematical work, and to develop mathematical literacy.

Pause for an Example

To illustrate the role of algebra in generalising and proving, look at the following problem.

Two numbers have sum equal to 1. Which will be larger, the square of the larger number plus the smaller, or the square of the smaller number plus the larger?

How did you approach the problem? By looking at an example? Which numbers did you choose? Would you reach the same conclusion if you used other number? How can algebra help to reach a conclusion which holds for any choice of numbers?

Changing the problem a little might show the power of algebra even clearer. Does it matter that the sum of the two numbers is 1? What if it is not 1? What conclusion can be drawn in that case? What is the same and what is different? Does it matter if the sum is larger than one or smaller than one? For further discussion and variations on this problem see (Mason, 2006b).

The historical roots of algebra go back to ancient times in connection with solving equations. Word problems involving one or more unknown quantities which cannot be determined by direct computing on the known quantities, can be found in many cultures. By introducing a symbol, usually a letter, for the unknown quantity (or quantities) the problem will often lead to an equation or a system of equations which can be solved. This is a different aspect of algebra which is not emphasised in what I have discussed above. This aspect is also not so salient in Kaput's list, except that it may come in as a final step for example in a modelling or a problem solving context. In generalising tasks the symbol (letter) that is introduced represents a generalised number (variable) and not an unknown quantity which can be determined. In order to develop the powers of generalising, arguing and proving it is important to foster the understanding of the *variable*.

The Four Dimensional Cube

In this final section I will work with an example where I intend to show the power of algebra in generalising. The example is not new but hopefully it is sufficiently unknown to create some interest and perhaps some surprise in the readers.

In his book "What is mathematics, *really?*" Reuben Hersh (1997) asks the question if a four dimensional cube exists, and he also asks in what sense can we say that a three dimensional cube exists? I will now discuss these questions in light of the theory that I have outlined in this article. If I regard the three dimensional cube as a mathematical concept it belongs in the top corner of Steinbring's (2006) epistemological triangle (Figure 1), and according to Duval (2006) it is only accessible through its representations. For the 3D cube there are several representations available – one could build a physical representation in wood or some other material, one could draw a sketch on a paper or on a computer, one could use the Schläfli symbol $\{4, 3\}$ (Coxeter, 1963/1973), or one could list the number of vertices ($V = 8$), edges ($E = 12$) and faces ($F = 6$).

Going back to the question of whether the 4D cube exists, this may be rephrased into the question "How could the 4D cube be represented?" Clearly it is not possible to build a physical representation but through a process of generalisation or abstraction from the 3D case it might be possible to come up with a representation in terms of the number of vertices, edges and faces, and to justify this representation. In this process it might be helpful to look at the "cube" in lower dimensions than three in order to identify a structure. The analogue in 2D will of course be the square, in 1D the segment and in 0D the point. A starting point for a generalisation into four dimensions could then be the following table.

Dimension	0	1	2	3	4
4D spaces (S)					1
3D spaces (R)				1	R_4
Faces (F)			1	6	F_4
Edges (E)		1	4	12	E_4
Vertices (V)	1	2	4	8	V_4

Figure 2.

It is natural to imagine that in the same way as the 3D cube contains 6 squares (2D cubes), the 4D cube will contain a certain number of 3D cubes (3D spaces) and that the 4D cube itself will have one 4D space. The question is what numbers should be put instead of R_4, F_4, E_4, and V_4? This problem is discussed for example in chapter 7 in the book by Coxeter (1963/1973). See also Wolfram MathWorld (`http://mathworld.wolfram.com/`) and search for hypercube. In his book Coxeter says that the Euclidian geometry in four or more dimensions can be approached in three ways: the axiomatic, the algebraic (or analytical), and the intuitive. About the intuitive way he says that "[t]his intuitive approach is very fruitful in suggesting what results should be expected. However, there is some danger of our being led astray unless we check our results with the aid of one of the other two procedures" (p. 119).

Although there is nothing mathematically new in this example I find it worthwhile to present it because it has proved to be a rich activity for generalising. In the cases that I have used it, both for students and for teachers with a generally strong mathematical background, it has turned out to work as a genuine inquiry process because the solution has not been known to them beforehand. The first observation is usually that the number of vertices seem to double from one dimension to the next, so the first conjecture is that $V_4 = 16$. This could be called an algebraic approach, using Coxeter's (1963/1973) terms. To get further it is often helpful to visualise how a 3D cube is obtained from a square. This could be called an intuitive approach. Imagine a square lying on the table. Now make a copy of this and lift the copy up. Then the number of vertices, edges and faces have doubled. To get a 3D cube, join the vertices in the copy and the original with new edges, and join the edges in the copy and the original with new faces. The result of this process will be

$$\begin{aligned} V_3 &= 2V_2 \\ E_3 &= 2E_2 + V_2 \\ F_3 &= 2F_2 + E_2. \end{aligned}$$

Imagine getting the 4D cube from the 3D cube in a similar way. Then the following conjecture, in addition to $V_4 = 2V_3$, will emerge:

$$\begin{aligned} E_4 &= 2E_3 + V_3 \\ F_4 &= 2F_3 + E_3 \\ R_4 &= 2R_3 + F_3. \end{aligned} \tag{1}$$

This gives the following table:

Dimension	0	1	2	3	4
4D spaces (S)					1
3D spaces (R)				1	8
Faces (F)			1	6	24
Edges (E)		1	4	12	32
Vertices (V)	1	2	4	8	16

Figure 3.

See also (Coxeter, 1969) where the entries for the 4D cube (called hypercube) can be found on page 414. On page 398 in the same book an attempt is made to draw a figure of the hypercube.

When I work with this example in a group, at some point I ask the participants to add one more line in the table. In this line I ask them to compute the sum of all the components in each dimension. For the dimensions 0-3 this gives the sequence 1, 3, 9, 27. If at this point there are still holes in the table for dimension 4 the conjecture that the sum of the components in dimension 4 should be 81 will readily appear.

Pause for Reflection

How can it be explained that adding all the components of a cube of a certain dimension seems to give 3 raised to the dimension? I will return to this question later.

On one occasion one of the participants in the group observed that the number of faces in dimension four, 24, also could be obtained by adding the number of faces in dimensions 2 and 3 and the number of edges in dimensions 1, 2, and 3, and that a similar approach could be used to obtain all the other numbers. This gave the conjecture below.

$$\begin{aligned} E_4 &= E_1 + E_2 + E_3 + V_0 + V_1 + V_2 + V_3 \\ F_4 &= F_2 + F_3 + E_1 + E_2 + E_3 \\ R_4 &= R_3 + F_2 + F_3. \end{aligned} \tag{2}$$

Clearly (2) gives the same result as (1) up to dimension 4 but do they really represent the same algebraic structure? By that I mean, will (1) and (2) give the same result if they are extended to dimensions higher than four? It is easy to generalise (1) and (2) to dimension n. To do this I will denote the components of the cube in dimension n by $S_i^{(n)}$, $i = 0, \ldots, n$, i.e. $S_0^{(n)}$ denotes the number of vertices, $S_1^{(n)}$ the number of edges and so on. Then (1) can be written

$$S_i^{(n)} = 2S_i^{(n-1)} + S_{i-1}^{(n-1)}, \quad i = 1, 2, \ldots, n-1, \tag{3}$$

and (2) can be written

$$S_i^{(n)} = \sum_{j=i}^{n-1} S_i^{(j)} + \sum_{j=i-1}^{n-1} S_{i-1}^{(j)}, \quad i = 1, 2, \ldots, n-1. \tag{4}$$

Both expressions can be applied from $n = 2$ on. The expression for the vertices will now be written

$$S_0^{(n)} = 2S_0^{(n-1)}.$$

In addition I take $S_i^{(i)} = 1, \ i = 0, 1, \ldots n$.

Open Question

Do (3) and (4) represent the same algebraic structure? I will leave this question for the moment. It could be a possible task to work on.

The Euler Formula

For polyhedra in three dimensions the Euler formula, $V - E + F = 2$, is well known. It is an interesting discussion to investigate for which polyhedra this formula really holds, and indeed, what should be understood by a polyhedron? This problem is thoroughly discussed in the book by Lakatos (1976) and I shall not go further into that here. It is not a problem to agree that the formula holds for the 3D cube and the interesting question here is what will be the analogue of this formula in higher dimensions?

A natural approach might be to look at the number 2 on the right hand side of the formula? What is this 2? Is it a constant? Is it perhaps $3 - 1$, where 3 represents the dimension? Or may be it is 2 times 1, or $1 + 1$? If so, what are the 1s? This is a situation where it is appropriate to ask the question what stays the same and what is changing when going from the particular 3D case to the general nD case.

Looking at the lower dimensions might give a clue. In dimension 2 the corresponding formula would be $V - E = 0$, or may be one should write it as $V - E + 1 = 1$, which could be interpreted as $V - E + F = 1$ since $F = 1$ in dimension 2. If faces are introduced in the 2D formula then it is natural to introduce 3D spaces in the 3D formula, and the original Euler formula should perhaps be written $V - E + F - R = 1$? If this is the case then the number 2 is really $1 + 1$, where one of the 1s is a constant and the other denotes the number of 3D spaces.

On the basis of this discussion it could be tempting to formulate the conjecture

$$\sum_{i=0}^{n} (-1)^i S_i^{(n)} = 1.$$

Checking for $n = 4$ it is easily seen that this holds. According to Coxeter (1963/1973, p. 165) the connection was known to hold in general for regular polytopes[1] by many people in the last half of the 19th century, and a proof was given by Poincaré in 1893.

Proofs of Conjectures and Observations

The Two Recursion Formulae are Equal

I will now return to the question whether (3) and (4) represent the same structure. For $n = 2$ the equation (3) gives

$$S_1^{(2)} = 2S_1^{(1)} + S_0^{(1)}$$

and (4) gives

$$S_1^{(2)} = S_1^{(1)} + S_0^{(0)} + S_0^{(1)} = 2S_1^{(1)} + S_0^{(1)},$$

using that $S_1^{(1)} = S_0^{(0)}$. Hence the two formulae are equal for $n = 2$. (I have already pointed out that they are equal for $n = 4$ but for completeness of the argument I choose to start "from the bottom" here.) The argument is now by induction, so assume that they are equal for $n = k$. Start by expressing $S_i^{(k+1)}$ using (3), and then use the assumption. This gives:

$$\begin{aligned} S_i^{(k+1)} &= 2S_i^{(k)} + S_{i-1}^{(k)} \\ &= 2\left(\sum_{j=i}^{k-1} S_i^{(j)} + \sum_{j=i-1}^{k-1} S_{i-1}^{(j)}\right) + S_{i-1}^{(k)} \\ &= 2\sum_{j=i}^{k-1} S_i^{(j)} + \sum_{j=i-1}^{k-1} S_{i-1}^{(j)} + \sum_{j=i-1}^{k-1} S_{i-1}^{(j)} + S_{i-1}^{(k)} \\ &= \sum_{j=i}^{k-1} S_i^{(j)} + \left(\sum_{j=i}^{k-1} S_i^{(j)} + \sum_{j=i-1}^{k-1} S_{i-1}^{(j)}\right) + \sum_{j=i-1}^{k} S_{i-1}^{(j)} \\ &= \sum_{j=i}^{k-1} S_i^{(j)} + S_i^{(k)} + \sum_{j=i-1}^{k} S_{i-1}^{(j)} \\ &= \sum_{j=i}^{k} S_i^{(j)} + \sum_{j=i-1}^{k} S_{i-1}^{(j)}. \end{aligned}$$

This shows that the two ways of expressing $S_i^{(k+1)}$ are equal, and the proof is complete.

[1]The word polytope is used by Coxeter to denote the general term of the sequence *point, segment, polygon, polyhedron* (p. 118).

The Occurrence of the Powers of 3

I will now discuss the observation made earlier that adding all the components of a cube of a certain dimension seems to give powers of 3? This can be stated more precisely as

$$\sum_{i=0}^{n} S_i^{(n)} = 3^n.$$

Why does the power of 3 occur? Since for $n = 0$ the sum is 1, another way of saying this is that the sum in dimension n is three times the sum in dimension $n - 1$. This can be seen by induction on the recursion formula (3). However, looking at a special case (going from the general to the particular) might give a better feeling for why the factor 3 occurs. In dimension 1 the sum is $1 + 2 = 3$ and in dimension 2 it is $1 + 4 + 4 = 9$. Investigating the two 4s in the last sum more closely, one will see, using the recursion formulae for edges and vertices, that the first 4 is $4 = 2 \cdot 1 + 2$ whereas the second 4 is $4 = 2 \cdot 2$. This leads to the expression

$$1 + 4 + 4 = 1 + (2 \cdot 1 + 2) + 2 \cdot 2 = 3(1 + 2),$$

and the factor 3 has appeared. Generalising to dimension n and using $S_i^{(i)} = S_j^{(j)}$ and $S_0^{(n)} = 2S_0^{(n-1)}$ one gets

$$\begin{aligned} \sum_{i=0}^{n} S_i^{(n)} &= S_n^{(n)} + (2S_{n-1}^{(n-1)} + S_{n-2}^{(n-1)}) + (2S_{n-2}^{(n-1)} + S_{n-3}^{(n-1)}) \\ &\quad + \ldots + (2S_1^{(n-1)} + S_0^{(n-1)}) + 2S_0^{(n-1)} \\ &= 3S_{n-1}^{(n-1)} + 3S_{n-2}^{(n-1)} + \ldots + 3S_1^{(n-1)} + 3S_0^{(n-1)} \\ &= 3\sum_{i=0}^{n-1} S_i^{(n-1)}. \end{aligned}$$

From this it can be seen that the factor 3 is really the sum of the coefficients 2 and 1 in the recursion formula (3).

Remark

In the last proof it is possible to see where the factor 3 comes from. An induction proof for the same result would establish the correctness of the result but it would not reveal much of the structure behind it. When presenting a proof it should be taken into account what is the purpose of the proof. I think the purpose of a proof is quite different in a teaching situation as opposed to when presenting a proof for a new research result in mathematics. In the latter situation the main purpose is to establish the truth of the result whereas in the former situation I will say that the main purpose is to explain the structure that lies behind. These two aspects of a proof can be expressed in the phrase *proofs that prove and proofs that explain*, quoting the title of an article by Gila Hanna (1989). Also in the case of research mathematics it is nice to have proofs that explain but sometimes one has to be content only with proofs that prove.

The Generalised Euler Formula

The generalised Euler formula $\sum_{i=0}^{n}(-1)^i S_i^{(n)} = 1$ can also be proved by induction, using (3) and the facts that $S_i^{(i)} = S_j^{(j)}$ and $S_0^{(n)} = 2S_0^{(n-1)}$.

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On Compact Topological Spaces

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Abstract. In this article, we make an attempt to motivate a basic topological property called Compactness.

1. Topological Spaces

Let X be a nonempty set and $(x, y) \in X \times X$. With each $(x, y) \in X \times X$ we associate a non-negative real number which we denote by $d(x, y)$. We want to identify $d(x, y)$ as the distance between the elements x, y in X. So it is natural to expect that

(M1) $d(x, y) = 0$ if and only if $x = y$;
(M2) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(M3) $d(x, y) \le d(x, z) + d(z, y)$, for all $x, y, z \in X$;

It is to be noted that to each element (x, y) in $X \times X$ we associate a unique element $d(x, y)$ in $\mathbb{R}^+ = [0, \infty)$. That is $d(x, y)$ is the image of $(x, y) \in X \times X$. Hence d is a function from $X \times X$ into $\mathbb{R}^+$. *ie.* $d : X \times X \to \mathbb{R}^+$.

If X is a nonempty set and $d : X \times X \to \mathbb{R}^+$ is a function satisfying the above conditions, then we say that d is a metric on X. In such a case, the pair (X, d) is called a metric space.

Let us fix $x \in X$. Now we want to collect all those elements of the space X which are not far away from x and we call this set as a neighborhood of x.

Well, What do you mean by "not far away from x"? The term "not far away" is a relative term. So we fix an $r > 0$ (in some sense radius of our neighborhood) and then take an element, say y from X. If the distance between y and x is strictly less than r, that is $d(x, y) < r$, then we say that y is in our neighborhood of x. Let us define

$$B(x, r) = \{y \in X : d(x, y) < r)\},$$

and call this set as one of our neighborhoods of x. If we change r, we get different neighborhoods of x. $B(x, r)$ is also known as the ball centered at x and radius r. When $X = \mathbb{R}^3$ and d, the distance function, is the usual Euclidean distance, *ie.* for any $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2},$$

then $B(x, r)$ is the usual Euclidean ball centered at x and radius $r > 0$.

Remark 1. *One can have different metrics on $\mathbb{R}^3$ (or $\mathbb{R}^n$, $n \ge 1$) and for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $r > 0$, $B(x, r)$ may be a cube or a solid sphere or an ellipsoid (excluding the points on the boundary) or a singleton $\{x\}$ or the whole space $\mathbb{R}^3$ under suitable metrics.*

Now consider a subset A of X. Suppose A has the property: If $x \in A$ then there exists at least one neighborhood of x say $B(x, r)$ which is contained in our set A. That is

$$x \in A \implies \exists\, r > 0 \text{ such that } B(x, r) \subseteq A.$$

(such an $r > 0$ depends on $x \in A$. *ie*. same r may not work for every $x \in A$. See Figure 1.)

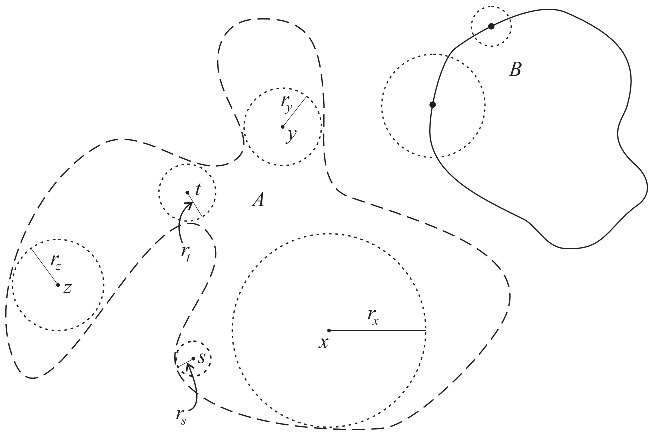

Figure 1. Set A has the stated property but B does not have.

Note: Our statement namely

$$x \in A \implies \exists\, r > 0 \text{ such that } B(x, r) \subseteq A$$

is a conditional statement. That is, we have two statements say p and q. Now consider the truth table

Table 1. *Truth Table for* $p \implies q$.

p	q	$p \implies q$
T	T	T
T	F	F
F	**T**	**T**
F	**F**	**T**

The so called empty set (or null set) ∅ is a subset of our space X. Whether empty set ∅ has the stated property? What is the stated property . . .

$$x \in \emptyset \implies \exists\, r > 0 \text{ such that } B(x, r) \subseteq \emptyset.$$

Where are we in the truth table . . . ? Whether there is $x \in \emptyset$? The answer is no. So our statement $x \in \emptyset$ is false. In such a case whether q is true or false it does not matter and $p \implies q$ is true. So the conclusion is that the empty subset ∅ of X has the above stated property.

What is important to realize here is, what is obvious is not obvious and that realization what makes us to understand the so called abstract concepts.

Well, now it is easy(if not obvious) to prove:

(a) X has the stated property
(b) $A, B \subseteq X$ such that A, B have the stated property then $A \cap B$ has the stated property.
(c) Consider a nonempty set J. Suppose for each $\alpha \in J$, $A_\alpha \subseteq X$ and A_α has the stated property. Then $\bigcup_{\alpha \in J} A_\alpha$ has the stated property.

Now let us consider a collection $\mathscr{M}$ of subsets of X defined below

$$\begin{aligned}\mathscr{M} &= \{U \subseteq X : U \text{ has the stated propery}\}\\ &= \{U \subseteq X : \forall\, x \in U,\ \exists\, r = r_x > 0\\ &\qquad \text{such that } B(x, r) \subseteq U\}\end{aligned}$$

The notation r_x makes it clear that the positive real number r depends on the element x in U. Then by the above discussions, we can conclude that $\mathscr{M}$ is a collection of subsets of X having the following properties:

(T1) $X \in \mathscr{M}$ and $\emptyset \in \mathscr{M}$
(T2) Arbitrary union of members of $\mathscr{M}$, is again a member of $\mathscr{M}$
(T3) Finite intersection of members of $\mathscr{M}$, is again in $\mathscr{M}$.

Let us consider the collection $\mathscr{B} = \{B(x, r) : x \in X, r > 0\}$. Also it is easy to verify that for any $x \in X$ and $r > 0$, $B(x, r) \in \mathscr{M}$. *ie*. $\mathscr{B} \subseteq \mathscr{M}$. Now let us restate the definition of $\mathscr{M}$ in another words as

$$U \subseteq X,\ U \in \mathscr{M} \text{ if and only if } \forall\, x \in U, \exists\, B \in \mathscr{B}$$
$$\text{such that } x \in B \subseteq U.$$

Let us call the members of $\mathscr{B}$ as *essential neighborhoods for X* or *basic open sets.* Members of $\mathscr{M}$ are called *open sets.* Then the collection $\mathscr{M}$ is said to be a *topology* generated by $\mathscr{B}$, the collection of all essential neighborhoods of X.

From the later definition of $\mathscr{M}$, it is clear that the collection $\mathscr{M}$ is completely determined by the collection of all essential neighborhoods $\mathscr{B}$ of X. Also notice that $\mathscr{B}$ depends on the metric d of X. That is for a given metric d on X, we are able to get a collection $\mathscr{B}$ known as essential neighborhoods for X which in turn induce a collection $\mathscr{M}$ of subsets of X that satisfies the conditions $(T1)$, $(T2)$, $(T3)$ mentioned above.

Suppose we are given a set X with out a metric. Now our aim is to find collections $\mathscr{B}$ and $\mathscr{T}$ of subsets of X that satisfy the following conditions

(1) $\mathscr{B} \subseteq \mathscr{T}$

(2) $\mathscr{T}$ satisfies the conditions $(T1), (T2), (T3)$

(3) $\mathscr{T} = \{U \subseteq X : \forall\, x \in U,\ \exists\, B \in \mathscr{B}$ such that $x \in B \subseteq U\}$.

In such a case, $\mathscr{T}$ is said to be a topology on X generated by the collection $\mathscr{B}$ and $\mathscr{B}$ is said to be a basis for the topology $\mathscr{T}$. The members of $\mathscr{T}$ are said to be open subsets of X and the members of $\mathscr{B}$ are called essential neighborhoods or basic open sets.

Since $X \in \mathscr{T}$, by the condition (3), for each $x \in X$ there exists $B \in \mathscr{B}$ such that $x \in B(\subseteq X)$. Note that if $B_1, B_2 \in \mathscr{B}$ then $B_1 \cap B_2 \in \mathscr{T}$(by conditions (1) and $(T3)$). Hence for any $x \in B_1 \cap B_2$, there exists $B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. Therefore $\mathscr{B}$ satisfies the properties

(B1) for every $x \in X$ there exists $B \in \mathscr{B}$ such that $x \in B$. *ie*. $X = \bigcup_{B \in \mathscr{B}} B$.

(B2) for any $B_1, B_2 \in \mathscr{B}$ and for each $x \in B_1 \cap B_2$, there exists $B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Suppose a collection $\mathscr{B}$, that satisfies the conditions $(B1), (B2)$. Will such a collection $\mathscr{B}$ generates (defining $\mathscr{T}$ by (3)) a topology on X? Yes, in fact such a $\mathscr{B}$ generates a unique topology $\mathscr{T}$ on X(Prove it).

Remark: It is to be noted that the collection $\mathscr{B}$ for $\mathscr{T}$ need not be unique. For example, let us consider a metric space (X, d). Let $\mathscr{T}$ be the topology generated by $\mathscr{B} = \{B(x, r) : x \in X,\ r > 0\}$. Consider the collection $\mathscr{B}_1 = \{B(x, \frac{1}{n}) : x \in X,\ n \in \mathbb{N}\}$. It is easy to see that $\mathscr{B}_1$ generates a topology, say $\mathscr{T}_1$. Now it is a simple exercise to prove $\mathscr{T} = \mathscr{T}_1$. Hence $\mathscr{B}, \mathscr{B}_1$ generates the same topology $\mathscr{T}$. Note that $\mathscr{B}_1 \subsetneqq \mathscr{B}$. Hence in practice, depends upon the need, we fix a collection $\mathscr{B}$ of essential neighborhoods and then find the topology $\mathscr{T}$ generated by $\mathscr{B}$. Such a collection $\mathscr{B}$ is called a basis for the topology $\mathscr{T}$.

If $\mathscr{T}$ is a topology generated by some $\mathscr{B}$, then in particular $\mathscr{T}$ itself serves as basis for the topology $\mathscr{T}$. In this case, the conditions (1), (3) mentioned above will become redundant and hence it is natural to define the notion of topology on a set X as follows:

Definition 1.1. *Let X be a set. A collection $\mathscr{T}$ of subsets of X ($\mathscr{T} \subseteq 2^X$) is said to be a topology on X if it satisfies the following:*

$(T1)'$ *whenever $\mathscr{A}$ is a subcollection of $\mathscr{T}$ (ie. $\mathscr{A} \subseteq \mathscr{T}$), then $\bigcup\limits_{A \in \mathscr{A}} A \in \mathscr{T}$*
ie. arbitrary union of members of $\mathscr{T}$, is again a member of $\mathscr{T}$

$(T2)'$ *whenever $\mathscr{F}$ is a finite subcollection of $\mathscr{T}$ (ie. $\mathscr{F}$ is a finite subset of $\mathscr{T}$) then $\bigcap\limits_{A \in \mathscr{F}} A \in \mathscr{T}$.*
ie. finite intersection of members of $\mathscr{T}$, is again in $\mathscr{T}$.

The set X together with a topology $\mathscr{T}$, denoted by $(X, \mathscr{T})$, is called a topological space. Members of $\mathscr{T}$ are called open sets in X.

Note that the property "$X \in \mathscr{T}$ and $\emptyset \in \mathscr{T}$" is not listed in the above definition of a topological space. Since $\emptyset \subseteq \mathscr{T}$, we can take $\mathscr{A} = \emptyset$, and $\mathscr{F} = \emptyset$. This implies that

$$\bigcup_{A \in \emptyset} A = \emptyset \in \mathscr{T} \quad \text{and} \quad \bigcap_{A \in \emptyset} A = X \in \mathscr{T}.$$

Exercise 1. *Prove the above statements.*

`Hint:` For $\emptyset, A_1, A_2, \ldots \in 2^X$ and $\mathscr{A}_1 = \{A_1\}, \mathscr{A}_2 = \{A_1, A_2\}, \mathscr{A}_3 = \{A_1, A_2, A_3\}, \ldots$ then

$$\bigcap_{A \in \mathscr{A}_3} A \subseteq \bigcap_{A \in \mathscr{A}_2} A \subseteq \bigcap_{A \in \mathscr{A}_1} A \subseteq \bigcap_{A \in \emptyset} A$$

Let us have some examples to illustrate the core theme mentioned above.

Example 1.1. *Let X be a nonempty set.*

(a) *Let $\mathscr{B} = \{\{x\} : x \in X\}$ and $\mathscr{T} = 2^X$, the set of all subsets of X. Then $\mathscr{B}$ generates the topology $\mathscr{T}$.*

(b) *Let $\mathscr{B} = \{X\}$ and $\mathscr{T} = \{\emptyset, X\}$. Then $\mathscr{B}$ generates the topology $\mathscr{T}$.*

Example 1.2. *Let $X = \{a, b, c, d\}, \mathscr{B} = \{\{a, c, d\}, \{a, b, d\}, \{a, d\}\}$. Then $\mathscr{B}$ generates the topology $\mathscr{T} = \{\emptyset, X, \{a, c, d\}, \{a, b, d\}, \{a, d\}\}$. Will $\mathscr{B}_1 = \{\{a\}, \{c\}, \{d\}\}$ generate $\mathscr{T}$?.*

Let $(X, \mathscr{T})$ be a topological space and $Y \subseteq X$ be a set. Consider a collection of subsets of Y given by $\mathscr{T}_Y = \{U \cap Y : U \in \mathscr{T}\}$. Then it is easy to verify that $\mathscr{T}_Y$ is a topology on Y and it is called the subspace topology on Y induced by $\mathscr{T}$. Elements of $\mathscr{T}_Y$ are called open sets of Y with respect to $\mathscr{T}_Y$. It is worth to pointing out that a $\mathscr{T}_Y$-open set in Y need not be open with respect to $\mathscr{T}$.

Problem 1. *Let* $(X, \mathscr{T})$ *be a topological space and* $\mathscr{B}$ *be a basis for* $(X, \mathscr{T})$. *Let* $Y \subseteq X$ *and* $\mathscr{B}_Y = \{B \cap Y : B \in \mathscr{B}\}$. *Prove that* $\mathscr{B}_Y$ *is a basis for the subspace topology* $\mathscr{T}_Y$.

Definition 1.2. *Let* $(X, \mathscr{T})$ *be a topological space. The topology* $\mathscr{T}$ *is said to be a Hausdorff topology if for any distinct points* $x, y \in X$ *there exist* $U, V \in \mathscr{T}$ *such that* $x \in U, y \in V$ *and* $U \cap V = \emptyset$

Problem 2. *Prove that every metric space is a Hausdorff topological space.*

Let $(X_1, \mathscr{T}_1)$ and $(X_2, \mathscr{T}_2)$ be topological spaces and $X = X_1 \times X_2$ be the cartesian product of X_1 and X_2. *ie*. $X_1 \times X_2 = \{(x_1, x_2) : x_1 \in X_1, x_2 \in X_2\}$. Let $\mathscr{B}_1 \subseteq \mathscr{T}_1, \mathscr{B}_2 \subseteq \mathscr{T}_2$ be collections of associated essential neighborhoods of X_1, X_2 respectively(*ie*. $\mathscr{B}_i$ is a basis for $\mathscr{T}_i$, $i = 1, 2$). Let $\mathscr{B}$ be the collection of subsets of $X_1 \times X_2$ given by

$$\mathscr{B} = \{U \times V \subseteq X_1 \times X_2 : U \in \mathscr{B}_1, V \in \mathscr{B}_2\}.$$

Since $\mathscr{B}$ satisfies the conditions $(B1)$, $(B2)$, we can define a topology $\mathscr{T}_p$ on $X_1 \times X_2$ generated by $\mathscr{B}$(called the *product topology* on $X_1 \times X_2$). The ordered pair $(X, \mathscr{T}_p)$ is called the product space. In a similar way, one can extend the above idea to define the product topology on a finite product of topological spaces. Some properties of product topological spaces are given in the next section.

Now the natural question arises that "Why can't the same idea be extended for an infinite product of topological spaces?". Well, with respect to the above concept we can define a topology and this topology is known as *box topology* (Why different name?).

2. Compact Spaces and their Properties

Suppose we are given a collection $\mathscr{B}$ of essential neighborhoods for X (*ie.* $\mathscr{B}$ is a basis for the given topology $\mathscr{T}$ on X). In general, $\mathscr{B}$ need not be a finite family. The next best thing one can expect is the possibility of finding a finite subfamily, say $\{B_1, B_2, \dots, B_n\}$, of $\mathscr{B}$ such that $X = \bigcup_{i=1}^{n} B_i$. (of course $n \in \mathbb{N}$ depends on $\mathscr{B}$).

If we are able to find such a finite subfamily for every basis $\mathscr{B}$ of the given topological space $(X, \mathscr{T})$, then we say that X is a compact topological space.

Remark: We have chosen a natural path (or should we say a difficult path) to introduce the concept of topology on a set X and hence we prefer to define the concept of compact topological space by the notion of $\mathscr{B}$ mentioned above.

Let $(X, \mathscr{T})$ be a topological space. As $X \in \mathscr{T}$, $\bigcup\limits_{A \in \mathscr{T}} A = X$ is always true. Also it is possible to have $\mathscr{A} \subseteq \mathscr{T}$ and $\bigcup\limits_{A \in \mathscr{A}} A = X$. Such a collection $\mathscr{A}$ is known as an *open cover* for X. If $\mathscr{A}$ is finite, then we say that it is a *finite open cover* for X.

Problem 3.

(1) *Let* $(X, \mathscr{T})$ *be a topological space and* $\mathscr{A} \subseteq \mathscr{T}$ *be a collection of subsets such that* $X = \bigcup_{A \in \mathscr{A}} A$. *Let* $\mathscr{B} = \{U \in \mathscr{T} : U \subseteq A$ *for some* $A \in \mathscr{A}\}$. *Then prove that* $\mathscr{B}$ *is a basis for the given topology* $\mathscr{T}$.

(2) *A subset* Y *of a topological space* $(X, \mathscr{T})$ *is compact if and only if:*
$\mathscr{A}$ is a collection of open sets in X such that $Y \subseteq \bigcup_{U \in \mathscr{A}} U$ *then there exists* $n \in \mathbb{N}$ *and* $U_1, U_2, \dots, U_n \in \mathscr{A}$ *such that* $Y \subseteq \bigcup_{i=1}^{n} U_i$

Let X be a set and $\mathscr{T}$ be a topology on X. Suppose $\mathscr{T}$ is a finite subset of 2^X, the set of all subsets of X, then $(X, \mathscr{T})$ is a compact topological space. Also if Y is a finite subset of a topological space $(X, \mathscr{T})$ then Y is a compact subset of X.

If we want to show a given topological space X is not compact, then we have to show the existence of an open cover for X which does not have any finite subcover. Summarily,

Let $(X, \mathscr{T})$ be a topological space which is not compact. Then there exists an open cover $\mathscr{A}$ for X (*ie.* $\mathscr{A} \subseteq \mathscr{T}$, $\cup_{U \in \mathscr{A}} U = X$) such that $\mathscr{A}$ does not have any finite sub collection $\mathscr{F} \subseteq \mathscr{A}$ such that $\cup_{U \in \mathscr{F}} U = X$

Example 2.1.

(1) *Let* X *be an infinite set with discrete topology* $\mathscr{D}$. *Then* X *is not compact. Since for each* $x \in X$, $\{x\} \in \mathscr{D}$ *and* $X = \cup_{x \in X}\{x\}$. *But this open cover* $\mathscr{A} = \{ \{x\} \in \mathscr{D} : x \in X\}$ *does not have a finite subcover for X, since X is infinite.*

(2) $\mathbb{R}$ *is not compact with respect to the usual topology(ie. the topology on* $\mathbb{R}$ *induced by the metric* d, *where* $d(x, y) = |x - y|$*), since* $\mathscr{A} = \{(-n, n) : n \in \mathbb{N}\}$ *is an open cover for* $\mathbb{R}$ *without having any finite subcover.*

3. Compactifications

Let $(X, \mathscr{T})$ be a topological space. We say a topological space $(X, \mathscr{T})$ satisfies the property (C) if there exists an element $x_0 \in X$ such that for every open set U containing x_0, U^c is a compact subset of X.

Note that any topological space having the property (C) will be a compact space. Conversely every compact space will have the property (C).

Exercise 2.

(1) *Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ with the topology generated by the metric d given by $d(x, y) = |x - y|$, for all $x, y \in X$. Then show that X has the property (C)($x_0 = 0$ in this case).*

(2) *Let X be a nonempty set and fix an element $x_0 \in X$. Let $\mathscr{T}$ be the collection of subsets of X given by $\mathscr{T} = \{A \subseteq X : x_0 \notin A\} \cup \{A \subseteq X : x_0 \in A, A^c$ is finite $\}$. Prove that $\mathscr{T}$ is a topology on X having the property (C). Also verify that $\mathscr{T}$ is a Hausdorff space.*

(3) *Let X be a nonempty set and $\mathscr{T} = \{U \subseteq X : U^c$ is finite$\}$ $\cup \{\emptyset\}$. Now $(X, \mathscr{T})$ is a topological space and $\mathscr{T}$ is called the co-finite topology on X. Prove that this topological space also satisfies the property (C).*

Suppose a given topological space $(X, \mathscr{T})$ is not a compact topological space. Then $(X, \mathscr{T})$ does not have the property (C). Now the natural question is

"Is it possible to add one additional point to X and define a suitable topology on it such that this topological space has the property (C)?"

That is, $(X, \mathscr{T})$ is a given topological space which is not compact. Let x_0 be a point not in X. Let $Y = X \cup \{x_0\}$. Now we want to define a topology on Y such that this topological space has property (C).

In a natural way one can define such a collection $\mathscr{T}_Y$ as

$$\mathscr{T}_Y = \{A \subseteq Y : A \in \mathscr{T} \text{ or } x_0 \in A \text{ and } A^c \text{ is a compact subset of } X\}$$

(where $A^c = Y \setminus A$). Check whether this collection $\mathscr{T}_Y$ form a topology on Y or not.

Problem 4. *Let $(X, \mathscr{T}_X)$ be a Hausdorff topological space which is not compact. Let x_0 be a point not in X. Let $Y = X \cup \{x_0\}$.*

$$\mathscr{T}_Y = \{A \subseteq Y : A \in \mathscr{T}_X \text{ or } x_0 \in A \text{ and } A^c = Y \setminus A \text{ is a compact subset of } X\}.$$

Then prove that $(Y, \mathscr{T}_Y)$ is a compact topological space.

If we assume that the started space has some more nice properties, then our compact space $(Y, \mathscr{T}_Y)$ will also have some nice properties.

"Is $(Y, \mathscr{T}_Y)$ a Hausdorff topological space?"

In general it need not be .... but if our topological space $(X, \mathscr{T}_X)$ has a nice property (known as locally compact) then $(Y, \mathscr{T}_Y)$ is a Hausdorff topological space.

Definition 3.1. (Locally compact space) *A topological space $(X, \mathscr{T})$ is said to be locally compact if for each $x \in X$ there exists a compact subset C_x of X and an open subset U_x of X such that $x \in U_x \subseteq C_x$*

Now $(Y, \mathscr{T}_Y)$ becomes a nice(compact Hausdorff) topological space known as the one point compactification of the given locally compact Hausdorff space $(X, \mathscr{T}_X)$.

4. Construction of New Compact Spaces

In this section, let us discuss how to construct new compact spaces from the given compact space X. We can easily see that a subset of a compact space need not be a compact space. But it is true if the subset is a closed one. (A subset Y is said to be a closed set if Y^c, the complement of Y in X, is an open set *ie.* $Y^c \in \mathscr{T}$.)

Theorem 4.1. *Let $(X, \mathscr{T})$ be a compact space and Y be a closed subset of X. Then Y is also a compact space.*

Choosing an open cover $\mathscr{A}$ for Y from $\mathscr{T}$ and consider $\mathscr{H} = \mathscr{A} \cup \{Y^c\}$, an open cover for X. Use the compactness of X to find a finite subcover for Y.

Theorem 4.2. *Every compact subset of a Hausdorff topological space is closed.*

Let Y be a compact subset of a Hausdorff space X. Take any point $x \in Y^c$. We have to prove the existence of an open set U such that $x \in U \subseteq Y^c$. For any point $y \in Y$, use the Hausdorff property to find open sets U_y, V_y such that $x \in U_y, y \in V_y$

and $U_y \cap V_y = \emptyset$. (See fig.) Now $\{V_y : y \in Y\}$ is an open cover for the compact space Y will imply the existence of a finite subcover, say $V_{y_1}, V_{y_2}, \ldots, V_{y_n}$, for Y. Take $U = U_{y_1} \cap U_{y_2} \cap \cdots \cap U_{y_n}$. Clearly U is an open subset containing x and complete the theorem by showing $U \subseteq Y^c$.

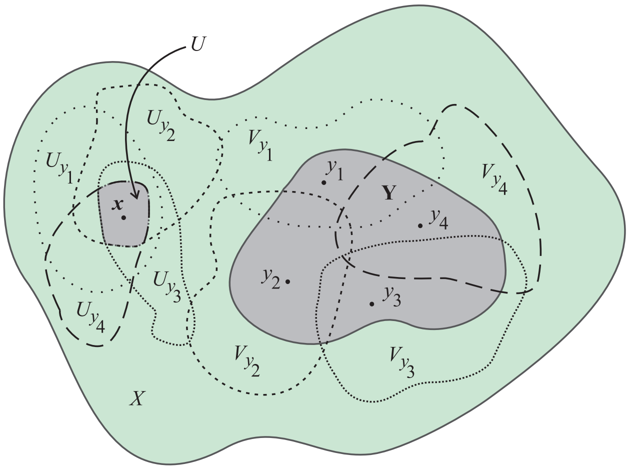

Figure 2. Compact subsets are closed in a Hausdorff space.

Remark: If a topological space $(X, \mathscr{T})$ is not Hausdorff then a compact subset of X need not be closed.

Let $(X, \mathscr{T})$, $(Y, \mathscr{H})$ be topological spaces and $f : X \to Y$ be a map. Then we say that $f : X \to Y$ is a continuous map if for each $V \in \mathscr{H}$, the inverse image of V under f, defined by $f^{-1}(V) = \{x \in X : f(x) \in V\}$, is in $\mathscr{T}$. *i.e.* $V \in \mathscr{H} \implies f^{-1}(V) \in \mathscr{T}$. Notice that this definition will coincide with our usual $\varepsilon - \delta$ definition of continuity in case of X, Y are metric spaces.

Theorem 4.3. *Continuous image of a compact space is compact.*

Let $(X, \mathscr{T})$ be a compact topological space and $(Y, \mathscr{U})$ be a topological space. Given $f : X \to Y$ is a continuous function. Note that the open cover $\mathscr{H} \subseteq \mathscr{U}$ for $f(X)$ will produce an open cover, namely $\mathscr{A} = \{f^{-1}(V) : V \in \mathscr{H}\} \subseteq \mathscr{T}$ for X. Use the compactness of X to find a finite subcover of $\mathscr{H}$ for $f(X)$.

Theorem 4.4. *Suppose X, Y are compact spaces. Then $X \times Y$ is a compact space.*

Before proving the above theorem, first let us prove the following lemma called *tube lemma.*

Lemma 4.1. **(Tube Lemma)** *Consider the product space $X \times Y$, where Y is a compact space. Let $x_0 \in X$ and if U is an open set containing $x_0 \times Y$, then U contains an open tube $W \times Y$, where W is a open set containing x_0 in X.*

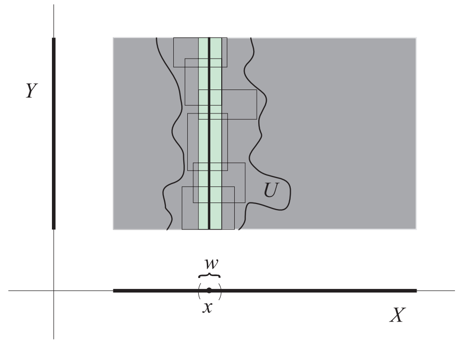

Figure 3. Tube Lemma.

Being a continuous image of Y, the set $x_0 \times Y$ is a compact subset of $X \times Y$. If $x_0 \times Y \subseteq U$, then U (also $x_0 \times Y$) can be covered by the collection $\mathscr{B} = \{U \times V : U \in \mathscr{B}_1, V \in \mathscr{B}_2\}$ of $X \times Y$ (in the figure, these essential neighborhoods are rectangles) where $\mathscr{B}_1, \mathscr{B}_2$ are collections of essential neighborhoods for X, Y respectively. Since $x_0 \times Y$ is compact, there exist a finite number of essential neighborhoods, say $U_1 \times V_1$, $U_2 \times V_2, \ldots, U_n \times V_n$ (for some n), which covers $x_0 \times Y$ (in the figure, we have six rectangles to cover $x_0 \times Y$). The set $W = U_1 \cap U_2 \cap \cdots \cap U_n$ will satisfy our requirements.

Note that W depends on the point $x_0 \in X$. Hence for each $x \in X$, we will able to find an open set $W_x \subseteq X$. Since X is compact and $\{W_x : x \in X\}$ is an open cover for X we will able to get a finite subcover, say $W_{x_1}, W_{x_2}, \ldots, W_{x_n}$, for X. Complete the proof of $X \times Y$ is compact by the fact $X \times Y$ can be covered by finite number of open tubes $W_{x_1} \times Y, W_{x_2} \times Y, \ldots, W_{x_n} \times Y$.

It is obvious that *Theorem 4.4* can be extended for any finite product of compact spaces. But the proof is not all that simple for an infinite product of compact spaces.

Let J be a nonempty set (called index set) and for each $\alpha \in J$, X_α is a set. Now let us define the arbitrary product of X_α, $\alpha \in J$.

$$X = \prod_{\alpha \in J} X_\alpha = \left\{ f : J \to \bigcup_{\alpha \in J} X_\alpha : f(\alpha) \in X_\alpha, \ \forall \alpha \in J \right\}$$

Consider the collection of topological spaces $(X_\alpha, \mathscr{T}_\alpha)$, $\alpha \in J$ where J is an index set and $X = \prod_{\alpha \in J} X_\alpha$.

$$\mathscr{B} = \left\{ \prod_\alpha U_\alpha : U_\alpha \in \mathscr{T}_\alpha, \ U_\alpha = X_\alpha, \text{ for each } \alpha \in J \setminus F, \right.$$
$$\left. \text{for some finite subset } F \subseteq J \right\}.$$

More specifically, for any $\prod_\alpha U_\alpha \in \mathscr{B},\ U_\alpha \in \mathscr{T}_\alpha$, for all $\alpha \in J$ and there exists a finite subset $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ of J such that $U_\alpha = X_\alpha$ for all $\alpha \notin \{\alpha_1, \alpha_2, \dots, \alpha_n\}$. The topology $\mathscr{T}$ generated by $\mathscr{B}$ is called the product topology on X. Note that every members of $\mathscr{B}$ are very large open sets in some sense. The main motivation to define the product topology in this way is to prove the following theorem known as Tychonoff's theorem.

Theorem 4.5. (Tychonoff's Theorem) *The product of compact topological spaces is compact with respect to the product topology.*

Remark: We are having two important topologies on the product space X namely the box topology $\mathscr{T}_B$ and the product topology $\mathscr{T}$. It is easy to verify that $\mathscr{T} \subset \mathscr{T}_B$ and when J a finite set then both these topologies will coincide.

5. Applications of Compact Spaces

Let Y be a compact subset of a metric space X. Now $\{B(y, 1) : y \in Y\}$ is an open cover for Y then there exist $y_1, y_2, \dots, y_n$ in Y such that $Y \subseteq \cup_{i=1}^n B(y_i, 1)$. Hence Y is a bounded subset of X.*ie.* there exists $k > 0$ such that $d(x, y) \leq k$, for all $x, y \in Y$. In a metric space X, every compact sets are bounded. In the following theorem, let us consider $\mathbb{R}$ with usual topology induced by the metric $d(x, y) = |x - y|$, for all $x, y \in \mathbb{R}$. Also we know that every bounded subset of $\mathbb{R}$ has least upper bound(lub) and greatest lower bound(glb) (Ref. [4]). If a bounded set is closed, we can see easily that the lub and the glb will be in the set.

Theorem 5.1. *Let $(X, \mathscr{T})$ be a compact topological space and $f : X \to \mathbb{R}$ be a continuous function. Then there exist $x_1, x_2 \in X$ such that*

$$f(x_1) \leq f(x) \leq f(x_2), \ \text{for all } x \in X.$$

Being a continuous image of a compact set X, $f(X)$ is a compact subset of $\mathbb{R}$. Since the topology generated by the metric d is Hausdorff, we have $f(X)$ is a closed bounded subset of $\mathbb{R}$. Then $\alpha = glb\ f(X), \beta = lub\ f(X)$ will exist (since $f(X)$ is bounded) and in $f(X)$ (since $f(X)$ is closed). Hence there exist $x_1, x_2 \in X$ such that $\alpha = f(x_1), \beta = f(x_2)$ such that $f(x_1) \leq f(x) \leq f(x_2)$, for all $x \in X$.

Fixed point theorems have number of applications for real life problems. The following theorem shows the existence of a unique fixed point for a map satisfying a property called contractiveness.

Theorem 5.2. *Let (X, d) be a compact metric space and $T : X \to X$ be such that*

$$d(Tx, Ty) < d(x, y), \ \text{for all } x, y \in X, x \neq y.$$

Then there exists a unique $x_0 \in X$ such that $Tx_0 = x_0$.

Proof. Define a map $f : X \to [0, \infty)$, by $f(x) = d(x, Tx)$, $x \in X$. It is easy to verify that f is a continuous map. Since X is compact, f attains its minimum at some point, say at x_0. If $x_0 \neq Tx_0$, then

$$f(Tx_0) = d(Tx_0, T(Tx_0)) < d(x_0, Tx_0) = f(x_0)$$

contradiction to the minimality of x_0. Hence $x_0 = Tx_0$ and the uniqueness will follow from the contractive condition.

6. Acknowledgement

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What is a Good Quadratic Form in Characteristic Two?*

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Abstract. Regular (or non-singular) quadratic forms are the good ones to study over a field of characteristic different from two (like the complex or real numbers). This article explains why the notion of regularity does not work for odd dimensions in characteristic two, how to redefine it not only for fields but also for commutative rings, and gives examples of how the resulting theory is satisfyingly analogous to that of regular forms in other characteristics.

1. Quadratic-and Symmetric Bilinear Forms

Let us begin by recalling the notion of a quadratic form q over a field k. It is a map $q : V \to k$ defined on a finite dimensional k-vector space V satisfying the following conditions:

1. $q(av) = a^2 q(v) \,\forall\, v \in V$ and $a \in k$ and
2. the map $b_q : V \times V \to k$ defined by

$$(v, v') \mapsto q(v + v') - q(v) - q(v')$$

is k-bilinear i.e., it is k-linear individually in v and in v'.

Note that b_q is a symmetric k-bilinear form on V. It is called the bilinear form associated to q. The analogue of a vector space over a field for a ring is a (left-) module. Thus, we may replace the field k in the definition above with a ring R (commutative with unity) and the vector space V with a left R-module M to get the definition of a quadratic form $q : M \to R$ defined on M over R.

Now consider the set $\mathrm{Quad}(V)$ of quadratic forms on V. This set has the natural structure of a k-vector space, since the sum of two quadratic forms as well as the product of a scalar and a quadratic form are again quadratic forms. Similarly, the set $\mathrm{SymBil}(V)$ of symmetric bilinear forms on V is also naturally a k-vector space. The association $q \mapsto b_q$ then gives us a k-linear map

$$\beta : \mathrm{Quad}(V) \to \mathrm{SymBil}(V).$$

If $b \in \mathrm{SymBil}(V)$, then the map $v \mapsto b(v, v)$ gives us a quadratic form on V, which we shall call q_b. Then the association $b \mapsto q_b$ gives a k-linear map

$$\alpha : \mathrm{SymBil}(V) \to \mathrm{Quad}(V)$$

such that $b_q \mapsto 2q$. Now if the characteristic of k is not two, then we could consider the k-linear map $\alpha/2 = 2^{-1}\alpha$ that would serve as an inverse to β. This is how quadratic forms are identified with symmetric bilinear forms over fields of characteristic different from two.

Given a basis $\{e_i\}$ of V over k, any element $b \in \mathrm{SymBil}(V)$ is determined by its values $b(e_i, e_j)$, or in other words, by the symmetric matrix $[b(e_i, e_j)]$. If we denote the k-vector space of symmetric matrices of size equal to the dimension n of V by $\mathrm{SymMat}(n, k)$, we then get a k-linear map

$$\mu : \mathrm{SymBil}(V) \to \mathrm{SymMat}(n, k) : b \mapsto [b(e_i, e_j)].$$

Given a matrix $[a_{ij}] \in \mathrm{SymMat}(n, k)$, an element b of $\mathrm{SymBil}(V)$ is uniquely determined by the requirements that $b(e_i, e_j) = a_{ij}$. The map μ is thus a k-linear isomorphism.

So when the characteristic of k is not two, quadratic forms on a vector space with a prescribed basis are represented by symmetric matrices of size equal to the dimension of the vector space.

Another representation of quadratic forms is by homogeneous quadratic polynomials in $n = \dim(V)$ variables $X_1, \ldots, X_n$, which form a k-vector space denoted by

$$\mathrm{HomQuadPoly}(X_1, \ldots, X_n).$$

If an element $v \in V$ is written (uniquely) as $v = \Sigma_i a_i e_i$, then it is easy to check that

$$q(v) = \Sigma_i\, q(e_i) a_i^2 + \Sigma_{i<j}\, b_q(e_i, e_j) a_i a_j.$$

In other words the effect of q on v is the same as evaluating the homogeneous quadratic polynomial

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$$p_q(X_1, \ldots, X_n) = \Sigma_i\, q(e_i) X_i^2 + \Sigma_{i<j}\, b_q(e_i, e_j) X_i X_j$$

at the point $(a_1, \ldots, a_n) \in k^n$ corresponding to v. This gives us an isomorphism of k-vector spaces

$$p : \text{Quad}(V) \to \text{HomQuadPoly}(X_1, \ldots, X_n) : q \mapsto p_q.$$

2. The Anomaly in Characteristic two and Martin Kneser's Way Out

In studying mathematical objects of a certain type, the following approach is natural. First we decide which of these objects are "good". Then we hope that these good objects have "good properties" which would help us in their classification. Thereafter, if we could somehow make sense of representing a bad (= not good) object as a limit of good objects, we could hope to use "limiting/degeneration techniques" from geometry to study the bad objects as well. This approach works well atleast for quadratic forms in low dimensions, and we shall demonstrate it for the 3-dimensional case. So our primary occupation now would be to define "good" quadratic forms.

Let us recall the definition of a regular quadratic form $q : V \to k$. The requirement is that the matrix $[b(e_i, e_j)]$ be invertible. It can be shown that this condition is independent of the basis $\{e_i\}$ chosen, and also that it is equivalent to the condition that the map

$$V \to \text{Hom}_{k\text{-linear}}(V, k) : v \mapsto (v' \mapsto b_q(v, v')),$$

which is a k-linear map, is an isomorphism. Sometimes in the literature, regular quadratic forms are also called non-singular forms or non-degenerate forms, but we shall stick to the term 'regular'. As we shall see later, regularity indeed gives us a well-behaved and useful notion of "good" form, as long as the dimension of V is even or the characteristic of k is not two. So what goes wrong in characteristic two?

First of all, over a field k of characteristic two, the map β does not admit an inverse because 2^{-1} does not exist. This map is not surjective, because its image is a subset of bilinear forms b such that $b(e_i, e_i) = 0$. It is neither injective, for the image of the "diagonal" form $\Sigma_i X_i^2$ under β is the zero form. Thus there is really no correspondence between quadratic forms and symmetric bilinear forms in characteristic two!

Secondly, let us look at the condition that q is a regular quadratic form in characteristic two. Thus $[b_q(e_i, e_j)]$ has to be invertible. We will show how this forces the dimension n of V to be even. Now if n were odd, then $[b_q(e_i, e_j)]$ being an alternating square matrix of odd size must necessarily have determinant zero, and hence it could not have been invertible to begin with. Hence regular forms do not exist on odd-dimensional vector spaces over fields of characteristic two! So we are left with the job of defining the notion of "good" quadratic forms over such vector spaces.

The solution to this anomaly is the notion of "semiregularity" due to the late Prof.Martin Kneser of the Mathematics Institute of Goettingen. It can be motivated as follows. We shall need to work with quadratic forms

$$q : M := R.e_1 \oplus \cdots \oplus R.e_n \to R.$$

Here R is a commutative ring with unity and M is an R-module with basis $\{e_i\}$ over R. As we have seen before, q is uniquely determined by the subset $\{q(e_i), b_q(e_i, e_j)\} \subset R$. If n is even, we could define q to be good if it is regular. The problem is when n is odd and R is a field of characteristic two, which is what we want to analyze.

Before we proceed, another notion that we would need is the "base change" of a quadratic form, defined thus. Given a quadratic form $q' : M' \to R$ and a homomorphism $\phi : R \to S$ of commutative rings (such that $\phi(1_R) = 1_S$), we get a quadratic form ${q'}_S = q' \otimes_R S$ on the S-module $M' \otimes_R S$ as follows. Note that ϕ makes S naturally into an R-module, by the multiplication $a.s := \phi(a)s$ for $a \in R$ and $s \in S$. Recall that $M' \otimes_R S$ is made up of elements of the form

$$\Sigma_i\, (m'_i \otimes s_i),\ m'_i \in M' \quad \text{and} \quad s_i \in S,$$

and that $m' \otimes s$ is bilinear in (m', s). Further the property that

$$a(m' \otimes s) = (am') \otimes s = m' \otimes (\phi(a)s)$$

makes $M' \otimes_R S$ firstly into an R-module. The image of an element $m' \in M'$ in $M' \otimes_R S$, under the natural map $M' \to M' \otimes_R S$ which is an R-module homomorphism, is denoted by $m' \otimes 1_S$. Further $M' \otimes_R S$ is also naturally an S-module as follows:

$$s.(\Sigma_i\, m'_i \otimes s_i) := \Sigma_i\, m'_i \otimes s s_i.$$

Now q'_S is defined by

$$q'_S(\Sigma_i\, m'_i \otimes s_i) := \Sigma_i\, \phi(q'(m'_i)) s_i + \Sigma_{i<j}\, \phi(b_{q'}(m'_i, m'_j)) s_i s_j.$$

If R and S were fields and ϕ were nonzero, then $q' \otimes_R S$ is just the quadratic form q' with scalars extended from R to S. With a quadratic form $q : M \to R$ defined on an R-module with a basis as above, the quadratic form q_S is thus defined by

$$q_S(e_i \otimes 1_S) := \phi(q(e_i)) \quad \text{and}$$
$$b_{q_S}(e_i \otimes 1_S, e_j \otimes 1_S) := \phi(b_q(e_i, e_j)),$$

it being uniquely determined since $\{e_i \otimes 1_S\}$ is a basis for $M \otimes_R S$ over S.

Now, let's go back to the idea of Kneser. Let n be any positive integer, not necessarily even. The first step is to take the polynomial ring

$$R_0 := \mathbb{Z}[X_i, X_{ij}]$$

in the $n(n+1)/2$ variables X_i, $1 \leq i \leq n$ and X_{ij}, $1 \leq i < j \leq n$. Now we consider the R_0-module M_0 given by

$$M_0 := R_0.E_1 \oplus \cdots \oplus R_0.E_n.$$

That is, M_0 is an R_0-module with basis $\{E_i\}$ over R_0. Then we consider the quadratic form

$$q_0 : M_0 \to R_0$$

uniquely determined by the requirements that $q_0(E_i) = X_i$ and $b_{q_0}(E_i, E_j) = X_{ij}$. The quadratic form q_0 is called the "universal quadratic form in n variables." This is because, firstly, given $q : M \to R$, under the ring homomorphism

$$\phi_q : R_0 \to R \text{ determined by } X_i \mapsto q(e_i),\ X_{ij} \mapsto b_q(e_i, e_j),$$

the base change $q_0 \otimes_{R_0} R$ is nothing else but q. Here we identify $E_i \otimes_{R_0} 1_R$ with e_i. Secondly, given any ring homomorphism

$$\phi : R_0 \to R \text{ determined by } X_i \mapsto a_i,\ X_{ij} \mapsto a_{ij},$$

the quadratic form $q_\phi : M \to R$ uniquely determined by the requirements $q_\phi(e_i) = a_i$, $b_{q_\phi}(e_i, e_j) = a_{ij}$ satisfies $\phi_{q_\phi} = \phi$. In other words, the mapping

$$\text{Quad}(M) \to \text{Hom}_{\text{Rings}}(R_0, R) : q \mapsto \phi_q$$

is a bijection with inverse $\phi \mapsto q_\phi$.

Now let us look at the requirement that the universal quadratic form q_0 is regular. It demands that the symmetric $(n \times n)$-matrix

$$[b_{q_0}(E_i, E_j)] = [X_{rs}]$$

(with the conventions that $X_{ii} = 2.X_i$ and $X_{rs} = X_{sr}$ for $r < s$), with entries in R_0, have for its determinant an invertible element of R_0. When n is odd, a computation of this determinant modulo 2 shows that there is a uniquely determined polynomial $P(X_i, X_{ij}) \in R_0$ such that

$$\det([b_{q_0}(E_i, E_j)]) = 2.P(X_i, X_{ij}).$$

Therefore it is but natural that when we evaluate $\det([b_q(e_i, e_j)])$ when R is a field of characteristic two and n is odd, we get

$$2.P(q(e_i), b_q(e_i, e_j)) = 0\,!$$

So it is this factor of 2 that is the source of the anomaly. Hence Kneser calls q to be "semiregular" if the quantity $P(q(e_i), b_q(e_i, e_j))$, called the "half-discriminant of q relative to the basis $\{e_i\}$" is an invertible element of R. It can be shown that this requirement is independent of the basis chosen, and further that when R is a field of characteristic different from two, semiregularity is the same as regularity.

3. What Could We Expect of Good Forms?

It happens that the notions of regularity (in even dimensions) and of semiregularity (in odd dimensions) are indeed good ones. We may ask why. Let us try to put down some properties that we could expect of good forms. Broadly we could expect the following:

1. the property of a form being good should behave well under various algebraic/geometric/topological operations.
2. the notion of good form should make easier (if not easy) the classification of good forms, and if possible also that of bad forms

In this section, we shall try to explain the first statement above. Soon we shall define the notion of a good quadratic form over a commutative ring. Given that this can be done, it is natural to expect that the base change of a good quadratic form is again good. We put this down as property 1:

P1: $q : M \to R$ good

$$\Rightarrow q \otimes_R S : M \otimes_R S \to S \text{ good } \forall\, \phi : R \to S.$$

Notice that we could have replaced $\Rightarrow$ by $\Leftrightarrow$ above, for we could take $S = R$ and ϕ the identity map on R. Now for the second property. We have at hand the notion of a good quadratic form over a field: we just require that it be regular if the vector space on which it is defined has even dimension, otherwise we require it to be semiregular. There must be a way of transporting this notion to forms defined on modules over commutative rings. Let us consider a module M over a commutative ring R (with 1_R, as always). Given a maximal ideal $\mathfrak{m} \subset R$, we get a field $\kappa(\mathfrak{m}) := R/\mathfrak{m}$, a vector space $M(\mathfrak{m}) := M \otimes_R \kappa(\mathfrak{m})$ over it, and the quadratic form $q(\kappa(\mathfrak{m}))$ over $M(\mathfrak{m})$ via base-change through the natural surjective ring homomorphism $R \to R/\mathfrak{m}$. Now as $\mathfrak{m}$ varies, we would like the vector spaces $M(\mathfrak{m})$ to have the same dimension. This shows that we need to put additional conditions on the module M. These conditions are: M should be finitely generated and projective of constant positive rank. Finitely-generated means that any element of the module can be written (though not necessarily uniquely) as an R-linear combination of a fixed finite set $B \subset M$ of elements of M. This ensures firstly that each vector space $M(\mathfrak{m})$ is finite dimensional. Next, the projectivity of constant positive rank ensures that all these vector spaces have the same positive dimension, which we shall refer to as the rank of the module M over R. A quadratic form q is said to be of rank $n > 0$ if the R-module on which it is defined is of rank n. Thus it is natural to expect:

P2: $q : M \to R$ good $\Leftrightarrow q(\kappa(\mathfrak{m}))$ good $\forall$ maximal $\mathfrak{m} \subset R$.

Note that the implication $\Rightarrow$ of **P2** follows from **P1**. Let us explain what **P2** means from the viewpoint of algebraic geometry. We consider the Maximal Spectrum of R, defined by

$$\text{MaxSpec}(R) := \{\mathfrak{m} \subset R : \mathfrak{m} \text{ is maximal}\}.$$

By declaring subsets of the form

$$Z(I) := \{\mathfrak{m} \in \text{MaxSpec}(R) : \mathfrak{m} \supset I\}$$

to be closed sets, where $I \subset R$ is an ideal, we get a topology on MaxSpec(R) called the Zariski topology. Then the module M over R can be thought of as a continuous family of vector spaces indexed by the points of MaxSpec(R), and is called a vector bundle on MaxSpec(R). Thus **P2** may be restated as "goodness over R" is equivalent to "goodness pointwise over MaxSpec(R)".

Before we proceed, let us define when a quadratic form $q : M \to R$ defined on a finitely generated projective R-module M of constant positive rank n is good. We consider the universal quadratic form q_0 and the polynomial

$$D(X_i, X_{ij}) := \det([b_{q_0}(E_i, E_j)]) \in R_0 = \mathbb{Z}[X_i, X_{ij}]$$

when n is even, or the polynomial

$$P(X_i, X_{ij}) = D(X_i, X_{ij})/2$$

when n is odd. We substitute $q(x_i)$ for the X_i and $b_q(x_i, x_j)$ for the X_{ij}, for any n-tuple of elements $(x_1, \ldots, x_n)$ of M, and consider the ideal of R generated by the resulting elements for all possible n-tuples. Then the requirement that q is good is that this ideal is all of R i.e., that it contains an invertible element of R. It can be checked that this reduces to the condition of regularity or semiregularity for modules that admit a finite basis, and so in particular for forms over fields. Let us remark that that the universal quadratic form becomes good after base-change to the quotient field (field of fractions) of R_0.

Next we come to the third property to expect of goodness of a form. This has got to do with the hope that once good forms have been classified and studied well, bad forms that can be represented somehow as limits of good forms could be studied. We can make sense of this in the Zariski topology. What we would like to have is, given a bad form $q_k : V \to k$ over a field k, we are able to find a suitable R and a form $q : M \to R$, alongwith a point $\mathfrak{m}_0 \in$ MaxSpec(R) such that $q(\kappa(\mathfrak{m}_0)) = q_k$, and such that it is a limit of good forms in a neighbourhood of $\mathfrak{m}_0$. For this, it is natural to expect that the subset of $\mathfrak{m} \in$ MaxSpec(R) where $q(\kappa(\mathfrak{m}))$ is bad is a closed subset, or in other words that goodness is an open condition:

P3: $\{\mathfrak{m} \in \text{MaxSpec}(R) : q(\kappa(\mathfrak{m}))$ good$\}$ is open.

We have the following results that justify that good forms have all the properties **P1**, **P2** and **P3** we expect of them.

Theorem 1. *Let $q : M \to R$ be a quadratic form defined on a finitely generated projective module M of constant positive rank over a commutative ring R. Then the following are equivalent:*

1. *q is good.*
2. *If $\phi : R \to S$ is any homomorphism of commutative rings, then the base-changed form $q \otimes_R S : M \otimes_R S \to S$ is good.*

3. $q \otimes_R \kappa(\mathfrak{m})$ *is good for each maximal ideal* $\mathfrak{m} \subset R$.
4. $q \otimes_R R_\mathfrak{m}$ *is good for each maximal ideal* $\mathfrak{m} \subset R$.
5. *There exists a faithfully flat homomorphism of commutative rings* $\phi : R \to S$ *such that* $q \otimes_R S$ *is good.*

We note that (1) $\Rightarrow$ (2) is **P1**, (1) $\Leftrightarrow$ (3) is **P2**. Condition (5) is rather technical: it says that "goodness is preserved under faithfully-flat descent". It is very useful in constructions involving quadratic forms in algebraic geometry, especially in scheme theory, while using cohomology theory. Condition (4) is connected with **P3** for which we have:

Theorem 2. *Let* $q : M \to R$ *be a quadratic form defined over a finitely generated projective module* M *of constant positive rank over a commutative ring* R. *Let* $\mathfrak{m} \subset R$ *be a maximal ideal of* R. *Then the following are equivalent:*

1. $q \otimes_R \kappa(\mathfrak{m})$ *is good.*
2. $q \otimes_R R_\mathfrak{m}$ *is good.*
3. *There exists an open neighbourhood* U *of* $\mathfrak{m}$ *in MaxSpec*(R) *such that* $q \otimes_R \kappa(\mathfrak{m}')$ *is good for each* $\mathfrak{m}' \in U$.

Now let us explain condition (4) of Theorem 1. The local geometry and topology of MaxSpec(R) at a point $\mathfrak{m}$ is captured by the local ring $R_\mathfrak{m}$. It contains information about all the neighbourhoods of the point $\mathfrak{m}$. Therefore the goodness of $q \otimes \kappa(\mathfrak{m})$ "spreads" to an open neighbourhood U, and this is reflected in the goodness of $q \otimes R_\mathfrak{m}$.

4. How Good Forms Help in Classification

We now explain the statement (2) of page 127, namely that goodness of forms should help in their study and classification.

How does one approach classification of mathematical objects of a certain type? One philosophy, which seems to work atleast for low-rank quadratic forms, is as follows. Firstly, we decide which are the good objects. Next we try to find invariants for these objects. These invariants would be other well-defined mathematical objects related in some way to the objects we want to study. They are called invariants because they would change only by isomorphisms if the original object is replaced with another one isomorphic to it.

It is precisely here that we hope that good objects would give rise to invariants with good and special properties which would help in our study. This accomplished, in order to complete the classification of good objects, all we need to do is to find enough invariants that together would completely determine an object upto isomorphism. Thereafter we could hope to study bad objects using limiting techniques from geometry provided such objects could be represented as limits of good objects.

Since our objects of study are quadratic forms, we shall have to look for invariants of quadratic forms. Two such invariants, that seem to work well atleast for low rank quadratic forms, are the Clifford algebra and the even-Clifford algebra. We now describe the construction of these invariants.

4.1 The Clifford-and Witt-invariants

The motivation behind the construction of the Clifford algebra for a quadratic form $q : M \to R$ may be given as follows. The simplest kind of quadratic form that one could think of on the simplest possible R-module—which is R itself—is given by the simplest possible homogeneous quadratic polynomial in one variable over R, namely X^2. For this quadratic form, one has $q(a) = a^2$. We would like to have such a thing happen for any quadratic form. For this, one needs to make sense of the square m^2 for $m \in M$. The solution is to embed M inside a larger R-module C via an injective map $i_M : M \to C$, such that C is also a ring with unity, the multiplication being compatible with multiplication by elements of R (such a C is called an R-algebra with unity); further we require that $i_M(m)^2 = q(m).1_C$ for each $m \in M$. It is not hard to construct such a C. We start with the tensor algebra of M over R:

$$T_R(M) := \oplus_{i\geq 0} T^i(M) \text{ where}$$
$$T^i(M) := M \otimes \cdots \otimes M \ (i \text{ factors}).$$

Of course, $T^0(M) := R$. Each $T^i(M)$ is an R-submodule that is a direct summand. Further there is an obvious multiplication in $T_R(M)$ which satisfies

$$T^i(M).T^j(M) \subset T^{i+j}(M).$$

In particular, $T^0(M) = R$ is a sub-algebra. We now consider the subset $S(M, q)$ of $T_R(M)$ given by

$$S(M, q) := \{m \otimes m - q(m).1_{T_R(M)} : m \in M = T^1(M)\}.$$

Notice that $m \otimes m \in T^2(M)$. The multiplication in the tensor algebra is not commutative if the rank of M is greater than 1. If M is an R-module with a basis $\{e_1, \ldots, e_n\}$, then we get a natural isomorphism

$$T_R(M) \cong R\{X_1, \ldots, X_n\},$$

where $\{X_1, \ldots, X_n\}$ is a basis of the dual module

$$M^* := \mathrm{Hom}_{R\text{-linear}}(M, R)$$

dual to the basis $\{e_1, \ldots, e_n\}$, and where $R\{X_1, \ldots, X_n\}$ denotes the non-commuting polynomial algebra over R in the n variables $X_1, \ldots, X_n$. Now we return to the case of a general M. We consider the two-sided ideal $I(M, q) \subset T_R(M)$ generated by the subset $S(M, q)$, which is simply the smallest two-sided ideal that contains this subset. Then we set

$$C(M, q) := T_R(M)/I(M, q)$$

and call it the Clifford algebra for (M, q). We get a natural map $i_M : M \to C(M, q)$ such that $i_M(m)^2 = q(m).1_{C(M,q)}$, by virtue of our very construction. That $C(M, q)$ is uniquely determined upto isomorphism for this property, and further properties such as it being finitely generated and projective as R-module of constant positive rank 2^n when M is finitely generated and projective of constant positive rank n, can be shown. Given an isomorphism of quadratic forms

$$g : (M, q) \xrightarrow{\cong} (M', q'),$$

i.e., an R-linear isomorphism $g : M \overset{\cong}{\to} M'$ such that $q = q' \circ g$, it can be shown that there is induced an R-algebra isomorphism

$$C(g) : C(M, q) \xrightarrow{\cong} C(M', q')$$

such that $C(g) \circ i_M = i_{M'} \circ g$. Thus the Clifford algebra is indeed an invariant, and given an isomorphism class of quadratic forms $[(M, q)]$, the corresponding isomorphism class of R-algebras $[C(M, q)]$ is called the Clifford invariant.

Another invariant we would like to introduce is called the Witt-invariant. For this, we notice first that the ideal $I(M, q)$ is not a homogeneous ideal, since its generators do not lie in the homogeneous components $T^i(M)$ of $T_R(M)$. However, we could break up the tensor algebra as

$$T_R(M) = T_{\mathrm{even}}(M) \oplus T_{\mathrm{odd}}(M),$$

where

$$T_{\mathrm{even}}(M) := \oplus_{\mathrm{even}\, i} T^i(M) \quad \text{and} \quad T_{\mathrm{odd}}(M) := \oplus_{\mathrm{odd}\, i} T^i(M)$$

and notice that if we were to relabel $T_{\mathrm{even}}(M)$ and $T_{\mathrm{odd}}(M)$ as $T_0(M)$ and $T_1(M)$ respectively, then the multiplication in the tensor algebra satisfies

$$T_i(M).T_j(M) \subset T_{(i+j) \bmod 2}(M).$$

In other words, $T_0(M)$ becomes a sub-algebra and $T_1(M)$ becomes a $T_0(M)$-bi-module, i.e., both a left-and a right $T_0(M)$-module. Now $I(M, q)$ is homogeneous with this new gradation (called the $(\mathbb{Z}/2\mathbb{Z})$-gradation on $T_R(M)$), and therefore $C(M, q)$ inherits a similar gradation, i.e., we have a direct sum decomposition

$$C(M, q) = C_0(M, q) \oplus C_1(M, q).$$

The sub-algebra $C_0(M, q)$ is called the even-Clifford algebra of (M, q) and $C_1(M, q)$ is called the Clifford bimodule (as it is a bimodule with respect to $C_0(M, q)$). Now given an isomorphism

$$g : (M, q) \cong (M', q'),$$

the induced algebra isomorphism $C(g)$ respects this gradation, i.e., it induces isomorphisms

$$C_i(g) : C_i(M, q) \cong C_i(M', q'),$$

showing that the even-Clifford algebra is indeed also an invariant. Its isomorphism class (as R-algebras) is called the Witt-invariant of (M, q).

4.2 Clifford-and Witt-Invariants and classification

Now that we have defined the Clifford-and Witt-invariants of a quadratic form, we would expect that these invariants possess special and good properties when the form is good, in line with our philosophy in page 129. This is in fact the case:

Theorem 3. *Let $q : M \to R$ be a good quadratic form defined on the finitely generated projective module M of constant positive rank n. Then*

1. *if n is even, $C(M, q)$ is an Azumaya R-algebra of rank 2^n.*
2. *if n is odd, $C_0(M, q)$ is an Azumaya R-algebra of rank 2^{n-1}.*

But then what's so special about these Azumaya R-algebras? Well, these are precisely those R-algebras A such that the underlying module is finitely generated and projective of constant rank l^2, and such that for each $\mathfrak{m} \in \mathrm{MaxSpec}(R)$, the $\kappa(\mathfrak{m})$-algebra $A \otimes_R \kappa(\mathfrak{m})$ has the special property that its further base change to the algebraic closure $\overline{\kappa(\mathfrak{m})}$ is isomorphic to an $(l \times l)$-matrix algebra. In other words, Azumaya algebras over R are the correct generalisations of matrix algebras over an algebraically closed field. Now how could these be useful? Using the properties of such algebras of rank 4, one can prove the following theorem

Theorem 4. *If q and q' are good quadratic forms over an R module M (finitely generated and projective of constant positive rank 3) such that they have isomorphic Witt-invariants i.e., there is given an isomorphism of R-algebras $\phi : C_0(M, q) \cong C_0(M, q')$, then a similarity $(g, \lambda) : (M, q) \sim (M, q')$ can be found such that $C_0(g, \lambda) = \phi$. In other words, the classification of good rank 3 quadratic forms upto similarity is achieved by the Witt-invariant.*

Here a similarity

$$(g, \lambda) : (M, q) \xrightarrow{\sim} (M', q')$$

is an R-linear isomorphism $g : M \cong M'$ together with another R-linear isomorphism $\lambda : R \cong R$ such that $\lambda \circ q = q' \circ g$. It can be shown that such a similarity also induces an isomorphism of the corresponding Clifford algebras and even-Clifford algebras. The point is that it is too much to expect an isomorphism of even-Clifford algebras to lift to an isomorphism of quadratic forms.

Now what about bad forms? One could study the set of all possible algebra structures, isomorphic to the $(l \times l)$-matrix algebra, defined on a vector space with dimension l^2 over an algebraically closed field from an algebraic-geometric point of view. It turns out that the geometry of the set of limits of such algebras has nice properties. Using these one can prove (see [2])

Theorem 5. *The theorem 4 above holds for bad forms as well.*

Thus we do get a satisfying theory of rank 3 quadratic forms (including the bad ones). At the moment of writing these notes, as such it seems to be very difficult to get a similarly satisfactory theory in rank 4, where there are problems classifying even good forms in a nice geometric way. Work is on and one hopes for the better in the near future.

References

[1] Knus Max-Albert, Quadratic and Hermitian forms over rings, Grundlehren der mathematischen Wissenschaften (Springer-Verlag) 294 (1991).

[2] Venkata Balaji Thiruvalloor Eesanaipaadi, Line-bundle-valued ternary quadratic forms over schemes, Journal of Pure and Applied Algebra (January 2007) (Elsevier)

RMS Pre-Conference Workshop on Harmonic Analysis

May 14–18, 2008

Venue:

Indian Institute of Technology Kanpur
Kanpur

This workshop is aimed at research scholars, university and college teachers.

The International Congress of Mathematicians (ICM) will be held in Hyderabad during August 2010. With the objective of training many researchers in modern areas of mathematics (so that they will be able to take active part in ICM 2010), Ramanujan Mathematical Society has embarked on a series of workshops in various branches of Mathematics. Last year, RMS organized a workshop on Complex Analysis at NIT Surathkal. This year, the Society will be organizing a 5-days instructional workshop on Harmonic Analysis. The programme will be directed by Alladi Sitaram (ISI and IISc., Bangalore) and Shobha Madan (IIT-K). (The other expected resource persons are S. Thangavelu, R. Rawat and E. K. Narayanan.) There will be four 50 minute lectures every day followed by discussions and tutorials. The topics to be covered are: Fourier Analysis on Euclidean Spaces, Representation Theory of Groups, and Harmonic Analysis on the Heisenberg Group.

Registration Form: Registration form can be downloaded at `http://www.ramanujanmathsociety.org/23conf2008.html`

Applications (with brief CV) for the workshop may be sent by either E-mail to Prof. Alladi Sitaram/Prof. E. K. Narayanan, at: rmswsmath.iisc.ernet.in or by post to:

Prof. Alladi Sitaram
Indian Statistical Institute
Bangalore 560 059, India
E-mail: rmsws@math.iisc.ernet.in

E. K. Narayanan
Department of Mathematics
Indian Institute of Science
Bangalore 560 012, India
Phone: + 91-080-2293 3270
E-mail: rmsws@math.iisc.ernet.in

on or before April 6, 2008. The Application Form for Registration may be downloaded from here. Selected participants will be informed (through E-mail) by April 10, 2008. The participants will be paid round trip train fare by AC III Tier. Local hospitality will be provided free of cost. All the selected participants are encouraged to stay on and attend the 23rd Annual Conference of the Ramanujan Mathematical Society during May 19–21, 2008, where more advanced lectures will be delivered. Local hospitality will be provided free of cost for this period also. There are no registration charges. Please indicate in your application whether, if selected, you will attend the conference or not.

For Updated Information Please Visit:

http://www.ramanujanmathsociety.org/23conf2008.html

23rd Annual Conference of Ramanujan Mathematical Society

May 19–21, 2008

Second Announcement

Venue:

Indian Institute of Technology Kanpur, Kanpur

There will be a Pre-conference Workshop on Harmonic Analysis during May 14–18, 2008.

The 23rd Annual Conference of Ramanujan Mathematical Society will be organized during May 19-21, 2008 in the Indian Institute of Technology Kanpur. During the conference, there will be Plenary Talks, Parallel Symposia and Paper Presentations. Plenary Talks: The Schedule of the talks is being worked out.

Symposia: There will be four symposia each with 8–10 speakers. The schedule is being worked out.

Paper Presentation: The Annual Conference of the Ramanujan Mathematical Society will provide an opportunity to all its members to contribute papers in special sessions. Each speaker will be given a time slot of 15 minutes for his/her presentation. If the Society receives more papers than can be accommodated in the schedule of the conference, then there will be a combination of poster sessions and talks. Those who wish to present papers, should send them to one of the following four persons, choosing the one closest to the topic of the paper. The papers should be received by April 10, 2008 to ensure consideration.

Registration Form:

Registration form can be downloaded at

http://www.ramanujanmathsociety.org/23conf2008.html

Please note that separate Registration Forms for the Workshop and the Conference must be sent to the following addresses:

Swagato K. Ray
Local Secretary, RMS Conference
Department of Mathematics and Statistics, I.I.T. Kanpur
Kanpur 208 016, India
Phone: + 91-512-2597972
E-mail: skray@iitk.ac.in

For Updated Information Please Visit:

http://www.ramanujanmathsociety.org/23conf2008.html

Spring School in Harmonic Analysis and PDE 2008

June 2–6, 2008

Venue: Helsinki University of Technology, Helsinki, Finland.

This spring school will bring together researchers of international stature as well as graduate students in the fields of harmonic analysis and partial differential equations.

All participants are asked to fill in the registration form and submit it before April 30th, 2008. There is no registration fee.

For More Information, Please See the Web Page:

`http://math.tkk.fi/conferences/pde2008/`

or contact:

`juha.kinnunen@tkk.fi`

Advanced Training in Mathematics Schools

Supported by

National Board for Higher Mathematics

Advanced Training School For Lecturers in Functional Analysis-I

Venue:

Indian Statistical Institute, Bangalore

June 2–13, 2008

Convener: T. S. S. R. K. Rao

Brief Description of ATM Schools: Advanced Training in Mathematics (ATM) Schools are a joint effort of more than 50 active researchers across the country with support from the National Board for Higher Mathematics. The programmes are conducted in reputed mathematics departments Summer and Winter each year. In these Schools, the emphasis will be on problems solving and on highlighting inter-relations of basic subjects in mathematics. Presently we invite applications for participation in

ATM School for Lecturers in **Functional Analysis**

Brief Description of the School: This program is meant for Lecturers in Colleges and Universities who do teaching of Postgraduate courses in Functional Analysis. The Principal speakers are leading experts in Functional Analysis and its applications.

They will be lecturing on topics leading to current areas of research and will develop ideas leading to some open problems in these areas.

Eligibility for Participation: Applications are invited from lecturers in mathematics who have passed NET/SET or equivalent examination and who are teaching at a college/university. Students doing M.Phil. may also be considered for the school. Teachers below the age of 30 will be given preference.

National Coordinating Committee		
Director	R. S. Kulkarni	IIT Bombay
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	Satya Deo	HRI, Allahabad
	S. A. Katre	Pune U., Pune
	Shobha Madan	IIT Kanpur
	I. B. S. Passi	Panjab U., Chandigarh
	R. A. Rao	TIFR, Mumbai

Financial Support: Selected participants will be paid III-AC return train fare from their place of work/home town to the venue by shortest route and provided with accommodation and local hospitality.

How to Apply: The syllabus, application forms and other information about the programme is available on the websites: **`http://www.math.iitb.ac.in/atm or www.bprim.org`** Applications may also be made on plain paper, giving the following information:

Name, Date of Birth, Age, Gender, Institute/Department, Areas of interest, Address for correspondence, E-mail address, City, State, Pincode, Academic Record: B.Sc./M.Sc./ with names of the Institutes, additional information (if any). These should be attested by Head/Principal of the institute.

Completed Application Forms should Reach:

T. S. S. R. K. Rao
Co-ordinator (ATML)
Head, Indian Statistical Institute
8th Mile Mysore Road,
R. V. College-PO, Bangalore 560 059

E-mail: tss@isibang.ac.in
Phone/fax: 91-80-28483001,
Fax: 91-80-28484265

by **Monday, 21st, April 2008**. List of selected candidates will be posted on the websites of ATM Schools on **Monday, 28th April 2008**.

Resource persons	
S. H. Kulkarni	I. I. T., Chennai
A. K. Roy, P. Bandyopadhyay	I. S. I., Kolkata
P. L. Muthuramalingam	I. S. I., Bangalore
B. V. Rajarama Bhat, A. Sitaram	I. S. I., Bangalore
T. S. S. R. K. Rao	I. S. I., Bangalore

Instructional Workshop on Differential Geometry

June 16–25, 2008

Venue:

Department of Mathematics, University of Mysore, Mysore

A ten-day Instructional Workshop sponsored by NBHM on Differentail Geometry is being organized at the Department of Mathemtics, University of Mysore, Manasagangothri, Mysore during 16th to 25th of June 2008. The objective of the workshop is to introduce basic concepts in Differential Geometry to the interested research students and young teachers in colleges and universities. It is planned to discuss in sufficient detail those results about curves and surfaces which are prototypes of more general results in higher dimension. Dr. C. S. Aravinda (TIFR, Bangalore) is the Convener of the Academic Coordination Committee for the Workshop. The Resources Persons for the Workshop include:

Dr. C. S. Aravinda (TIFR, Bangalore)
Dr. G. Santhanam (I.I.T. Kanpur)
Dr. Ravi Aithal (University of Mumbai)
Dr. Anandateertha Mangasuli
(Bhaskaracharya Pratisthana, Pune)
Dr. K. V. Srikanth (I.I.T. Guwahati)

Applications are invited from interested and highly motivated research students and young teachers from colleges and universities. All selected participants will be provided with travel support (Train Sleeper Class/Bus) and local hospitality. Duly filed application form in the proforma given below should be sent to:

Prof. H. N. Ramaswamy
Organizing Secretary
Department of Mathematics
University of Mysore, Mysore 570 006
E-mail: hnrama@gmail.com

Deadline for receiving the application for the workshop is 30th April 2008. Selected participants will be informed by E-mail by 10th of May, 2008.

Proforma for Application for the Instructional Workshop on Differential Geometry

1) Name and Affiliation :
2) Gender and Age :
3) Address/Telephone No./E-mail :
4) Address for communication :
5) Educational qualifications :
6) Research experience :

Note: Please enclose a recommendation letter from a Senior faculty/Guide.

Place and Date Signature

International Workshop and Conference on Surface Mapping Class Groups and Related Topics

June 16–28, 2008

Venue:

Department of Mathematics,
North-Eastern Hill University
NEHU Campus, Shillong 793 022,
Meghalaya, India

Main Topics to be Covered:

1. Mapping Class Groups:
   - (i) Basic hyperbolic geometry of surfaces
   - (ii) Basics of Mapping Class Groups
   - (iii) Teichmuller theory for surfaces
2. Curve Complexes and Variants:
   - (iv) Basics of Curve complexes and variants
   - (v) Heegaard Distances
3. Hyperbolic Three Manifolds:
   - (vi) Basics of hyperbolic three manifolds
   - (vii) Teichmuller Theory and Kleinian groups
   - (viii) Lamination and Foliation, ends of 3-manifolds, tameness, density and ending lamination conjectures.

Nature of Participation: The pre-requisites assumed will be basic manifold theory (classification of surfaces) and algebraic topology.

Application Procedure: Those desirous of participating in the workshop and conference are requested to fill in the application form and send by post/E-mail attachment to:

Prof. Himadri Kumar Mukerjee,
Head, Department of Mathematics, NEHU,
Shillong 793 022, Meghalaya, India
E-mail: himadri@nehu.ac.in; himadrinehu@gmail.com
Phone: (+91364) 2722714, 2550070(R), 2726714(R)
Fax: (+91364) 2550076, 2721000

For More Details Visit:

www.nehu.ac.in/; http://www.nehu.ac.in/

Advanced Training in Mathematics Schools

Supported by

National Board for Higher Mathematics

Advanced Instructional School in Complex Analysis

Venue: Bhaskaracharya Pratishthana, and University of Pune, Pune

June 5–July 2, 2008

Conveners: Dinesh Thakur and S. A. Katre

Brief Description of ATM Schools: Advanced Training in Mathematics (ATM) Schools are a joint effort of more than 50 active researchers across the country with support from the National Board for Higher Mathematics. The programmes are conducted in reputed mathematics departments in Summer and Winter each year. In these Schools, the emphasis will be on problems solving and on highlighting inter-relations of basic subjects in mathematics. The schools are offered mainly for Ph.D. students and lecturers. At the initial stage, ATM Schools consist of two Annual Foundation Schools (AFS I & II) in basic topics such as algebra, analysis, and topology. At a later stage, Advanced Instructional Schools in different topics in Mathematics are organised especially for students who wish to pursue research in related areas.

Advanced Instructional School in Complex Analysis:

In this school, after introductory lectures on Complex Analysis, some advanced topics such as Introduction to Riemann surfaces, several variables theory, Teichmuller Theory will be discussed. There will be problem sessions in the afternoon and UM lectures. There will be emphasis on understanding various approaches and viewpoints to the subject matter, basic techniques, applications to Number Theory and Physics. Emphasis on examples, calculations as well as proofs. Several lectures should lead to live research topics.

We expect that there will be some follow-up workshops to this material in the next two years.

A special feature of this AIS programme is the inclusion of "Unity of Mathematics" Lectures on applications and interconnections with other parts of mathematics and sciences.

National Coordinating Committee		
Director	R. S. Kulkarni	IIT Bombay
Secretary	J. K. Verma	IIT Bombay
Members	S. D. Adhikari	HRI, Allahabad
	Satya Deo	HRI, Allahabad
	S. A. Katre	Pune U., Pune
	Shobha Madan	IIT Kanpur
	I. B. S. Passi	Panjab U., Chandigarh
	R. A. Rao	TIFR, Mumbai

Eligibility for Participation: The school will admit 40 students in their first and second years of Ph.D. programme, and a few young university lecturers and college teachers. Students who have attended AFS-I/II before will be given preference to attend this school.

Financial Support: Selected participants will be paid III-AC return train fare from their place of work/home town to the venue by shortest route and provided with accommodation and local hospitality.

How to Apply: The syllabus, application forms and other information about the programme is available on the websites:

http://www.math.iitb.ac.in/atm or www.bprim.org

Applications may also be made on plain paper, giving the following information: Name, Date of Birth, Age, Gender, Institute/ Department, Areas of interest, Address for correspondence, E-mail address, City, State, Pincode, Academic Record: B.Sc./M.Sc. with names of the Institutes. These should be attested by Head/Principal of the institute.

Completed Application Forms should Reach:

Professor S. A. Katre
AIS-Complex analysis
C/o, Bhaskaracharya Pratishthana
56/14, Erandavane, Damle Path
Off Law College Road, Pune 411 004
E-mail: sakatre@bprim.org
Phone/Fax: 91-22-25434547,
Phone: 91-22-25410724

by **Saturday, April 12, 2008**. List of selected candidates will be posted on the website of ATM Schools on **Saturday, April 19, 2008**.

Resource persons		
Sudhir Ghorpade	S. A. Katre	Ravi Kulkarni
A. R. Shastri	R. R. Simha	Dinesh Thakur
	Kaushal Verma	

Unity of Mathematics Lectures	
R. Balasubramanian	Ashok Raina

Analysis, PDES and Applications

On the Occasion of the 70th birthday of Vladimir Maz'ya

June 30–July 3, 2008

For More Details Visit:

http://www.mat.uniroma1.it/~mazya08/

International Conference on Nonlinear Dynamical Systems and Turbulence

July 17–22, 2008

Venue:

Faculty Hall
Indian Institute of Science
Bangalore

The IISc Mathematics Initiative (IMI), Indian Institute of Science, Bangalore will be organizing an International Conference on Nonlinear Dynamical Systems and Turbulence from July 17–22, 2008.

This conference will consist of plenary talks, invited lectures and poster sessions. There will be research talks by experts on all aspects of nonlinear dynamical systems and turbulence. The topics to be covered include: bifurcation, attractors and ergodic theory, elements of fluid mechanics and turbulence, nonlinear dynamics, burgers turbulence.

There would be about 30 outstation participants (including initial graduate students, young and interested faculty, scientists and industry participants).

A maximum of 10 seats will be reserved for scientists from organizations, which finance the theme programme.

Students and faculty from IISc who desire to participate in the theme programme will need to register.

Note: This web page will be modified continuously with updated information on paper submission and status, session themes, registration status and other details. So, please keep checking.

Deadline: May 31, 2008

Contact Number: +91-80-22933217, 22933218

For More Details Visit:

http://math.iisc.ernet.in/ imi/Conf-NDST.htm

Function Spaces, Differential Operators and Nonlinear Analysis

August 25–29, 2008

The conference will be organized between August 25–29, 2008 in Helsinki, Finland. The conference is preceded by a workshop, August 22–24, aimed especially at younger researchers.

The main topics of interest are harmonic analysis, nonlinear PDE's and applications. The program consists of invited lectures.

For Details Visit Home Page:

http://mathstat.helsinki.fi/fsdona

Geometry and Analysis

August 25–29, 2008

This conference, to be held at the Royal Institute of Technology in Stockholm, brings together leading specialists on nonlinear wave equations, geometric analysis and general relativity. Among the topics to be covered are Ricci flow, conformal geometry, nonlinear wave equations, black hole uniqueness and stability, as well as aspects of the constraint equations and the initial value problem for the Einstein equations.

There is no conference fee.

Registration is compulsory, before June 30, 2008.

A grant proposal has been submitted to the NSF for travel support for U.S. participants; those interested should contact Greg Galloway at galloway@math.miami.edu

Participants are expected to find their own accommodation. For local information, see http://www.math.kth.se/Howtoget/

Participants are advised that early booking of hotels is a necessity, since it is sometimes hard to find a place to stay in Stockholm at short notice.

The conference is sponsored by the Göran Gustafsson Foundation, the Swedish Research Council and the Wenner-Gren Foundations.

For More Information Visit: http://www.math.kth.se/ag08/

International Conference of Numerical Analysis and Applied Mathematics (ICNAAM 2008)

September 16–20, 2008

Venue:

Hotel Kypriotis Village-Kypriotis
Panorama–Kypriotis
International Conference Center, Kos, Greece

The aim of ICNAAM 2008 is to bring together leading scientists of the international Numerical and Applied Mathematics community and to attract original research papers of very high quality.

Topics to be Covered Include (but are not limited to):

All the research areas of Numerical Analysis and Computational Mathematics
All the research areas of Applied and Industrial Mathematics

Important Dates: Early Registration ends (i.e. fees paid and a Bank Slip is arrived in the fax (++ 30210 94 20 091, ++ 30 2710 237 397) of Secretary of ICNAAM or a Visa-Master-American Express Card has been charged): April 30, 2008

Normal Registration ends (i.e. fees paid and a Bank Slip is arrived in the fax (++ 30210 94 20 091, ++ 30 2710 237 397) of Secretary of ICNAAM or a Visa-Master-American Express Card has been charged): June 15, 2008

Late Registration ends (i.e. fees paid and a Bank Slip is arrived in the fax (++ 30210 94 20 091, ++ 30 2710 237 397) of Secretary of ICNAAM or a Visa-Master-American Express Card has been charged): July 25, 2008

Submission of Extended Abstract: July 15, 2008

Final Date Notification of Acceptance: July 22, 2008

Submission of the source files of the camera ready extended abstracts to American Institute of Physics (AIP Conference Proceedings): July 31, 2008 – Final Date

Submission of the full paper for consideration for publication in the journals: September 30, 2008 – January 31, 2009

Secretary ICNAAM (Mrs Eleni Ralli-Simou)
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tsimos.conf@gmail.com
Web: http://www.icnaam.org/

International Conference on Applied Mathematics and Approximation Theory

October 11–13, 2008

Venue:
University of Memphis,
Memphis, Tennessee.

Conference Topics:

Topics of interest include, but are not limited to:
(1) Abstract Approxmiation Theory
(2) Applied Mathematics

Standard Fee Schedule:
Standard Registration (payment deadline 15 September 2008): $230 U.S. Dollars

Late Registration (payment received after 15 September 2008): $250 U.S. Dollars

Conference Dinner (Sunday evening, October 12): $40 U.S. Dollars

(Note: We will need your Reservation for the dinner by 15 September 2008.)

Refund Policy: $200 refund if registration cancelled by 15 September 2008

Contact Information:

Professor George Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis TN 38152 USA
Phone Numbers: 901.678.3144
901.751.3553
Fax: 901.678.2480
E-mail correspondence to ganastss@memphis.edu

For More Information Visit:
http://www.msci.memphis.edu/AMAT2008/

Fractional Differentiation and Its Applications

November 05–07, 2008

Venue:

Cankaya University
Ogretmenler Caddesi No. 14 Balgat
Ankara, Turkey

Description: The scope of the workshop is to present the state of the art on fractional systems, both on theoretical and application aspects. The growing research and development on fractional calculus in the areas of mathematics, physics and engineering, both from university and industry, motivates this international event gathering and unifying the whole community. Main Areas: Representation tools; modeling vibration insulation; analysis tools; identification filtering synthesis tools; observation pattern recognition simulation tools; control edge detection.

The following disciplines are thus mainly concerned:

- Electrical engineering;
- Electronics;
- Electromagnetism;
- Electrochemistry;
- Thermal engineering;
- Mechanics;
- Automatic control;
- Robotics;
- Signal processing;
- Image processing;
- Biology;
- Physics;
- Mathematics;
- Economy

Important Dates:

Draft Papers Submission	:	15, April 2008
Notification of acceptance	:	6, June 2008
Early Registration Deadline	:	25, July 2008
Workshop date	:	05–07, November 2008

Contact Addresses:

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For More Information Visit:

http://www.cankaya.edu.tr/fda08/

International Conference on Graph Theory and It's Applications

December 11–13, 2008

Organised By:

Department of Mathematics
AMRITA VISHWA VIDYAPEETHAM Ettimadai,
Coimbatore 641 105, Tamil Nadu, India

About the Conference: This will be a three-day Conference in Graph Theory, Graph Algorithms and its applications. It will be focusing on the subareas in graph theory that has applications in Optimization, Computing Techniques, VLSI Design and Testing, Image Processing, and Network Communications. The goal of this conference is to bring top researchers in this area to Amrita to foster collaboration and to expose students to important problems in the growing field. The meeting will stimulate joint work among researchers both India and abroad and attract students and postdoctoral fellows.

The Conference will cover a broad range of topics in Graph Theory. The topics include, but are not limited to:

- Graph Theory
- Algebraic Graph Theory
- Algorithms and Computing Techniques
- Graph Optimization
- VLSI Design and Testing
- Image Processing
- Networks
- Communications and Control Theory

Registration: Interested participants should send in their application in the enclosed proforma along with the prescribed registration fee in the form of demand draft drawn in favor of "The Coordinator, Graph Theory Conference 2008", payable at Coimbatore, India.

Registration Fee: Participants from India – Rs. 500/- (Rupees Five Hundred only)

Participants from other countries – $250/- (US dollar Two Hundred and Fifty only)

Important Dates:

Full paper to be sent before	:	15th September 2008
Announcement of acceptance	:	5th October 2008
Registration for Accommodation	:	15th November 2008
Conference Date	:	11–13 December 2008

Contact Address:

Dr. K. Somasundaram,
Associate Professor
Department of Mathematics,
Amrita Vishwa Vidyapeetham
Coimbatore 641 105, India
Phone:+91-422-2656422 Extension: 281
E-mail: icgta08@amrita.edu
s_sundaram@ettimadai.amrita.edu
Web: http://www.amrita.edu/icgta08/

INFOSYS
Prize for Mathematics

Call for Nominations:

Infosys Technologies Limited (Infosys) and the National Institute of Advanced Studies (NIAS), Bangalore announce the establishment of a new Prize in mathematics called the INFOSYS Prize for Mathematics.

The Prize will be awarded to a person who, in the opinion of the award jury, has made outstanding contributions fundamental or applied in any field of mathematics including in the areas of pure mathematics, mathematical foundations of computer science and applied mathematics in natural, life and social sciences. The Prize is to recognize contributions of extraordinary depth and influence to the mathematical sciences, and which raise the status of mathematics in society and stimulate the interest of students in mathematics.

The Prize will consist of a cash prize of Rs. 10 Lacs (Rupees Ten Lacs) and a medal. If more than one person wins the award in a given year, the prize money will be equally shared and each will be awarded a separate medal.

Candidates should have held a permanent position in India for at least five years as on the year of the award.

Candidates should not have attained the age of 45 as on 31 December of the year of the award.

The Jury for this Prize will consist of eminent mathematicians from around the world with representations from different fields.

Candidates must be nominated. Self-nomination is not accepted. The nominator must send a brief report highlighting the significant achievements of the nominee along with a CV and a list of publications.

The nomination must be posted to "Infosys Prize for Mathematics", National Institute of Advanced Studies, Indian Institute of Science Campus, Bangalore 560 012.

The last date for receiving nominations is:

April 30, 2008.

International Congress of Mathematicians, 2010 (ICM 2010)

Satellite Conferences

The **International Congress of Mathematicians** (ICM), which is the largest mathematical conference covering all areas of mathematics, is held once every four years. The next ICM will be held in **Hyderabad, India** during **August 19–27, 2010**.

It is customary to organize several international conferences in emerging areas of mathematical research, close to the dates of the ICM, in the region where it is being held taking advantage of the presence of delegates to the ICM from all over the world. The Executive Organizing Committee (EOC) has set aside some funds for supporting such conferences held during **August–September, 2010**. Conferences receiving such support will be labelled "ICM Satellite Conferences" by the EOC and will be listed as such on the ICM 2010 web-site and a link will be provided to the conference web-site, if there is one. Conference organizers who may not need finacial support of the EOC may also apply for this listing and link provision.

The **International Mathematical Union** (IMU), which is responsible for the conduct of all ICMs, has designated 20 areas of mathematics in which parallel sessions would be held during ICM 2010. They are as follows.

1. Logic and Foundations,
2. Algebra,
3. Number theory,
4. Algebraic and complex geometry,
5. Geometry,
6. Topology,
7. Lie theory and generalizations,
8. Analysis,
9. Functional analysis and applications,
10. Dynamical systems and ordinary differential equations,
11. Partial differential equations,
12. Mathematical physics,
13. Probability and Statistics,
14. Combinatorics,
15. Mathematical aspects of computer science,
16. Numerical analysis and scientific computing,

17. Control theory and optimization,
18. Mathematics in science and technology,
19. Mathematics education and popularization of mathematics, and
20. History of Mathematics.

For a detailed description of these sections see the ICM 2010 conference web-site **www.icm2010.org.in**.

Those who are interested in organizing satellite conferences in the areas mentioned above are encouraged to submit a proposal to the **Subcommittee for Satellite Conferences, ICM 2010** containing the following information:

- Title of the conference
- Brief description of the theme and scope of the conference
- Section(s) (see the IMU classification above)
- Name of the organizer(s)
- Affiliation(s)
- Address for correspondence
- Phone number(s)
- Fax
- E-mail
- Venue of the conference
- Dates of the conference
- Composition of the organizing committee
- Probable list of invited speakers
- Expected number of participants
- Budget
- Funds requested from the EOC, ICM 2010
- Funds requested/available from other sources
- Name, Designation and Address of the authority to whom funds should be sent in case the proposal is approved.

The Proposals Should be Sent To:

Professor S. Kesavan,
Convenor, Committee for Satellite Conferences, ICM 2010,
The Institute of Mathematical Sciences,
CIT Campus, Taramani,
Chennai 600 113.

and a soft copy (**in PDF format**) should also be sent via E-mail to **kesh@imsc.res.in** so as to reach before **September 30, 2008**.

Organizers of conferences that have been short listed for financial support will be notified directly by the EOC, ICM 2010. The information will also be available at the ICM 2010 web-site:

(**www.icm2010.org.in**) by **January 31, 2009**.

Chennai Mathematical Institute

H1, SIPCOT IT Park, Padur P.O.,
Siruseri 603 103

A University under Section 3
of the UGC Act, 1956

Admissions, 2008

CMI is a centre of excellence for research and teaching in the mathematical sciences. CMI conducts B.Sc. (Hons.) programmes in Mathematics & Computer Science (integrated) and Physics, and M.Sc. and Ph.D. programmes in Mathematics and in Computer Science.

All courses at CMI are taught by active researchers who draw on their professional expertise to offer new insights into the subject material.

CMI graduates have been admitted with full financial aid at the best academic institutions in India and abroad. These include Caltech, MIT, Princeton, and Yale in USA, ENS Paris in France, the Max Planck Institutes in Germany and the IITs, IMSc and ISI in India. CMI students have also been placed in leading companies.

CMI's programme is fully residential and is housed in a beautiful campus with excellent facilities. All students receive handsome scholarships that cover all expenses. To obtain an application form for the session starting in August 2008, send a DD for Rs. 300/- in favour of Chennai Mathematical Institute payable at Chennai to the address above. Indicate your name, address and the programme(s) you are applying for. Application forms can also be downloaded from http://www.cmi.ac.in. The last date for submitting forms is April 21, 2008. Enquiries may be sent to admissions@cmi.ac.in.

Thompson and Tits Win 2008 Abel Prize

John Griggs Thompson, Graduate Research Professor, University of Florida, and Jacques Tits, Professor Emeritus, Collége de France, have been awarded the 2008 Abel Prize "for their profound achievements in algebra and in particular for shaping modern group theory." In the prize citation, the Abel Committee writes that "Thompson revolutionized the theory of finite groups by proving extraordinarily deep theorems that laid the foundation for the complete classification of finite simple groups, one of the greatest achievements of twentieth century mathematics." In 1963, Thompson and Walter Feit proved that all nonabelian finite simple groups were of even order, work for which they both won the Frank Nelson Cole Prize in Algebra from the AMS in 1965. Thompson also won a Fields Medal in 1970. In the Abel citation for Tits, the committee writes that "Tits created a new and highly influential vision of groups as geometric objects. He introduced what is now known as a Tits building, which encodes in geometric terms the algebraic structure of linear groups." The committee noted the link between the two winners' work: "Tits's geometric approach was essential in the study and realization of the sporadic groups, including the Monster." Tits received the Grand Prix of the French Academy of Sciences in 1976, and the Wolf Prize in Mathematics in 1993. The Abel Prize is awarded by the Norwegian Academy of Science and Letters for outstanding scientific work in the field of mathematics. The prize amount is 6,000,000 Norwegian kroner (over US$1,000,000). Thompson and Tits will receive their prize in a ceremony in Oslo on May 20, 2008. See the Abel Prize website (`http://www.abelprisen.no/en/`) for more information about the laureates, their work, and the prize.

Thanks to the Family of Late Professor S. K. Lakshman Rao

Ramanujan Mathematical Society receives contribution from the family of the late Prof. S. K. Lakshman Rao.

RMS gratefully acknowledges the generous contribution of Rs. 5,00,000/- (Rupees five lakhs only) made by Ms. S. K. Vijayalakshamma, Mr. K. C. Sekhar and Ms. K. Padmavathy, of Bangalore, in memory of their brother, distinguished mathematician and scholar, the late Prof. S. K. Lakshman Rao.

The readers may download the Mathematics Newsletter from the RMS website at
`www.ramanujanmathsociety.org`