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Why Do We Call them Projective Spaces?

Satya Deo
Harish–Chandra Research Institute
Chhatnag Road, Jhusi, Allahabad 211 019
E-mail: sdeo@mri.ernet.in, vcsdeo@yahoo.com

1. Introduction

In this article, the symbol $\mathbb{R}^n$, $n \geq 0$, will denote the real vector space of dimension n with the Euclidean topology or the usual metric topology. Some of its subspaces like the n-disk $\mathbb{D}^n$ and the $(n-1)$-sphere $\mathbb{S}^{n-1}$ are the simplest kind of topological spaces which we use frequently.

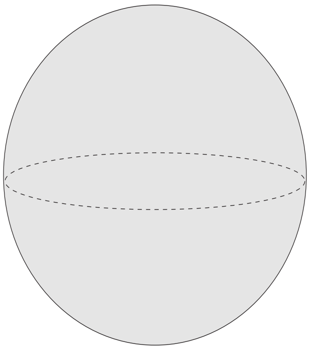

Figure 1. 3-disk

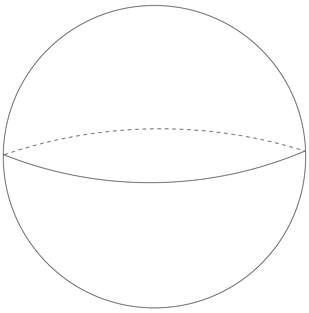

Figure 2. 2-sphere

These are the faithful pictures of given spaces which are subsets of the 3-dimensional Euclidean space. We can really see them because they are sitting in our 3-space. The pictures of 4-disks and 3-spheres cannot be drawn faithfully because these spaces are lying in 4-dimensional Euclidean spaces and are not homeomorphic to any subspace of $\mathbb{R}^3$. This last statement

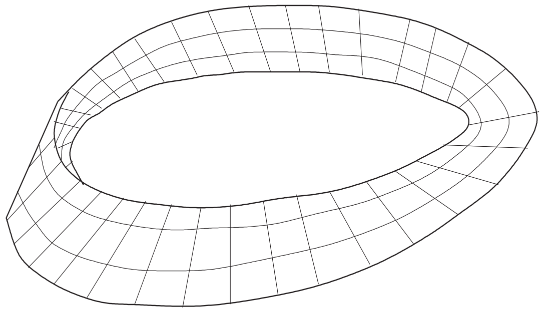

Figure 3. Moebius band

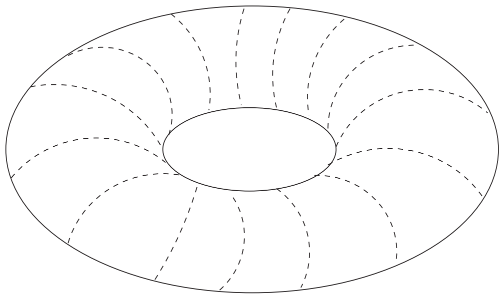

Figure 4. Torus

needs a proof which can be given but we are omitting it here because the proofs are not relevant to this elementary article. The same statements are true for all n-disks and all $(n-1)$-spheres for any $n > 3$.

The next class of simple and useful examples of topological spaces are the Moebius band (this is like a band and was discovered by A. F. Moebius), the torus, the projective plane and the Klein bottle (this appears like a bottle and was studied by Felix Klein).

We are not drawing any pictures of the projective plane or the Klein bottle because these spaces cannot be embedded as subspaces of the 3-space. This is, of course, a subtle result, but can be proved using the methods of algebraic topology, and as mentioned earlier we take it for granted. This is essentially the reason that none of the books on topology display a picture

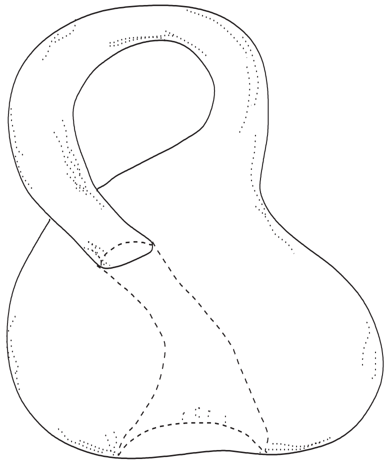

Figure 5. Klein bottle

of a projective plane. Some books do try to draw a picture of the Klein bottle, but that picture is neither faithful nor correct. The picture usually presented only stresses the fact that if we try to draw a figure of the Klein bottle in 3-space, then there are bound to be self-intersections in the bottle as shown in Figure 5.

The projective spaces are amongst the most important kind of topological spaces, and yet their elaborate description is usually not given in the elementary books on topology. The purpose of this article is to explain the construction and salient properties of various kinds of projective planes and their generalizations called projective spaces explaining, on the way, why the word "projective" is used to describe them.

2. Projective Planes

We first explain several ways to define a real projective plane and then discuss some other interesting topological properties of these spaces normally covered in any first course on algebraic topology (see [3] or [4] for details).

2.1 Real Projective Plane

It is well-known that Euclidean geometry starts with some undefined terms followed by a set of axioms given by Euclid. Based on these axioms we are able to deduce theorems of Euclidean geometry which are very often visually clear and appealing. This geometry takes into account basically the Euclidean metric, e.g., the measurements of lengths, the angles etc. There is an equally important kind of geometry called Projective geometry in which such measurements are absent, but yet for which there are interesting geometrical properties which follow from the axioms. This geometry, which was discovered much later in 19th century (see [2] for details) as compared to Euclidean geometry, also starts with some undefined terms like "points", "lines" and "planes" etc., and a set of axioms to build a kind of noneuclidean geometry. To get an idea of projective geometry, let us take a point and project from this point a circle lying on a plane not containing that point onto another plane. The resulting image of a circle may not be a circle, it could be an ellipse and really depends on the position of the projecting point as well as on the location of the two planes. Similarly, the projection of two parallel lines may result in non-parallel lines. However, it is interesting to learn that there are certain essential geometrical properties which remain intact upon such projections. We remark that in projective geometry points, lines and planes don't look like the points, lines or planes of Euclidean geometry and so their visual pictures are not what we are accustomed to see. Therefore if we have to consider a projective line or a projective plane, we only consider a suitable "model" of them and make sure that points and lines describing them satisfy the axioms of projective geometry. We may or may not be able to visualize them. We give below several models of the "real projective plane" along with its topology. However, before we do that, let us briefly recall the axioms and definitions of these geometries:

Let X be a set of "points" $x \in X$ and let $\mathbb{L}$ be a collection of subsets of X, called "lines" of X. Then the pair $(X, \mathbb{L})$ is said to be a plane geometry if the following two conditions are satisfied.

(a) Given any two distinct points P, Q of X there is a line $l \in \mathbb{L}$ such that l passes through P and Q.

(b) Given a line l and a point P not lying on l, any line l' passing through P intersects l in at most one point.

If, in addition to (a) and (b), the following is also satisfied then the geometry $(X, \mathbb{L})$ is called an Euclidean plane geometry.

(c) **(Parallel Postulate)** Given a line l and a point P not lying on l, there exists a unique line which does not intersect l.

If (c) is not satisfied, then the geometry $(X, \mathbb{L})$ is said to be a noneuclidean geometry. If, in place of (c), the following (c′) is satisfied, then the geometry $(X, \mathbb{L})$ is said to be a projective plane geometry.

(c′) Given a line l and a point P not lying on l, every line through P intersects l in a unique point.

If $(X, \mathbb{L})$ is a projective plane geometry then the set X is called a projective plane. Observe that in a projective plane there are no parallel lines. Here is the motivating description of a projective plane:

Example: Consider the 2-sphere $\mathbb{S}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ as the set of "points". Declare great circles of $\mathbb{S}^2$ as "lines". Then clearly, (a) is true but (b) is not true. To make (b) also true we modify our set $\mathbb{S}^2$ and consider any pair of antipodal points x and $-x$ to be a single point. Suppose X is the resulting quotient set whose elements we will call "points". Now we declare the image of great circles of $\mathbb{S}^2$ as "lines" of the new set X. Then one can check that $(X, \mathbb{L})$ is a projective geometry and therefore the set X of "points" (the equivalence classes consisting of pairs of antipodal points) with new "lines" is called a projective plane of the projective geometry.

When we consider only the topology of $\mathbb{S}^2$ and take the quotient topology on X, the new topological space X is called the real projective plane in topology and this space is denoted by $\mathbb{R}P^2$. This explains the reason why we call $\mathbb{R}P^2$ a projective plane. Once it is clear why we call this space a projective plane, then any other description of this plane or even any generalization of this can be called a projective space regardless of how we arrive at that space. Below, we will arrive at the same topological space through several other methods yielding different but equivalent construction of the real projective plane $\mathbb{R}P^2$. We will only emphasize how we get the set of points and the topology on that.

Let us consider the Euclidean 3-space $\mathbb{R}^3$ and imagine the set of all straight lines of this space passing through the origin. These lines are clearly distinct. If we identify each of these lines with a single point, then since all these lines have a single point in common, the resulting quotient space is just a singleton. To avoid this collapsing we consider the punctured 3-space $\mathbb{R}^3 - (0, 0, 0)$ in which the lines are now mutually disjoint. In fact each line now is the union of two components along opposite directions of the origin. Now identify each of these broken lines with a point. Then we get a quotient space X of $\mathbb{R}^3 - (0, 0, 0)$ whose points are still in 1-1 correspondence with the set of all lines of the 3-space passing through the origin, but the space itself is something new. Take any 2-dimensional plane passing through the origin and declare the equivalence classes lying in this plane as the "lines". Then again one gets a projective plane satisfying the axioms of projective plane geometry. Hence this quotient space is also called the real projective plane and is denoted by $\mathbb{R}P^2$(see [2] for details).

There is another way to describe the same real projective plane which is quite interesting. Now we don't care about the "points", "lines" and the axioms; we care only about the topological space $\mathbb{R}P^2$ and so any space X which is homeomorphic to $\mathbb{R}P^2$ will be called a projective plane. The antipodal map $x \mapsto -x$ from the 2-sphere to itself defines an obvious action of the group $G = \mathbb{Z}_2$ on $\mathbb{S}^2$ whose orbit space is clearly what we have described above as the real projective plane. We will see later on that it is this description of the real projective plane which yields an easy generalization of the concept of real projective plane to complex projective plane, quaternionic projective plane, Cayley plane and their higher dimensional versions providing us with a very attractive and useful class of topological spaces.

There is one more method of defining the real projective plane which is also frequently useful. Consider the unit 2-disk $\mathbb{D}^2$. Its boundary is clearly the unit circle $\mathbb{S}^1$. Identify each point x of the boundary circle to its antipodal point $-x$. Then the resulting quotient space of the 2-disk is again homeomorphic to the real projective plane. Another view of this construction, which is quite common, is to take a square $ABCD$ and identify the directed line segments BC with DA and AB with CD. Then the resulting quotient space is the real projective plane.

Finally, there is yet another method of defining the real projective space using the concept of "attaching spaces". Once

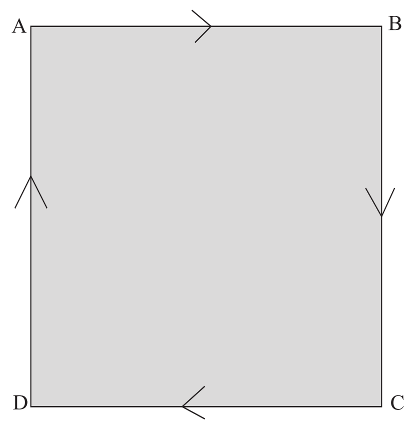

Figure 6. Projective plane

again we consider a 2-sphere $\mathbb{S}^2$ and fix a small disk $\mathbb{D}^2$ lying on $\mathbb{S}^2$. Remove the interior of this disk. Take a Moebius band $\mathbb{M}$ and note that since the boundary $\partial\mathbb{M}$ is homeomorphic to a circle $\mathbb{S}^1$, there is obviously a homeomorphism, say h from the boundary of the Moebius band to the boundary of the removed 2-disk which is still a part of $\mathbb{S}^2$. Now we attach the Moebius band $\mathbb{M}$ to the remaining part of $\mathbb{S}^2$ via this map h. It can now be shown that the quotient space obtained by this attaching is independent of the disk chosen and also of the homeomorphism chosen, and the resulting space is indeed homeomorphic to the real projective plane.

Using the definition of quotient topology, one can easily prove that all the descriptions of the projective plane mentioned above produce homeomorphic topological spaces. The proof is omitted for the simple reason that the technical details of the proof dealing with quotient topology will only interrupt the otherwise easy appreciation of the objectives of this article.

We may remark that, unfortunately, none of the definitions of the real projective plane mentioned above help us in faithfully visualizing the real projective plane, and as mentioned earlier, we really cannot do this, but one description could be definitely preferable to the other in having a better idea of the real projective plane as a topological space. Using the fact that certain properties of topological spaces easily pass on to the quotient spaces, we can easily see that the real projective plane is a compact, Hausdorff and path-connected space. The more significant and deeper property of the real projective plane is that it is a non-orientable 2-dimensional manifold without boundary and hence is a closed surface. Its Euler characteristic is 1.

2.2 Complex Projective Plane

Having understood the definition and properties of the real projective plane, it is natural to ask if there is an analogue of the real projective plane in the setting of complex numbers. The answer is readily "yes" and it can be easily described. What we have to observe is that the real numbers have to be carefully replaced by the complex numbers everywhere. Since the complex numbers $\mathbb{C}$ have a multiplication so that $\mathbb{C}$ becomes a field, we can consider the complex vector space $\mathbb{C}^3$ (in place of the real vector space $\mathbb{R}^3$) which has a good topology viz. the topology of the Euclidean space $\mathbb{R}^6$. Also, we can talk of "complex lines", "complex 2-sphere" etc. Thus we think of all complex lines passing through the origin and identify each line of the space $\mathbb{C}^3 - \{0\}$ with a point, so that the quotient space is defined to be the complex projective plane $\mathbb{C}P^2$. The following description of the complex projective plane is sometimes better: We start with the 2-dimensional complex sphere:

$$\{(z_0, z_1, z_2) \in \mathbb{C}^3 : |\Sigma|z_i|^2 = 1\}$$

The unimodular complex numbers form the well-known circle group $\mathbb{S}^1$, and there is an obvious action of this group on the complex 2-sphere defined as follows:

$$z.(z_0, z_1, z_2) = (z.z_0, z.z_1, z.z_2)$$

Now it is easily seen that the orbit space resulting from this group action is precisely our complex projective plane. Notice that the complex projective plane is a compact Hausdorff simply connected manifold of complex dimension 2, but of real dimension 4.

Visualizing faithfully the complex line or the complex 2-sphere etc is not possible and therefore one has no idea of how a complex line or complex plane looks like, but the algebraic picture of complex line as 1-dimensional subspace and complex plane as 2-dimensional subspace of $\mathbb{C}^n$is quite clear and we depend on that algebraic picture. We don't worry anymore about the correct geometric pictures of these objects, and this is now going to be our pattern when we define other projective spaces later on. In fact given any field $\mathbb{F}$ with a topology (may be even discrete) one can take the vector space $\mathbb{F}^3$ and consider all of its one-dimensional subspaces. Then identify each such subspace in $\mathbb{F}^3 - (0, 0, 0)$ with a point. The resulting quotient space is then defined to be the projective plane over the field $\mathbb{F}$ and is denoted by $\mathbb{F}P^2$.

2.3 Quaternionic Projective Plane

Extending the idea of real projective plane, complex projective plane, one is tempted to say that likewise one can define a multiplication in $\mathbb{R}^3$, $\mathbb{R}^4$, ... so that these are all fields and then define projective planes corresponding to these fields in an analogous manner. However, it is well known that there is no multiplication in any of these vector spaces which can make them into a field. It is a deep result proved by the methods of algebraic topology that we can define a multiplication in $\mathbb{R}^n$, $n \geq 3$ making the vector space into a division algebra only for $n = 4$ and $n = 8$ (see [3] p.173). For $n = 4$, there is a well known multiplication defined by William Hamilton (in 1943) so that $\mathbb{R}^4$ becomes a skew field $\mathbb{H}$, and then the elements of the corresponding vector space are called quaternions. For example if we have two quaternions $q_1 = (a_1, a_2, a_3, a_4)$ and $q_2 = (b_1, b_2, b_3, b_4)$, then their product is defined as

$$q_1q_2 = (a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4, a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3, a_1b_3 + a_3b_1 + a_2b_4 - a_4b_2, a_1b_4 + a_4b_1 + a_2b_3 - a_3b_2)$$

Now we observe that we can talk of quaternionic lines in $\mathbb{H}^3$ as its 1-dimensional right subspaces. Then, as before, we consider all quaternionic lines of $\mathbb{H}^3 - \{0\}$ passing through origin and identify each line to a point. Then the quotient space so obtained is called the quaternionic projective plane and is denoted by $\mathbb{H}P^2$. We can also define these planes by considering the group $\mathbb{S}^3$ of unit quaternions acting on the right on the 2-dimensional quaternionic sphere contained in $\mathbb{H}^3$ as follows:

$$(q_0, q_1, q_2) \cdot q = (q_0q, q_1q, q_2q)$$

Then the corresponding orbit space can be easily seen to be homeomorphic to the quaternionic projective plane $\mathbb{H}P^2$. We remark that this projective plane is a compact connected manifold of quaternionic dimension 2, but its real dimension is 8.

2.4 Octonionic Projective Plane or Cayley Plane

Finally, let us consider the vector space $\mathbb{R}^8$ and introduce the well-known multiplication defined by Arthur Cayley (in 1845) in this vector space so that it becomes a division algebra. Then the elements of this vector space, denoted by Cay, are called Cayley numbers or Octonions. The product of Cayley numbers is defined as follows:

$$(a, b).(c, d) = (ac - \bar{d}b, da + b\bar{c})$$

Here a Cayley number is represented by a pair of quaternions and $\bar{a} = a_0 - a_1i - a_2j - a_3k$ denotes the quaternionic conjugate of $a = a_0 + a_1i + a_2j + a_3k$. Following the pattern of quaternions one can now easily define the 1-dimensional subspaces of Cay^3 and then identify each of these subspaces in $Cay^3 - \{0\}$ to obtain the Cayley plane $CayP^2$. We may remark here that the orbit space description of the Cayley plane is not possible since the set of unit octonions $\mathbb{S}^7$ has a multiplication which is not associative and so a suitable group action cannot be defined on the 2-dimensional octonionic sphere. The Cayley plane is a compact Hausdorff simply connected manifold of real dimension 16. This is the story of classical projective planes.

3. Higher Dimensional Projective Spaces

The notion of real projective plane can be easily generalized to obtain the so called n-dimensional real projective space $\mathbb{R}P^n$ for any $n \geq 0$. Consider the Euclidean $(n+1)$-dimensional space $\mathbb{R}^{n+1}$ and think of all lines passing through the origin of this space. Observe that in the space $\mathbb{R}^{n+1} - \{0\}$, these lines are all disjoint. We identify each one of them with a single point and go to the quotient space. This quotient space, denoted by $\mathbb{R}P^n$, is called the real projective space of dimension n. The definition of real projective space of dimension n can also be formulated in terms of the orbit space of the n-sphere $\mathbb{S}^n$ lying in $\mathbb{R}^{n+1}$. For this we just note that the antipodal map on $\mathbb{S}^n$ defines an action of the group $G = \mathbb{Z}_2$ on $\mathbb{S}^n$ and the orbit space is precisely the real projective space $\mathbb{R}P^n$. It is easy to see that this projective space is a compact Hausdorff connected manifold of real dimension n, whose topology is very well understood. We will now explain what is meant by saying that its topology is well-understood.

At this point, it is appropriate to mention some basic topological invariants of a topological space X. The fundamental group $\pi_1(X)$, the homology groups $H_q(X; G)$, G is the coefficient group. $q \geq 0$, and the cohomology groups $H^q(X; G)$, $q \geq 0$, of a space X are the simplest and well-understood (see [3] for details) topological invariants in the sense that if two spaces X and Y are homeomorphic, then the fundamental group, the homology groups, the cohomology groups of X and Y are isomorphic in all dimensions. Therefore, knowing precisely these groups for a space X in fact yields a fairly accurate understanding of the topology of the space X. Higher homotopy groups of a topological space are yet another important class of topological invariants, but their determination has been a difficult proposition even for as simple spaces as spheres! It is in this respect as well as in respect of cellular structure decomposition of these spaces that the topology of projective spaces which we have discussed so far is well-understood.

The fundamental group $\pi_1(\mathbb{R}P^n)$ is the cyclic group of order two. This follows from the fact that $\mathbb{S}^n$ is a double covering of $\mathbb{R}P^n$. Since these projective spaces are compact Hausdorff manifolds of dimension n, these can be embedded in the Euclidean space $\mathbb{R}^{2n+1}$. These spaces carry a nice finite CW structure on them such that there is one k-cell in each dimension k for $0 \leq k \leq n$. This structure provides a very good handle for dealing with these spaces and proving results about them. For example, their cellular homology with integer coefficients is quite easy to calculate and is given by

$$H_i(\mathbb{R}P^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i = 0 \text{ and } i = n \text{ odd} \\ \mathbb{Z}_2 & \text{for } i \text{ odd, } 0 < i < n \\ 0 & \text{otherwise} \end{cases}$$

Since these spaces are compact polyhedra, their cellular homology is the same as singular or Cech homology. The most important property of these spaces is that their cohomology algebra with coefficients in $\mathbb{Z}_2$ can be calculated without much difficulty and the result is

$$H^*(\mathbb{R}P^n, \mathbb{Z}_2) = \mathbb{Z}_2[a]/(a^{n+1}),$$

where a is a homogeneous element of degree 1. Following the pattern of real projective spaces, one can easily define the complex projective spaces $\mathbb{C}P^n$ for all $n \geq 1$.The definition using the notion of group actions is direct. One considers the complex n-dimensional sphere given by

$$\{(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} | \Sigma |z_i|^2 = 1\}$$

which is really a real $(2n+1)$-dimensional sphere $\mathbb{S}^{2n+1}$. We also note that the circle group $\mathbb{S}^1$ acts freely on $\mathbb{S}^{2n+1}$ by coordinatewise multiplication. Then the orbit space so obtained is the complex projective space $\mathbb{C}P^n$ of real dimension $2n$. Topological properties of these complex projective spaces are very interesting because they also carry a very nice CW structure having a $2k$-cell in each even dimension $2k \leq 2n$. $\mathbb{C}P^n$ is simply connected and its cellular homology is given by

$$H_i(\mathbb{C}P^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i = 0, 2, 4, \ldots, 2n \\ 0 & \text{otherwise} \end{cases}$$

As in the case of real projective spaces its cohomology algebra with integer coefficients is given by

$$H^*(\mathbb{C}P^n, \mathbb{Z}) = \mathbb{Z}[a]/(a^{n+1}),$$

where a is a homogeneous element of degree 2.

Taking the clue from the definition of complex projective spaces one can also define the quaternionic projective space $\mathbb{H}P^n$ as the orbit space of the real $(4n+3)$-dimensional sphere when the group $\mathbb{S}^3$ of unit quaternions acts freely on it on the right. These spaces are also compact Hausdorff simply connected manifold of real dimension $4n$ and they carry a nice CW-structure having one k-cell in each dimension which is a multiple of 4. The homology of $\mathbb{H}P^n$ with integer coefficients is given by

$$H_i(\mathbb{H}P^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i = 0, 4, 8, \ldots, 4n \\ 0 & \text{otherwise} \end{cases}$$

Its cohomology algebra is also computed using the same method as for complex projective spaces and the result is

$$H^*(\mathbb{H}P^n, \mathbb{Z}) = \mathbb{Z}[a]/(a^{n+1}),$$

where a is a homogeneous element of degree 4.

Finally, one can also define the octonionic projective space using the first definition of real projective spaces as the quotient space of lines passing through the origin. Then its cohomology can be computed as well. The interested reader is referred to [3] for further details.

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A Dozen Integrals: Russell-Style

Tewodros Amdeberhan
Department of Mathematics
MIT, Cambridge, MA 02139
E-mail: tewodros@math.mit.edu

Victor H. Moll
Department of Mathematics
Tulane University, New Orleans, LA 70118
E-mail: vhm@math.tulane.edu

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On June 15, 1876, the Proceedings of the Royal Society of London published the paper [1] by Mr. W. H. L. Russell entitled *On certain integrals*. The paper starts with *The following are certain integrals which will, I hope, be found interesting*. The rest of the paper is a list of 12 definite integrals starting with

$$\int_0^\infty dz\, e^{-(r+1)z+xe^{-z}} = e^x(1 - r/x + r(r-1)/x^2 - \cdots)/x.$$

In honor of Russell notable achievements in the evaluation of integrals, we hope the reader will find the next list interesting:

$$\int_0^\infty x\left(\frac{\gamma \sinh \gamma x}{\cosh^2 \gamma x} e^{-x^2/\pi^2} + \frac{\sqrt{\pi}\sinh x}{\cosh^2 x} e^{-\gamma^2 x^2}\right) dx = \int_0^\infty \frac{e^{-x^2/\pi^2}\, dx}{\cosh \gamma x} \tag{1}$$

$$\int_0^\infty x\left(\frac{1}{\pi} e^{-x^2/\pi^2} + \frac{1}{\sqrt{\pi}} e^{-x^2}\right)\frac{\sinh x\, dx}{\cosh^2 x} = \int_0^\infty \frac{e^{-x^2}\, dx}{\cosh \pi x} \tag{2}$$

$$\int_0^\infty x\left(e^{-x^2/\pi} + 2e^{-16x^2/\pi}\right)\frac{\sinh 2x\, dx}{\cosh^2 2x} = \int_0^\infty \frac{e^{-4x^2/\pi}\, dx}{\cosh 4x} \tag{3}$$

$$\int_0^\infty \left(\frac{\sinh x}{\cosh^2 x} + \frac{\pi^{3/2}\sinh \pi x}{\cosh^2 \pi x}\right) x e^{-x^2}\, dx = \int_0^\infty \frac{e^{-x^2}\, dx}{\cosh x} \tag{4}$$

$$\int_0^\infty x e^{-x^2/\pi}\frac{\sinh x\, dx}{\cosh^2 x} = \int_0^\infty \frac{e^{-4x^2/\pi}\, dx}{\cosh 2x} \tag{5}$$

$$\int_0^1 \frac{x^{-\ln x}\, dx}{1+x^2} = \int_0^\infty \frac{e^{-4x^2/\pi}\, dx}{\cosh 2\sqrt{\pi}x}\, dx \tag{6}$$

$$\int_0^\infty \frac{x^2 e^{-x^2}\, dx}{\cosh \sqrt{\pi}x} = \frac{1}{4}\int_0^\infty \frac{e^{-x^2}\, dx}{\cosh \sqrt{\pi}x} \tag{7}$$

$$\int_0^\infty \left(e^{-x^2/\pi^2} + \pi^{5/2}e^{-x^2}\right)\frac{x^2\, dx}{\cosh x} = \frac{\pi^2}{2}\int_0^\infty \frac{e^{-x^2/\pi^2}\, dx}{\cosh x} \tag{8}$$

$$\int_0^\infty \left(\sqrt{\pi} e^{-x^2/3} + 9\sqrt{3}\pi^{-2} e^{-3x^2/\pi^2}\right)\frac{x^2\, dx}{\cosh x} = \frac{3\pi\sqrt{3}}{2}\int_0^\infty \frac{e^{-3x^2}\, dx}{\cosh \pi x} \tag{9}$$

$$\int_0^\infty \left(\pi^5 e^{-\pi^3 x^2/G} + G^{5/2} e^{-Gx^2/\pi}\right)\frac{x^2\, dx}{\cosh \pi x} = \frac{\pi G^{3/2}}{2}\int_0^\infty \frac{e^{-Gx^2/\pi}\, dx}{\cosh \pi x} \tag{10}$$

$$\int_0^\infty x(3 - 4\pi x^2)\frac{e^{-\pi x(x+1)}\, dx}{\sinh \pi x} = \frac{1}{2\pi} \tag{11}$$

$$\int_0^\infty \frac{\sin^2 x}{\cosh x + \cos x}\frac{dx}{x^2} + \frac{2}{\pi}\int_0^{e^{-\pi/2}} \frac{\tan^{-1} x}{x}\, dx = \frac{\pi}{4} \tag{12}$$

Here,

$$\gamma := \lim_{n\to\infty} \sum_{k=1}^n \frac{1}{k} - \ln n \tag{13}$$

is Euler's constant and

$$G := \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^2} \tag{14}$$

is Catalan's constant. We note the fact that both of these constants have so far resisted all attempts at proofs of irrationality.

Every formula can be checked by observing that if $f(y) = 1/\cosh(y)$ and M is the transformation

$$M(f)(y) = \int_0^\infty e^{-x^2} f(xy)\,dx,$$

then we have the elementary relation

$$\frac{df}{dx} = -xf(y)\sqrt{1-f^2(y)}$$

and also

$$yM(f)(y) = \sqrt{\pi}M\left(f\left(\frac{\pi}{y}\right)\right).$$

Applying these to the left hand side integrands produces the right-hand side. In the last few calculations, the role of the functions $1/\sinh y$ and e^{-x^2} has to be interchanged.

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Multiply Connected Domains as Circular-Slit Annuli

R. R. Simha

simhahome@yahoo.com

1. Introduction

The aim of these talks is to show that every domain of finite connectivity in the complex plane $\mathbb{C}$ is biholomorphic (i.e. conformally equivalent) to a special kind of domain, namely, the complement of a finite set in the complex plane or the unit disc, or the complement of a finite union of closed arcs (which could be singletons) lying on circles $|z| = r$, in an annulus $A_R := \{z \in \mathbb{C} : 1 < |z| < R\}$. We first explain what the above-said means in more detail.

Recall that a domain in $\mathbb{C}$ is a non-empty connected open subset of $\mathbb{C}$. Two domains Ω and Ω' are said to be *bihlomorphc* to each other (or conformally equivalent) if there exists a biholomorphic (i.e. holomorphic one-one onto) map $f : \Omega \to \Omega'$; $f^{-1} : \Omega' \to \Omega$ is then automatically holomorphic (compare with Algebra where one-one onto homeomorphisms are isomorphisms, and topology, where one-one onto continuous maps need not be homeomorphisms).

Biholomorphic domains have the same "function-theoretic" properties. For example, Liouville's Theorem, which says that for a bounded domain Ω, every holomorphic map $f : \mathbb{C} \to \Omega$ is constant, remains valid if Ω is replaced by a (possibly unbounded) domain Ω' biholomorphic to a bounded domain Ω (if $g : \Omega' \to \Omega$ is biholomorphic, or even just non-constant, $g \circ f$ is constant by Liouville, hence so is f).

There are other examples of the above kind. For instance, it is obvious that the *upper half plane*, $\mathbb{H} := \{z \in \mathbb{C} : Imz > 0\}$ is *homogeneous i.e.* given any two points p and q in $\mathbb{H}$, there exists a biholomorphic map $f : \mathbb{H} \to \mathbb{H}$ such that $f(p) = q$ (if $q = b+ai$, then $f(z) := az+b$ maps i to q). This homogeneity property is therefore true for all domains biholomorphic to $\mathbb{H}$, but is not so obvious to see even for the unit disc D. On the other hand, D admits the rotations $z \to e^{i\alpha}z$, $\alpha \in \mathbb{R}$ as automorphisms, while it is less obvious that $\mathbb{H}$ admits a "circle" of automorphisms.

Thus, it is useful to know that an arbitrary given domain Ω is biholomorphic to a standard domain Ω_0 (like D or $\mathbb{H}$ above) whose function-theoretic properties are easier to study. In these talks, we want to give one set of such standard models for all domains of finite connectivity in $\mathbb{C}$.

Recall that a domain Ω in $\mathbb{C}$ has *connectivity* n ($n \geq 1$ an integer) if $\mathbb{C} \setminus \Omega$ has $n-1$ bounded (hence compact) connected components. For $n = 1, 2$ and 3, one says that the domain is simply-connected, doubly-connected and triply-connected, respectively. A domain of finite connectivity can of course have infinitely many unbounded components, and this sometimes causes technical difficulties. These difficulties can be avoided by regarding domains as lying in the Riemann sphere: a domain is n-connected if and only if its complement in the Riemann sphere has n connected components. However, the proof of the equivalence of the two definitions of n-connectivity needs a little basic Set Topology.

Examples.

(i) $\mathbb{C} \setminus \mathbb{Z}$ has 'infinite connectivity', *i.e.* it is not of finite connectivity.

(ii) D and $\mathbb{H}$ are both simply connected, so is an infinite strip $\{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ or the domain $\mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Re} z \in \mathbb{Z}, Im\, z \geq 0\}$

(iii) $D^* = D \setminus \{0\}$, $A_R := \{1 < |z| < R\}$ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ are doubly connected.

(iv) If Ω is n-connected and p is any point of Ω, then $\Omega \setminus \{p\}$ is a domain of connectivity n+1. Conversely if Ω is n-connected and q is an isolated point of $\mathbb{C} \setminus \Omega$ (so that $n \geq 2$), then $\Omega \cup \{q\}$ is an $(n-1)$-connected domain.

We now state again the main result we wish to prove:

Main Theorem: *Let $\Omega \neq \mathbb{C}$ be an n-connected domain such that $\mathbb{C} \setminus \Omega$ has no isolated points. Then*

(i) *if $n = 1$ then Ω is biholomorphic to the unit disc D.*

(ii) *if $n = 2$ then Ω is biholomorphic to an annulus $A_R := \{z \in \mathbb{C} : 1 < |z| < R\}$ for a unique $R \in (1, \infty]$.*

(iii) *if $n \geq 3$ the Ω is biholomorphic to a* circular-slit annulus. *$A_R \setminus K$, where K is a finite union of closed circular arcs lying on circles with the origin as centre (there may be several slits on the same circle).*

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2. Proof of the Main Theorem. Preliminaries

The case $n = 1$ is the familiar Riemann Mapping Theorem, (RMT), which we shall assume and repeatedly use to bring our n-connected domain ($n \geq 2$) to a preliminary standard form. We shall also need a version of RMT which is valid for domains in the extended compex plane (or the Riemann sphere) $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$:
$(\text{RMT})_\infty$: Let Ω be a domain (*i.e.* non-empty connected open set in $\hat{\mathbb{C}}$) such that $\hat{\mathbb{C}} \setminus \Omega$ is connected and not a singleton. Then Ω is biholomorphic to the unit disc D.

This is equivalent to (RMT) (and a little Set Topology):

$(\text{RMT})_\infty \implies$ (RMT). Suppose $\Omega \subset \mathbb{C}$ is a domain such that each connected component F_α of $\mathbb{C} \setminus \Omega$ is unbounded. Then the closure $\bar{F_\alpha}$ of F_α in ($\hat{\mathbb{C}}$) is precisely $F_\alpha \cup \{\infty\}$, and of course is also connected. Thus $\hat{\mathbb{C}} \setminus \Omega$, which is the union of the $\bar{F_\alpha}$, is also connected, and $(\text{RMT})_\infty$ applies to Ω.

(RMT) $\implies (\text{RMT})_\infty$. Let $\Omega \subset \hat{\mathbb{C}}$ be a domain such that $K := \hat{\mathbb{C}} \setminus \Omega$ is connected and not a singleton. Choose any $p \in K$ consider the map $g(z) = \frac{1}{z-p}$. This is a biholomorphic map of $\hat{\mathbb{C}}$ onto itself (by the definition of holomorphy in $\hat{\mathbb{C}}$), and $g(\Omega) = \hat{\mathbb{C}} \setminus g(K)$ is now a plane domain Ω'; also $\mathbb{C} \setminus \Omega' = g(K) \setminus \{\infty\}$. To apply (RMT) to Ω', we need to know that $g(K) \setminus \{\infty\}$ does not acquire *compact* connected components. This needs the following basic result from Set Topology.

Proposition 1. *Let X be a closed subset of $\mathbb{R}^2$ (or more generally any locally compact Hausdorff space), and C a compact connected component of X. Then C is the intersection of all the sets containing it which are simultaneously open in X* and *compact.*

Corollary 1. *Let K be a compact connected space, and $p \in K$. Then $K \setminus \{p\}$ has no compact connected components.*

Proof. Otherwise, $K \setminus \{p\}$ would also contain a non-empty *compact* open set F by Proposition 1., and this F would be open and closed in K (and $\neq K$).

Remark. There exists a (non-locally compact) subset X of $\mathbb{R}^2$ which is connected , and a point p of X such that $X \setminus \{p\}$ is totally disconnected.

Corollary 2. *Let X be any locally compact space. Then the union of* all *the compact connected components of X is open in X.*

Proof. According to Proposition 1, the union of in question is also the union of all the *compact open* subsets of X.

Proof of Proposition 1 Let $U \supseteq C$ be any open set in X such that $L := \bar{U}$ (the closure of U in X) is compact. Then C is of course a connected component of the smaller space L, and we see that it is enough to prove the proposition for the compact space L (subsets of U open in L are also open in X).

Thus consider $C' = \cap\{F \subset L : F \text{ open in } L \text{ and compact}\}$. Our claim is that $C' = C$. Otherwise $C \subsetneqq C'$ is a connected component of C', hence C' is not connected. Hence we can write $C' = A \cup B$, where A and B are non-empty, disjoint and compact. Hence there exist disjoint open sets V, W in L such that $A \subset V$, $B \subset W$. Let $C \subset A$. By the definition of C' and the usual compactness argument, there exists $F \subset V \cup W$ such that F is compact, open and contains C, but does not contain C'. This contradicts the definition of C'

After this long digression into Set Topolgy, we return to the proof of the Main Theorem.

Proof of the Main Theorem: Method of Proof

We follow Ahlfors [1] fairly closely in the construction of the mapping function. The idea is that, if ϕ maps our multiply connected domain biholomorphically onto a circular slit annulus, then $\log|\phi|$ is a harmonic function in the domain with constant boundary values, at least if we assume that the domain has 'good' boundary. Conversely, we may hope that a harmonic function in the domain with suitable constant boundary will the logarithm of the absolute value of a holomorphic function ϕ which will be the candidate for the mapping function. The verification that the function ϕ which we construct does map our domain biholomorphically onto a slit annulus is surprisingly hard, and this part of our proof differs completely from (and is, we hope, 'cleaner' than) that in [1].

To motivate the construction of the mapping function, we first consider the doubly connected case. Thus let Ω be a domain such that $\mathbb{C} \setminus \Omega$ has a single compact connected component K, which is not a singleton. Then we claim that $\Omega' := \Omega \cup K$ is a simply-connected domain. Indeed, $\mathbb{C} \setminus \Omega'$, being the union of all the unbounded components of $\mathbb{C} \setminus \Omega$ is a closed set by Corollary 2 of Proposition 1. Hence Ω' is an open set. It is connected since the boundary of K is contained in $\bar{\Omega}$, and its complement has no compact components by construction. This proves our claim.

Thus by (RMT) there exists a biholomorphic map $f : \Omega'' \to D$, under which Ω is mapped onto $D \setminus f(K) =: \Omega_1$.

We now consider $\Omega'' := \hat{\mathbb{C}} \setminus f(K)$. This is a domain in $\hat{\mathbb{C}}$, whose complement $f(K)$ in $\hat{\mathbb{C}}$ is connected and not a singleton. Hence $(\text{RMT})_\infty$ applies to Ω'' and we have a biholomorphic map $g : \Omega'' \to D$ under which our original domain becomes $\Omega_2 = g(\Omega_1) = g(f(\Omega)) = D \setminus g(\hat{\mathbb{C}} \setminus D$. Observe that $\hat{\mathbb{C}} \setminus D$ is "biholomorphic" to $\{|z| \leq 1\}$ via the map $z \to 1/z$. Thus we may assume that $\Omega = D \setminus L$ where L is a compact connected subset of D biholomorphic to the closed unit disc.

We shall now *assume* that there exists a biholomorphic map $\phi : \Omega \to A_R$ and try to pin down R and ϕ. First since ϕ is in particular a homeomorphism, it is clear that $|\phi(z)|$ tends to 1 or R as $z \in \Omega$ tends to $\partial\Omega$ (*i.e.* for every sequence (z_n) converging to a point $\zeta \in \partial\Omega$ every convergent subsequence $\{|\phi(z_n)|\}$ has either 1 or R as limit).In fact, because the boundary of Ω is so 'good', elementary Set Topology will imply that $|\phi|$ extends as a continuous function to $\bar{\Omega}$; we shall not prove this since we are only motivating the proof.

Thus $h = \log|\phi|$ is continuous function on $\bar{\Omega}$, harmonic in Ω and with constant boundary values 0 and $\log R$ on the two connected components, say, C_0 and C_1, of $\partial\Omega$ (= boundary of Ω). By using the automorphism $z \mapsto R/z$ of A_R if necessary, we may assume that $h = 0$ on $C_0 := \{|z| = 1\}$ and $log R_1$ on $C_1 := \partial L$. By the Schwarz Reflection Principle for harmonic functions as proved in Ahlfors ([1], p. 170–171), h extends harmonically to a neighbourhood of $\bar{\Omega}$, hence so does ϕ. It is in order to secure this regularity of the mapping function 'upto the boundary' that we replaced our original domain by one with 'analytic' boundary

Finally, to pin down R_1 (hence h), hence ϕ upto a constant factor of modulus one, we observe that, locally in Ω, there is a holomorphic function $\log\phi$ with real part $= h$, say $\log\phi = h + i * h$ ($*h$ is unique upto an *additive constant*). Then

$$\frac{\phi'}{\phi} = \frac{\partial h}{\partial x} + i\frac{\partial}{\partial x}(*h)$$

$$= \frac{\partial h}{\partial x} - i\frac{\partial h}{\partial y} \qquad \text{Cauchy – Riemann for } \log\phi$$

If we regard C_0 and C_1 as paths oriented in the usual way with respect to Ω, and assume hopefully that ϕ maps C_0 bijectively onto the circle $C_0' := \{|w| = 1\}$, then we get

$$-1 = \frac{1}{2\pi i}\int_{|w|=1}\frac{dw}{w} = \frac{1}{2\pi i}\int_{C_0}\frac{\phi'}{\phi}dz$$

$$= \frac{1}{2\pi i}\int_{C_0}\left(\frac{\partial h}{\partial x} - i\frac{\partial h}{\partial y}\right)(dx + i dy)$$

$$= \frac{1}{2\pi i}\int_{C_0}\left(\frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy\right)$$

$$+ \frac{1}{2\pi}\int_{C_0}\left(-\frac{\partial h}{\partial y}dx + \frac{\partial h}{\partial x}dy\right)$$

$$= 0 + \frac{1}{2\pi}\int_{C_0}\left(\frac{\partial * h}{\partial x}dx + \frac{\partial * h}{\partial y}dy\right)$$

Note that though $*h$ is only locally defined, $\frac{\partial *h}{\partial x} = -\frac{\partial h}{\partial y}$ and $\frac{\partial *h}{\partial y} = \frac{\partial h}{\partial x}$ are well defined in a neighbourhood of $\bar{\Omega}$. We denote $\frac{\partial *h}{\partial x}dx + \frac{\partial *h}{\partial y}dy = -\frac{\partial h}{\partial y}dx + \frac{\partial h}{\partial x}dy$ by $*dh$. And our conclusion is that the boundary value $b_1 := \log R_1$ for h on C_1 should be so chosen that

$$\int_{C_1} *dh = -\int_{C_0} 2dh = 2\pi$$

This does pin down h, since we have already assumed that $h = 0$ on C_0; thus h is the unique continuous function on $\bar{\Omega}$ which is $\equiv 0$ on C_0 and $\equiv b_1$ on $\mathbb{C}_1$ and harmonic on Ω, where b_1 is the unique choice that makes $\int_{C_1} *dh = 2\pi$.

Proof of the Main Theorem

We shall now carry out the proof of the Main Theorem by the method suggested by the preceeding considerations. Thus let Ω be an $(n+1)$-connected domain, and $K_1, \dots, K_n$ the compact connected components of $\mathbb{C} \setminus \Omega$.

Step 1. Ω is biholomorphic to a domain of the form $D \setminus (L_1 \cup \dots \cup L_n)$, where the L_i are disjoint (compact connected) subsets which are biholomorphic to the closed disc $\{|z| \le 1\}$.

The proof is by induction on n - note that we have already proved the assertion for $n = 1$. Observe now that $\Omega \cup K_n$ is an n-connected domain (we are assuming $n \ge 2$), hence by induction biholomorphic to a domain of the form $D \setminus \{L_1 \cup \dots \cup L_{n-1}\}$, where the L_i are disjoint compact sets of the desired kind, so Ω has the form $D \setminus \{L_1 \cup \dots \cup L_{n-1} \cup K\}$ where $K \subset D$ is compact connected and disjoint from the L_i. We now apply $(\text{RMT})_\infty$ to the domain $\hat{\mathbb{C}} \setminus K$; then, under the biholomorphic map $g : \hat{\mathbb{C}} \setminus K \to D$, Ω takes the form $D \setminus \{g(L_1) \cup \dots g(L_{n-1}) \cup g(\hat{\mathbb{C}} \setminus D)\}$, which is of the desired form. Thus Step 1 is complete.

Step 2. We assume that the $(n+1)$-connected domain Ω ($n \ge 1$) is of the form $D \setminus \{L_1 \cup \dots \cup L_n\}$ with the L_i as in Step 1. We denote the unit circle by C_0, and the boundary of L_i (which is a biholomorphic image 0f $|z| = 1$), by C_i. We also regard the C_i as closed analytic Jordan curves, oriented in the usual manner with respect to Ω, so that the full oriented boundary of Ω is $C_0 + \dots + C_n$. We set $b_0 = 0$ for uniformity and, for any $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{R}^n$, construct the unique continuous function $h_{\mathbf{b}}$ on $\bar{\Omega}$, harmonic in Ω, and $\equiv b_i$ on C_i, $0 \le i \le n$. The existence and uniqueness of $h_{\mathbf{b}}$ is proved in Ahlfors ([1], pp. 244–246). As noted before, since the boundary $\partial\Omega$ of Ω is analytic, and the boundary values are locally constant on $\partial\Omega$, $h_{\mathbf{b}}$ extends harmonically to a neighbourhood of Ω.

We claim that the linear map $\mathbb{R}^n \to \mathbb{R}^n$ defined by

$$(0, b_1, \dots, b_n) \longrightarrow \left(\int_{C_1} d * h_{\mathbf{b}}, \dots, \int_{C_n} d * h_{\mathbf{b}}\right)$$

is injective (and hence also surjective). Indeed, if $\mathbf{b}$ is in the kernel of this map, the integral of $*dh_{\mathbf{b}}$ along *any* closed curve in Ω vanishes, since $C_1, \dots, C_n$ are a *basis for cycles* in Ω, *i.e.* any cycle (*i.e.* integral combination of closed curves in Ω) is "homologous" to an integral linear combination of the C_i (Ahlfors [1], p. 146). Thus there would be a harmonic function $*h$ on the whole of Ω such that $\frac{\partial *h}{\partial x} = -\frac{\partial h}{\partial y}$, $\frac{\partial *h}{\partial y} = \frac{\partial h}{\partial x}$. Obviously, $*h$ will also be harmonic in a neighbourhood of $\bar{\Omega}$, hence $f = h + i * h$ would be a holomorphic function in a neighbourhood of Ω. If $\mathbf{b} \neq 0$, f will be non-constant, so $f(\Omega)$ will be a *bounded* domain whose boundary is contained in the union of the lines $\Re w = b_i$, $0 \le i \le n$. Obviously, no such bounded domain exists.

Thus our linear map is indeed injective, and hence surjective. Hence there exists a unique $\mathbf{b}$ such that

$$\int_{C_1} *dh_{\mathbf{b}} = 2\pi, \qquad \int_{C_i} *dh_{\mathbf{b}} = 0 \quad \text{for} \quad i > 1$$

We denote this $h_{\mathbf{b}}$ simply by h from now on. This h is the harmonic function which we expect to prove is $\log|\phi|$, where ϕ is the mapping function we are looking for.

Step 3. Consider the function $g =: \frac{\partial h}{\partial x} - i\frac{\partial h}{\partial y}$. This is *holomorphic* in a neighbourhood of Ω, since h is harmonic (Cauchy–Riemann equations!). An easy calculation that we have already seen shows that $g(z)\,dz = dh + i * dh$. Hence, by our construction of h, the integral of $g(z)$ along C_i, and hence along any closed curve in Ω, is an integral multiple of $2\pi i$. Hence

$$\phi(z) := \exp\left(\int_{z_0}^{z} g(w)dw - h(z_0)\right)$$

is independent of the path from z_0 to z and defines a holomorphic function ϕ on Ω. Clearly, ϕ extends holomorphically to a neighbourhood of $\bar{\Omega}$, since h does so harmonically. It is clear that $|\phi| = e^h$. Thus $|\phi|$ has the constant values $R_i := e^{b_i}$ on the C_i. We expect to prove that $R_1 > R_0 = 1$ and that ϕ maps Ω biholomorphicaly on to A_{R_1} with slits on the circles $|w| = R_i$, $i \geq 2$.

What we can say immediately of $\phi(\Omega)$ is that it is a bounded domain in $\mathbb{C}^*$ whose boundary is contained in the union of the circles $\{|w| = R_i\}$ $0 \leq i \leq n$ (by the Open Mapping Theorem). We can also say that ϕ maps C_0 *onto* $C_0' := \{|w| = R_1\}$ and C_1 onto $C_1' := \{|w| = R_1\}$, since

$$\frac{\phi'(z)}{\phi(z)}dz = dh + i * dh$$

so that the winding numbers of the curves $\phi(C_0)$ and $\phi(C_1)$ around the origin are non-zero (in fact ± 1).

Assuming for the moment that $R_1 > 1$, we can also say that $\phi(\Omega)$ contains all the circles $\{|w| = r\}$, $r \in (1, R_1)$ except possibly when r is one of the R_i, $i \geq 2$. This is because $|\phi|$ takes all values between 1 and R_1 in Ω, so $\phi(\Omega)$ meets each of these circles, and the circles not contained in Ω must have a boundary point of Ω. But this happens at the most for the circles $|w| = R_i$.

Thus Area $\phi(\Omega) \geq \pi(R_1^2 - 1)$.

Step 4. We have an obvious upper bound for Area$\phi(\Omega)$, namely

$$\text{Area}\,\phi(\Omega) \leq \int_{\Omega} |\phi(z)|^2 dx\,dy$$

This is a lemma from Real Analysis. Assuming this for the moment, we see that equality holds above only if ϕ is one-one in Ω; in fact if equality holds, we cannot have $z_1 \in \Omega$, $z_2 \in \bar{\Omega}$, $z_1 \neq z_2$, such that $\phi(z_1) = \phi(z_2)$. Otherwise, we could remove a (small) closed disc from Ω to get a new domain Ω' which ϕ still maps onto $\phi(\Omega)$, violating the inequality for Ω'. Observe however that equality does not imply that ϕ is one-one on the closure of Ω

So we compute $\int_\Omega |\phi'(z)|^2\,dx\,dy$. We do this by applying Green's Theorem (= Stokes Theorem in $\mathbb{R}^2$) to the form

$$\bar{\phi}(z)\phi'(z)\;dz = \bar{\phi}(z)\phi'(z)\;dx + i\bar{\phi}(z)\phi'(z)\;dy$$

We get

$$\begin{aligned}
&\int_{\partial\Omega} \bar{\phi}(z)\phi'(z)\;dz \\
&\quad = \int_{\Omega} \left\{ \frac{\partial}{\partial x}(i\bar{\phi}(z)\phi'(z)) - \frac{\partial}{\partial y}(\bar{\phi}(z)\phi'(z)) \right\} dxdy \\
&\quad = 2i \int_{\Omega} \frac{1}{2} \left\{ \frac{\partial}{\partial x}(\bar{\phi}(z)\phi'(z) + i\frac{\partial}{\partial y}(\bar{\phi}(z)\phi'(z)) \right\} dxdy
\end{aligned}$$

Now, the operator $\frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$ is the familiar Cauchy-Riemann operator $\frac{\partial}{\partial \bar{z}}$: it vanishes on holomorphic functions, and $\frac{\partial \bar{f}}{\partial z} = \bar{f'}(z)$ if f is holomorphic. Thus we get:

$$\int_{\Omega} |\phi'(z|^2\;dxdy = -\frac{i}{2}\int_{\partial\Omega} \bar{\phi}(z)\phi'(z)\;dz$$

In our case, $\phi' = \phi(\frac{\partial h}{\partial x} - i\frac{\partial h}{\partial y})$, so the integrand of the right side simplifies to $|\phi|^2(dh + i * dh)$, as we have seen before. Hence the right side becomes $\frac{1}{2}\sum R_i^2 \int_{C_i} *dh = \pi(R_1^2 - 1)$

In particular, since the left side is certainly positive, we get $R_1 > 1$, as we had assumed.

Thus we have equality in the Area Inequality, showing that ϕ is one-one in Ω, and that $\phi(\Omega)$ is contained in A_{R_1}. And $A_{R_1} \setminus \Omega$ is a disjoint union of finitely many arcs contained in the union of the circles $\{|w| = R_i\}$; these arcs are at the most $n-1$ in number, since their union is $\cup_{i\geq 2}\phi(C_i)$.

We claim that $A_{R_i} \setminus \Omega$ is in fact the union of $n-1$ disjoint arcs, and no fewer.

Thus, suppose $A \setminus \phi(\Omega)$ has m connected components $E_1, \ldots, E_m$ $(m \leq n-1)$. We choose disjoint open sets $V_i \supset C_i$ in $\mathbb{C}$, and consider $\Omega \setminus \cup V_i = \bar{\Omega} \setminus \cup V_i = K$. This is a compact subset of Ω, hence $\phi(K)$ is a compact subset of $\phi(\Omega)$. Since $\phi(\Omega)$ is a circular-slit domain, it is clear that there are disjoint open sets $W_j \supset E_j$, and $U \supset C'_i$ $i = 1, 2$, disjoint from $\phi(K)$ such that the $U_i \cap \phi\Omega$ and $W_j \cap \phi(\Omega)$ are connected. Then ϕ^{-1} maps each $U_i \cap \phi(\Omega)$ or $W_j \cap \phi(\Omega)$ into a single V_l , hence surjectivity of ϕ^{-1} would be violated if m were less than $n - 1$. Thus the E_j are indeed the $\phi(C_j)$, $j \geq 2$.

Remark. The above argument only used the fact that $\phi^{-1} =:$ ψ, say, was a continuous map of $\phi(\Omega)$ onto Ω such that $\psi^{-1}(K)$ $(= \phi(K)$ in our case) was compact for each compact $K \subset \Omega$, *i.e.* that $\psi : \phi(\Omega) \to \Omega$ was a *proper onto map*. Thus, what we have seen (with a slight change of notation) is that if Ω, Ω' are domains of finite connectivity, and $f : \Omega \to \Omega'$ is a continuous *proper* map of Ω onto Ω', then connectivity $(\Omega') \leq$ connectivity (Ω). Our proof above used the fact that Ω has "good" boundary.

It remains to prove the area estimate that we have used. Clearly it is sufficient to prove

Proposition. Let Ω be an open set in $\mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}^n$ a C^1 map with Jacobian determinant $J > 0$ everywhere in Ω. Then

$$\text{vol } f(\Omega) \leq \int_\Omega J\phi \, dx_1 \cdots dx_n$$

Remark. By the Inverse Function Theorem, f is a local C^1 diffeomorphism. Hence $f(\Omega)$ is an open subset of $\mathbb{R}^n$, and vol $f(\Omega)$ makes sense.

Proof. Let $K \subset f(\Omega)$ be any compact set. Since f is in particular an open map, there exists a compact subset L of Ω such that $f(L) \supset K$. Enlarging L if necessary, we may assume that L is a finite union of cubes C_i in $\mathbb{R}^n$ whose interiors are disjoint. Since L is covered by open sets in Ω restricted to which f is a diffeomorphism, we may by subdividing the C_i into smaller cubes if necessary assume that each C_i is contained in one such open set. Then vol $(f(C_i)) = \int_{C_i} J dx_1 \cdots dx_n$ by the change of variable formula for multiple integrals (Rudin [5], pp. 153–154). Now $K \subset f(L) = \cup f(C_i)$ and hence

$$\text{vol } (K) \leq \sum \text{vol } f(C_i) = \sum_{C_i} \int_{C_i} J \, dx_1 \cdots dx_n$$

$$= \int_L J \, dx_1 \cdots dx_n$$

(since the C_i are almost disjoint)

$$\leq \int_\Omega J \, dx_1 \cdots x_n$$

Since this formula holds for every compact $K \subset f(\Omega)$, we are done.

Thus our Main Theorem is proved. It is easy to take care of isolated points in the complement of the domain Ω. Indeed, if Ω is n-connected and $\mathbb{C} \setminus \Omega$ has k $(\leq n)$ isolated points $p_1, \ldots, p_k$, then $\Omega \cup \{p_1, \ldots, p_k\}$ is an $(n - k)$-connected domain. Thus Ω becomes biholomorhic to a standard $(n-k)$-connected domain as in the Main Theorem, with k punctures. For example a triply-connected domain with one isolated point in its complement would be biholomorphic to an annulus A_R with one puncture. The puncture could be brought to the positive real axis by a rotation and then into the interval $(1, \sqrt{R}]$ by using the map $z \to R/z$ if necessary; it is then uniquely determined by the biholomorphic class of Ω Thus the set of triply-connected domains with one isolated point on the boundary is parametrised in a one-one manner by the set $\{(x, y) \in \mathbb{R}^2 : x > 1, \ 1 < y \leq \sqrt{x}\}$!.

Applications of the Main Theorem

1. **Theorem.** *Two domains of finite connectivity in $\mathbb{C}$ are homeomorphic iff they have the same connectivity.*

   In view of the Main Theorem this needs to be checked only for standard models, and the proof is a matter of elementary Geometric Topology in $\mathbb{R}^2$. We leave the proof to the reader.
2. **Theorem.** *If $n \geq 3$ the group* Aut (Ω) *(of biholomorphic maps $\phi : \Omega \to \Omega$) of a domain of connectivity n is finite*.

   Again, this needs to be proved only for the standard domains, see Hemasundar [2] for the details.
3. From the main theorem, we can see that there is a *connected* set in some Euclidean space which parametrises all domains of given finite connectivity, (perhaps with repetitions). This permits one to prove some property for *all* domains of finite connectivity by showing that the set of points in the parameter space whose corresponding domains have the property in question is non empty, open and closed. In fact this method has been used to prove many important results.

Note that a circular-slit annulus is determined by its outer radius, and the locations and lengths of the slits. Assuming that no slit is a singleton, we see that this involves $3(n-2)+1$ real parameters if the connectivity is $n \geq 2$. However, rotations do not change the biholomorphic class of the slit annulus, though they change the domain when $n \geq 3$. Thus the number of 'free' parameters is $3(n-2)$ if $n \geq 3$. These are not mutually non-biholomprphic, but each biholomorphic class occurs (upto rotations) only a finite (in fact bounded) number of times in the list. Thus it may safely be said that the dimension of the space of biholomorphic classes of n-connected plane domains without punctures is $3(n-2)$ for $n \geq 3$.

For another confirmation of this dimension count, see Kulkarni [3] and Nag [4].

Planar Riemann Surfaces

We now make some comments of a less elementary nature for readers who are familiar with the language of Riemann surfaces. A natural question to ask about Riemann surfaces is: when is a given (abstract) Riemann surface X (biholomorphic to) a domain in the Riemann sphere? For example, suppose Ω is a domain (i.e. connected open set) in X with compact closure, whose boundary consists of finitely many disjoint analytic Jordan curves. When can we say that Ω is biholomorphic to a circular-slit annulus?

An examination of our proof of the Main Theorem above shows that the only way we have used the fact that the domain in question is a domain in the complex plane (once its boundary has been made analytic) is to assert that the boundary curves of the domain generate its homology and are subject only to the one obvious relation that their sum is homologous to zero. Indeed, Perron's method of solving the Dirichlet Problem (construction of a harmonic function with given boundary values) is tailor-made for generalisation to Riemann surfaces, and everything else we have said is valid verbatim on Riemann surfaces.

Following Koebe (who made fundamental contributions to mapping theorems for plane domains and Riemann surfaces during the period 1880–1920), we call a Riemann surface *planar* (from the German 'schlichtartig') if it satisfies a certain purely topological condition which must be satisfied for a Riemann surface X to be *homeomorphic* to an open subset of the Riemann sphere. One way of stating this condition is that, for every Jordan curve (i.e. homeomorphic image of S^1) γ in X, the set $X \setminus \gamma$ should be disconnected. A more modern equivalent condition is that, for every closed 1-form ω *with compact support* on X, the integral $\int_\gamma \omega$ should vanish for every closed curve γ on X. It is obvious that every open subset of a planar Riemann surface is planar. It is not very hard to show that, for the domains Ω considered above (with compact closure and good boundary), the condition that the boundary curves generate the homology is equivalent to the planarity condition.

On the other hand, using for example Perron's Theorem on the solvability of the Dirichlet Problem and Sard's Theorem, one can show that every non-compact Riemann surface X is the increasing union of a sequence of domains Ω_n which have compact closure in X and whose boundaries are finite disjoint unions of *analytic* Jordan curves. Thus, if X is planar, then, by our earlier considerations, each of these domains is biholomorphic to a domain in $\mathbb{C}$. Another elegant theorem of Koebe (assering the normality of the family of normalised one-one holomorphic functions on a domain in the plane) then implies that X itself is biholomorphic to a domain in $\mathbb{C}$. The case of a *compact* planar Riemann surface X can be reduced to the non-compact case by considering the complement of a point in X; it then turns out easily that X is the Riemann sphere.

In conclusion, the mapping theorem we have proved for plane domains generalises to the case of planar domains, and then plays a crucial role in the proof of the so-called Generalised Uniformisation Theorem of Koebe:

A Riemann surface is biholomorphic to a domain in the Riemann sphere if and only if it satisfies the planarity condition.

For this theorem, see also Simha [6].

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An Introduction to Complex Dynamics

Anand P. Singh
Department of Mathematics, University of Jammu
Jammu 180 006, India

1. Introduction

Complex dynamics discusses the behaviour of analytic functions under iteration. The field originated with the Newton-Raphson iteration method of approximating roots. The foundation of the modern theory of complex dynamics was laid by Fatou and Julia in their long memories [15], [16], [19], and using Montel's theory of normal families which had just emerged. The theory of complex dynamics met with a set back due to the premature death of Fatou, and also due to the fact that there were not sufficient tools to solve some of the major problems that the earlier mathematicians encountered with.

The introduction of quasiconformal mappings, topological concepts, hyperbolic geometry, ergodic theory, computer graphics, the number system, local and global geometry, Klenian groups, polynomial automorphisms of several variables, such as the complex Henon maps, growth and factorization of entire and meromorphic functions, have now become tools in the study and in trying to solve some of the central and related conjectures of complex dynamical system such as the density of hyperbolicity conjecture, MLC conjecture, Robust conjecture, the no-invariant line fields conjecture [21] and problems relating to dynamics of entire and meromorphic functions. In this notes we plan to give an introduction to the subject concentrating on rational maps. Towards the end we shall mention about the dynamics of entire and meromorphic functions. Some of the books and survey articles connected with the subject can be found in [4, 9, 10, 14, 18, 22, 23, 29].

2. Extended Complex Plane

Let $\mathbb{C}$ be the complex plane and $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let S be the unit sphere. Let $\pi : \hat{\mathbb{C}} \to S$ be the stereographic projection defined by $\pi(z) = Z$, (the point of intersection of the sphere with the line joining $(0, 0, 1)$ to z), for every $z \in \mathbb{C}$ and $\pi(\infty) = (0, 0, 1)$. Clearly π is a bijection map.

Define σ on $\hat{\mathbb{C}}$ by $\sigma(z, w) = |\pi(z) - \pi(w)|$, then clearly

$$\sigma(z, w) = \frac{2|z - w|}{(1 + |z|^2)^{\frac{1}{2}}(1 + |w|^2)^{\frac{1}{2}}}$$

for $z, w \in \mathbb{C}$ and

$$\sigma(z, \infty) = \lim_{w\to\infty} \sigma(z, w) = \frac{2}{(1 + |z|^2)^{\frac{1}{2}}}.$$

Then σ is a metric called *chordal metric* on $\hat{\mathbb{C}}$.

We can also define another metric σ_0 called the *spherical metric* as follows:

Definition 2.1. *The spherical distance $\sigma_0(z, w)$ is the Euclidian length of the shortest path of S between $\pi(z)$ and $\pi(w)$, where π is the projection mapping defined above.*

Note. If the chord joining $\pi(z)$ to $\pi(w)$ subtends an angle θ at the origin, then $\sigma_0(z, w) = \theta$, and further σ and σ_0 satisfy the following inequality:

$$\frac{2}{\pi}\sigma_0(z, w) \le \sigma(z, w) \le \sigma_0(z, w).$$

And so we may use σ and σ_0 interchangeably (changing some constants, if necessary).

Definition 2.2. *A rational map is a function of the form $R(z) = \frac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are polynomials. We assume $P(z)$ and $Q(z)$ are relatively prime and we define* $\deg(R) = \max\{\deg P, \deg Q\}$.

A function f is said to be defined in a neighbourhood of ∞ if it is defined on some set $\{|z| > r\} \cup \{\infty\}$. Further f is said to be holomorphic at ∞ if the map $z \to f(\frac{1}{z})$ is holomorphic near 0.

Theorem 2.3. *If R is a rational map of positive degree d, then for every $w \in \hat{\mathbb{C}}$, $R(z) = w$ has precisely d solutions. (In this case we say that R is a d-fold map of $\hat{\mathbb{C}}$ onto itself).*

Definition 2.4. *Let (X, σ) be a metric space and let $f : X \to X$ be a mapping. Then f is said to satisfy Lipschitz condition on X if there exists a constant M such that*

$$\sigma(f(x), f(y)) \leq M\sigma(x, y)$$

for every $x, y \in X$.

Theorem 2.5. *A rational function R satisfies Lipschitz condition. Further if $R(z) = \frac{az+b}{cz+d}$ where $ad - bc = 1$, then R satisfies the Lipschitz condition*

$$\sigma_0(R(z), R(w)) \leq (|a|^2 + |b|^2 + |c|^2 + |d|^2)\sigma_0(z, w).$$

Definition 2.6. *A Möbius map is a rational map of the form $R(z) = \frac{az+b}{cz+d}$, $ad - bc \neq 0$, where as usual, by convention, we take $R(\infty) = \frac{a}{c}$, and $R(\frac{-d}{c}) = \infty$, if $c \neq 0$, and $R(\infty) = \infty$ if $c = 0$.*

Conjugacy plays an important role in Complex dynamics.

Definition 2.7. *Two rational maps R and S are said to be conjugate if and only if there exists a Möbius map g such that $S = gRg^{-1}$.*

Conjugacy is an equivalence relation and the equivalence classes of rational maps are called *conjugacy classes* of rational maps.

Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be rational map. Throughout the notes unless otherwise stated, we define $R^0(z) = z$, $R^n(z) = R(R^{n-1}(z))$, $n = 1, 2, \ldots$.

Note. If R and S are conjugate then $\deg(R) = \deg(S)$. Further if $S = gRg^{-1}$ where g is Möbius, then $S^n = gR^ng^{-1}$. Thus we can transfer the problem concerning the iteration of R to the problem concerning the iteration of S.

Example 2.8. Let $P(z) = Az^2 + Bz + C$, be any quadratic polynomial. Define

$$Q(z) = z^2 + \left(AC + \frac{2B - B^2}{4}\right)$$

and

$$g(z) = Az + \frac{B}{2}.$$

Then clearly

$$(g^{-1}Q^ng)(z) = P^n(z) \text{ for all } n = 1, 2, \ldots .$$

Thus in order to study the dynamics of $P(z)$ it is sufficient to study the dynamics of $Q(z)$, (i.e. a quadratic polynomial of the form $z^2 + c$, for some constant c).

3. Two Examples

Example 3.1. Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the rational map defined by $R(z) = z^2$. Then clearly $R^n(z) \to \infty$ as $n \to \infty$ on $| z |> 1$ and $R^n(z) \to 0$ as $n \to \infty$ on $| z |< 1$. Let us now consider the behaviour of $R^n(z)$ as $n \to \infty$ on $| z |= 1$.

Clearly $R^n(e^{i\theta}) = e^{i2^n\theta}$ and so if z is of the form $e^{i\pi\frac{r}{2^m-1}}$ for some positive integers r and m then $R^m(z) = 1$ and $R^{m+p}(z) = 1$ for all positive integers p. Thus any number on $| z |= 1$ of the form $e^{i\pi\frac{r}{2^m-1}}$ will converge to 1 and after m iteration will remain there, after that. A number on $| z |= 1$ not of the above form will not converge and will always be on the unit circle. Also the numbers of the form $e^{i\pi\frac{r}{2^m-1}}$ are dense on the unit circle. Thus the complex plane can be divided into two types of regions:

(i) Fatou set: where the limiting behaviour is smooth viz. $(| z |< 1) \cup (| z |> 1)$

(ii) Julia set: where the limiting behaviour is chaotic viz. $| z |= 1$.

The Julia set $| z |= 1$ for $R(z) = z^2$ is probably the simplest structure. If the function is slightly changed, the structure of the Julia set can be quite complicated. We also note that the points where the iteration R^n converges are not arbitrary in the sense all the three points viz., $0, 1, \infty$ are "fixed points" of R.

Definition 3.2. *Let R be a rational map. A point z_0 will be called a fixed point of R if $R(z_0) = z_0$.*

Definition 3.3. *Let z_0 be a fixed point of R. Then z_0 will be called attracting fixed point, repelling fixed point, or indifferent fixed point according as $|R'(z_0)| < 1$, $|R'(z_0)| > 1$ or $|R'(z_0)| = 1$ respectively.*

Some Questions:

(i) In the above example $R(z) = z^2$, the points $0, \infty$ are attracting fixed point of R and both lie in "Fatou set". On the other hand the fixed point 1 is repelling fixed point, which lies in "Julia set". One may question whether this is always true. In fact this is true which we shall show subsequently. Looking at the Julia set ($|z| = 1$) we can frame several questions. For instance we can frame the following:

(ii) We observe that the Julia set ($|z| = 1$) is bounded. Is this always true?

(iii) The Julia set ($|z| = 1$) has no "thickness". Is this always true?
(iv) The Julia set ($|z| = 1$) has infinite number of elements. Is this always true?
(v) The Julia set ($|z| = 1$) has no isolated points. Is this always true?
(vi) The Julia set ($|z| = 1$) is connected. Is this always true?

Thus we observe that several questions can be framed looking at this simple diagram of Julia on set. We shall try to give answers to these questions and others, and also on Fatou sets and its components.

Example 3.4. In this example we shall look at the behaviour of Möbius map under iteration. Now a Möbius map can have at most two fixed points in $\hat{\mathbb{C}}$. If the Möbius Map $R(z) = \frac{az+b}{cz+d}$, $ad - bc \neq 0$, has a unique fixed point ξ, finite or ∞ then $R^n(z) \to \xi$ as $n \to \infty$.

If R has two fixed points, then either $R^n(z)$ converge to one of the fixed points of R or they move cyclically through a finite set of points, or they form a dense subset of some circle. Thus we see that the iteration of a Möbius map behave in a simple way and hence throughout our discussion we shall always assume that the rational map under consideration is of degree *greater* than one.

4. Valency and Critical Points

Let f be analytic at z_0 in $\mathbb{C}$. Then writing Taylor series in a neighbourhood of z_0 we get

$$f(z) = a_0 + a_k(z - z_0)^k + a_{(k+1)}(z - z_0)^{k+1} + \cdots$$

where k is the first natural number such that $a_k \neq 0$. Thus $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{(z-z_0)^k} = a_k \neq 0$. We denote this k by $v_f(z_0)$ and call it the *valency* of f at z_0. More precisely, we have the following definition.

Definition 4.1. *Let f be analytic at z_0. If $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{(z-z_0)^k}$ exists, is finite and nonzero, then k is called valency of f at z_0.*

Exercise 4.2. Let $f(z) = z^2 - 4z + 8$. Calculate $v_f(2)$, $v_f(4)$, $v_f(\infty)$.

Exercise 4.3. Show that the valency function satisfies the chain rule

$$v_{f(g)}(z_0) = v_f(g(z_0))v_g(z_0).$$

Exercise 4.4. Show that $v_f(z_0) = 1$ if and only if f is one to one in some neighbourhood of z_0.

Exercise 4.5. Let R be a rational map of degree d. For any $w \in \hat{\mathbb{C}}$ show that $\sum_{z\in R^{-1}\{w\}} v_R(z) = d$.

Exercise 4.6. A rational map of degree d ($d > 1$) has precisely $d + 1$ fixed points.

Exercise 4.7. If R and S are rational maps and $S = g^{-1}Rg$ where g is a Möbius map, then show that $\deg(S) = \deg(R)$ and $\sum(v_S(z) - 1) = \sum(v_R(z) - 1)$.

Definition 4.8. *A point z_0 is called a critical point of a rational map R if R fails to be one to one in every neighborhood of z_0. The value $R(z_0)$ is called critical value of R.*

Note. If z_0 is a critical point of R, then $v_R(z_0) > 1$. Thus if $R'(z_0) = 0$ then z_0 is a critical point of R.

Let R be of degree d. If w is not a critical value, then $R^{-1}(\{w\})$ consists of precisely d distinct points, say $z_1, z_2, \ldots, z_d$, and there exist neighbourhood $N_1, \ldots, N_d$ of $z_1, \ldots, z_d$ and a neighbourhood N of w such that R acts as a bijection map from N_j onto N. Let R_j be the restriction map $R_{/N_j}$. Thus R_j has a restriction map $R_j^{-1} : N \to N_j$. We call these R_j^{-1} as *branches* of R^{-1} at w.

The following theorem known as *Riemann Hurwitz relation*, relates the valency of rational maps with the degree of the rational map.

Theorem 4.10. *For any nonconstant rational map R*

$$\sum(v_R(z) - 1) = 2\deg(R) - 2.$$

Corollary 4.11. *A rational map of degree d has at most $2d-2$ critical points in $\hat{\mathbb{C}}$. A polynomial of degree d has at most $d - 1$ critical points in $\mathbb{C}$.*

5. Fatou and Julia Sets

Definition 5.1. *A family $\mathcal{F}$ of maps from a metric space (X, d) into the metric space (X_1, d_1) is equicontinuous at $x_0 \in X$ if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$d(x, x_0) < \delta \text{ implies } d_1(f(x), f(x_0)) < \epsilon$$

for all $x \in X$ and for all $f \in \mathcal{F}$. Further $\mathcal{F}$ is equicontinuous on $A \subset X$ if $\mathcal{F}$ is equicontinuous at each point of A.

Clearly if $\mathcal{F}$ is equicontinuous on each subset D_α of X, then $\mathcal{F}$ is also equicontinuous on $\cup D_\alpha$. This immediately leads to the following consequences:

(i) Let $\mathcal{F} = \{f : (X, d) \to (X_1, d_1)\}$ be a family of maps. Then there exists a maximal open subset of X on which $\mathcal{F}$ is equicontinuous.

(ii) Let $f : (X, d) \to (X, d)$. There exists a maximal open subset of X on which the family of iterates $\{f^n\}$ is equicontinuous.

This leads us to define the following:

Definition 5.2. *Let R be a nonconstant rational map. Then the Fatou set $F(R)$, of R is the maximal open subset of $\hat{\mathbb{C}}$ on which $\{R^n\}$ is equicontinuous, and Julia set, $J(R)$ of R is the complement of Fatou set in $\hat{\mathbb{C}}$.*

Note. Fatou set is also called as *set of normality*. By definition, $F(R)$ is open and $J(R)$ is closed and consequently compact, being closed subset of the compact space $\hat{\mathbb{C}}$.

We now give some properties of Julia set and Fatou set.

Theorem 5.3. *For any nonconstant rational map R and any positive integer p, $F(R^p) = F(R)$ and $J(R^p) = J(R)$.*

Theorem 5.4. *Let R be a rational map and g be a Möbius map. Let $S = gRg^{-1}$. Then $F(S) = g(F(R))$ and $J(S) = g(J(R))$.*

Exercise 5.5. Let R be a rational map and let the iterates R^n converge uniformly to some constant on a domain D. Prove that $D \subset F(R)$. Show that repelling fixed points of a rational map R lie in Julia set of R where as attracting fixed points of R lie in Fatou set of R.

Definition 5.6. *Let $f : X \to X$ be a map and $E \subset X$. Then E is said to be*

(i) *forward invariant if $f(E) = E$*

(ii) *backward invariant if $f^{-1}(E) = E$*

(iii) *completely invariant if $f(E) = E = f^{-1}(E)$.*

Note.

(i) If f is surjective, then $f(f^{-1}(E)) = E$, and so the concept of backward invariance and complete invariance coincide. Also this result need not be true if f is not surjective. For instance under the mapping $f(z) = e^z$, $f^{-1}(\mathbb{C}) = \mathbb{C}$ and $f(\mathbb{C}) = \mathbb{C} \setminus \{0\}$.

(ii) Since rational maps are surjective on $\hat{\mathbb{C}}$, it follows that backward invariance coincide with complete invariance for rationals.

Theorem 5.7. *Let R be a rational map of degree at least 2. Let E be a finite set which is completely invariant. Then E has at most two elements, and the result is best possible, as for the function $R(z) = \frac{1}{z^2}$, $\{0, \infty\}$ is completely invariant set.*

Proof. Let E be a completely invariant set having k elements. Then $R(E) = E$ and so R is a permutation. Consequently for some q, R^q is an identity function on E.

Let R^q be of degree d say. Then for every $w \in E$, the equation $R^q(z) = w$ has d solutions, and all the solutions are at w. Also there are k elements in E and so using Riemann Hurwitz relation we get

$$2d - 2 = \sum_{z \in \hat{\mathbb{C}}} (v_{R^q}(z) - 1) \geq \sum_{z \in E} (v_{R^q}(z) - 1) \geq k(d - 1).$$

Thus $k \leq 2$.

We list below some of the properties of completely invariant sets. These are true in any topological space X.

Let $g : X \to X$ be surjective.

(i) If E is completely invariant under g and h is a bijection of X onto itself, then $h(E)$ is completely invariant under the conjugation $h \circ g \circ h^{-1}$.

(ii) Intersection of a family of completely invariant sets is itself completely invariant.

(iii) Let $E_0 \subset X$. Let $E = \cap \{F | F$ is completely invariant set which contains $E_0\}$. Then E is the smallest completely invariant set containing E_0. We say E_0 *generates* E.

For $x, y \in X$ define $x \sim y$ if and only if there exist non-negative integers n and m such that $g^n(x) = g^m(y)$. Then $\sim$ is an equivalence relation. We denote the equivalence class containing x by $[x]$ and call it as *orbit* of x.

(iv) Let $g : X \to X$ be any surjective map. Then $[x]$ is completely invariant set generated by $\{x\}$.

(v) Let X be any topological space and $g : X \to X$ be any continuous surjective open map. Let E be completely invariant under g. Then $X \setminus E$, E^o, ∂E, $\overline{E}$ are also completely invariant.

Theorem 5.8. *Let R be a rational map. Then the Fatou set $F(R)$ and the Julia set $J(R)$ are completely invariant under R.*

Theorem 5.9. *Let P be a polynomial of degree at least two. Then $\infty \in F(P)$ and the component F_∞ of $F(P)$ containing ∞ is completely invariant under P.*

6. Normal Families

The concepts of Julia set and Fatou set can also be defined using the definition of normal families.

Definition 6.1. *A sequence $\{f_n\}$ of maps from a metric space (X_1, d_1) to a metric space (X_2, d_2) is said to converge locally uniformly on X_1 to some map f if each $x \in X_1$ has a neighbourhood on which $\{f_n\}$ converges uniformly to f.*

Definition 6.2. *A family $\mathcal{F}$ of functions from (X_1, d_1) to (X_2, d_2) is said to be normal family in X_1 if and only if every infinite sequence of functions from $\mathcal{F}$ contains a subsequence which converges locally uniformly on X_1.*

The following theorem known as *Arzela-Ascoli* theorem gives a relation between the equicontinuous family of functions and normal family.

Theorem 6.3. *Let D be a subdomain of the complex sphere. Let $\mathcal{F}$ be a family of continuous maps of D into the sphere. Then $\mathcal{F}$ is equicontinuous in D if and only if $\mathcal{F}$ is normal family in D.*

A criteria to determine the normality of a family of holomorphic maps is the following theorem of Montel.

Theorem 6.4. *Let D be a domain on $\hat{\mathbb{C}}$. Then the family $\mathcal{F}$ of all analytic maps $f : D \to \hat{\mathbb{C}} \setminus \{0, 1, \infty\}$ is normal in D.*

7. Exceptional Set and Properties of Julia Sets

Definition 7.1. *A point z is said to be exceptional point of a rational map R, if the orbit $[z]$ is finite. The set of all exceptional points of R is called exceptional set of R and will be denoted by $E(R)$.*

Definition 7.2. *For any $z \in \hat{\mathbb{C}}$, the backward orbit of z, denoted by $O^-(z)$ is defined to be the set given by*

$$O^-(z) = \{w | R^n(w) = z \text{ for some } n \geq 0\} = \cup_{n\geq 0} R^{-n}\{z\}$$

The points of $O^-(z)$ are called *predecessors* of z. Clearly $O^-(z) \subset [z]$ and further $O^-(z)$ is finite if and only if z is exceptional. Also if $O^-(z)$ is finite then $O^-(z) = [z]$.

Theorem 7.3. *A rational map of degree at least two has at most two exceptional points. If $E(R) = \{\xi\}$, then R is conjugate to a polynomial with ξ corresponding to ∞. If $E(R) = \{\xi_1, \xi_2\}$ where $\xi_1 \neq \xi_2$ then R is conjugate to some map $z \to z^d$ where ξ_1 and ξ_2 correspond to 0 and ∞.*

We now give some properties of Julia set. We first state the following theorem.

Theorem 7.4. *If a family of rational functions $\{R_n\}$ converge uniformly on the complex sphere to a function R then R is rational function. And for sufficiently large values of n,* $\deg(R_n) = \deg(R)$.

Theorem 7.5. *If* $\deg(R) \geq 2$*, then $J(R)$ is infinite.*

Proof. We first show $J(R) \neq \phi$. For suppose $J(R) = \phi$, then $F(R) = \hat{\mathbb{C}}$, and so $\{R^n\}$ is normal on the entire complex sphere. Using Theorem 7.4, there exist some subsequence say $\{R^{n_k}\}$ which converges to a rational function say S, and $\deg(R^{n_k}) = \deg(S)$ for all $n_k \geq N$. Since $\deg(R^{n_k}) = [\deg(R)]^{n_k}$, it follows that $[\deg(R)]^{n_k}$ is a constant, which is possible only if $\deg(R) = 1$. This contradiction proves $J(R) \neq \phi$.

Let $x \in J(R)$. Since $J(R)$ is completely invariant, $O^-(x) \subset J(R)$ and so if $J(R)$ is finite then $O^-(x)$ is finite and consequently x is exceptional point and so, must belong to $F(R)$. This contradicts $F(R) \cap J(R) = \phi$. Thus $J(R)$ must be an infinite set.

Theorem 7.6. *Let R be a rational map with* $\deg(R) \geq 2$. *Let A be any closed, completely invariant subset of the complex sphere. Then either*

(i) *A has at most two elements and $A \subset E(R) \subset F(R)$ or*
(ii) *A is infinite and $A \supset J(R)$.*

An immediate consequence of the above theorem is that the, *Julia set is the smallest closed completely invariant set with at least three points*. This is known as the *minimality property* of Julia set.

Theorem 7.7. *For a rational map R of degree ≥ 2, $J(R)$ has an empty interior or else $J(R) = \hat{\mathbb{C}}$.*

Proof. Let J and F denote the Julia set and Fatou set of R. Then $\hat{\mathbb{C}} = F \cup J = F \cup J^o \cup \partial J$. Also since J is completely invariant, it follows that J^o, ∂J are also completely invariant. Now if $J = \hat{\mathbb{C}}$, there is nothing left to be proved. So let $J \neq \hat{\mathbb{C}}$. Then $F \neq \phi$ and $F \cup \partial J$ is closed completely invariant, and in view of Theorem 7.6, it is also an infinite set. Thus by minimality property of J, $J \subset F \cup \partial J$. Since $F \cap J = \phi$, it follows that $J \subset \partial J$, and so $J^o = \phi$.

Note. It is known that the derived set $d(J)$ is an infinite, closed and completely invariant subset of J. By minimality of J, $J \subset d(J)$. Since $d(J)$ is closed, it follows that $J = d(J)$, so that J has no isolated points. Baire's Category theorem then gives that J is uncountable.

The next theorem gives the expanding property of a neighbourhood of Julia point.

Theorem 7.8. *Let R be a rational map of degree at least two. Let W be any non-empty open set which meets J. Then*

(i) $\cup_{n=o}^{\infty} R^n(W) \supset \hat{\mathbb{C}} \setminus E(R)$,
(ii) *for sufficiently large n, $R^n(W) \supset J$.*

Definition 7.9. *A point z_0 is called a periodic point of R if $R^n(z_0) = z_0$ for some $n \in \mathbb{N}$.*

Theorem 7.10. *Let $\deg(R) \geq 2$. Then J is contained in the closure of the set of periodic points of R.*

Theorem 7.11. *Let R be a rational map with $deg(R) \geq 2$.*

(i) *If z is not exceptional then $J \subset \overline{O^-(z)}$.*
(ii) *If $z \in J$ then $J = \overline{O^-(z)}$.*

Theorem 7.12. *Let R and S be rational maps of degree ≥ 2. Let $R \circ S = S \circ R$ then $J(R) = J(S)$.*

There exist rational maps whose Julia set is $\hat{\mathbb{C}}$. One of the examples is $R(z) = \frac{(z^2+1)^2}{4z(z^2-10)}$, which has $J(R) = \hat{\mathbb{C}}$. There are certain criteria to find some functions which have Julia set as $\hat{\mathbb{C}}$.

Theorem 7.13. *If every critical point of a rational map R is pre-periodic, then $J(R) = \hat{\mathbb{C}}$.*

Theorem 7.14. *$J(R) = \hat{\mathbb{C}}$ if and only if there exist some z whose forward orbit $\{R^n(z) : n \geq 1\}$ is dense in $\hat{\mathbb{C}}$.*

8. Components of Fatou Set and Euler Characteristic

Theorem 8.1. *Let $\deg(R) \geq 2$. Let F_0 be a completely invariant component of $F(R)$. Then*

(i) $\partial F_0 = J(R)$
(ii) *F_0 is either simply connected or infinitely connected*
(iii) *all other components of $F(R)$ are simply connected*
(iv) *F_0 is simply connected if and only if $J(R)$ is connected.*

Definition 8.2. *Let S be a compact or a bordered surface. A triangulation T of S is a partition of S into a finite number of mutually disjoint subsets called vertex , edge, face with the following properties:*

(i) *each vertex is a point of S*
(ii) *for each edge e , there exists a homeomorphism ϕ : $[a, b] \to S$ which maps the open interval (a, b) onto e and end points a and b to the vertices of T.*
(iii) *for each face f, there exists a homeomorphism $\psi : \triangle \to S$, (where $\triangle$ is a triangle), which maps the edges and vertices of $\triangle$ to the edges and vertices of T and such that f is the ψ-image of interior of $\triangle$.*

The triangulation T partitions S into mutually disjoint subsets of S. Each such subset is either vertex, edge or a face and each of them is called *simplex* of T of dimension 0, 1, 2 respectively. Further for any simplex s of dimension m we define *Euler characteristic* by $\chi(s) = (-1)^m$. Thus $\chi(vertex) = 1$, $\chi(edge) = -1$, $\chi(face) = 1$, and if a triangulation of S consists of V vertices, E edges and F faces then $\chi(S) = V + F - E$, and also $\chi(S)$ is independent of particular triangulation used.

Exercise 8.3. Prove that

(i) $\chi(S_1 \cup S_2) + \chi(S_1 \cap S_2) = \chi(S_1) + \chi(S_2)$.
(ii) $\chi(S) = \chi(S^o) + \chi(\partial S)$.
(iii) $\chi(S) = 2$ if and only if $S = \hat{\mathbb{C}}$.
(iv) $\chi(S) = 1$ if and only if S is simply connected but not $\hat{\mathbb{C}}$.
(v) $\chi(S) = 0$ if and only if S is doubly connected, and in all other cases (i.e., other than (iii), (iv) and (v)), $\chi(S)$ is negative.

Definition 8.4. *For any $z \in \hat{\mathbb{C}}$, the deficiency of R at z is defined by $\delta_R(z) = v_R(z) - 1$, and the total deficiency of R over a set A is defined by $\delta_R(A) = \sum_{z \in A} \delta_R(z)$.*

Exercise 8.5. If A and B are disjoint sets then show that $\delta_R(A \cup B) = \delta_R(A) + \delta_R(B)$.

Exercise 8.6. Use Riemann Hurwitz relation to show that $\delta_R(\hat{\mathbb{C}}) = 2d - 2$ where d is the degree of the rational map R, and for a polynomial P of degree d, $\delta_P(\mathbb{C}) = d - 1$.

The next theorem gives a relation between the Euler Characteristic, deficiency and the degree of the polynomial.

Theorem 8.7. *Let V be a domain bounded by a finite number of mutually disjoint Jordan curves. Let U be a component of $R^{-1}(V)$. Let there be no critical values of R on ∂V. Then there exists a positive integer m such that R is an $m-$fold map of U onto V and*

$$\chi(U) = \delta_R(U) = m\chi(V).$$

Theorem 8.8. *Let F_0 and F_1 be components of Fatou set $F(R)$ of a rational map R.Suppose R maps F_0 into F_1. Then for some integer mm, R is an $m-$fold map of F_0 onto F_1 and*

$$\chi(F_0) + \delta_R(F_0) = m\chi(F_1).$$

Theorem 8.9. *The Fatou set F of R contains at most two completely invariant components, and if there are two then each is simply connected.*

Proof. Let $\deg(R) = d$. Let $F_1, \ldots, F_k$ be completely invariant components. If $k < 2$ nothing left to be proved. So let $k \geq 2$. As F_1 is completely invariant, all other components viz., $F_2, F_3, \ldots, F_k$ are simply connected. Similarly since F_2 is completely invariant all other components viz., $F_1, F_3, F_4, \ldots, F_k$ are simply connected. Thus $F_1, F_2, \ldots, F_k$ are all simply connected and consequently $\chi(F_i) = 1$ for all $i = 1, 2, \ldots, k$. And so by the above theorem , since $R(F_i) = F_i$, it follows that

$$\delta_R(F_i) = d\chi(F_i) - \chi(F_i) = d - 1.$$

Thus

$$k(d-1) = \sum_{i=1}^{k} \delta_R(F_i) \leq \delta_R(\hat{\mathbb{C}}) = 2d - 2.$$

Thus $k \leq 2$. Also the above argument has already shown that each F_i is simply connected.

Theorem 8.10. *A Fatou set $F(R)$ of a rational map R has either* 0, 1, 2 *or infinitely many components.*

Proof. If $\deg(R) = 1$ then the result is trivially true. So let $\deg(R) \geq 2$. Suppose $F(R)$ has only finitely many components say $F_1, F_2, \ldots F_k$. Then each F_j is completely invariant under R^m for some m. Hence the Fatou set of R^m (and hence of R) contains at most two completely invariant domains. Thus $k \leq 2$.

9. Components of Julia Set

Theorem 9.1. *If $J(R)$ is disconnected then it has infinitely many components.*

Proof. Suppose $J(R)$ has only finitely many components , say $J_1, J_2, \ldots J_n$. Since $J(R)$ is infinite set without loss of generality we can assume J_1 is an infinite set. Also each J_k and in particular J_1 is completely invariant under R^m for some m. Also components of $J(R)$ are closed and hence by the minimality of Julia set, $J(R) = J(R^m) \subset J_1 \subset J(R)$. Thus $J(R) = J_1$ and so $J(R)$ is connected. This contradiction proves the theorem.

Note. A little more elaborate argument can be used to prove a more general theorem.

Theorem 9.2. *If $J(R)$ is disconnected then $J(R)$ has uncountably many components and each point of $J(R)$ is an accumulation point of infinitely many distinct components of $J(R)$.*

Recall Definition 3.1 for fixed point. We shall make some further classification in the fixed points

Definition 9.3. *Let z_0 be a fixed point of R. Then z_0 is called*

(i) *super attracting fixed point, if* $|R'(z_0)| = 0$
(ii) *attracting fixed point if* $0 < |R'(z_0)| < 1$
(iii) *repelling fixed point if* $|R'(z_0)| > 1$
(iv) *rationally indifferent fixed point if $R'(z_0)m$ is a root of unity*
(v) *irrationally indifferent fixed point if* $|R'(z_0)| = 1$ *but $R'(z_0)$ is not a root of unity.*

Definition 9.4. *A point z_0 is a periodic point of a rational function R if z_0 is a fixed point of some iterate R^m of the rational function R. In such a case we can find a positive integer m such that $z_0, R(z_0), \ldots, R^{m-1}(z_0)$ are all distinct and $R^m(z_0) = z_0$. The set $\{z_0, R(z_0), \ldots, R^{m-1}(z_0)\}$ is called cycle of z_0, and m is called period of z_0.*

If z_0 is repelling periodic point then every point of the cycle $\{z_0, R(z_0), \ldots R^{m-1}(z_0)\}$ is repelling and consequently we can speak of a repelling cycle.

Theorem 9.5. *Let* $\deg(R) \geq 2$. *Then*

(i) *every repelling cycle of* R *lies in* $J(R)$
(ii) *every rationally indifferent cycle of* R *lies in* $J(R)$.

Theorem 9.6. *A forward invariant component* F_0 *of the Fatou set* $F(R)$ *is one of the following types.*

(a) *an attracting component:* F_0 *contains an attracting fixed point* z_0 *of* R
(b) *a super attracting component:* F_0 *contains a super attracting fixed point* z_0 *of* R
(c) *a parabolic component or a Leau domain: there is a rationally indifferent fixed point* z_0 *of* R *on the boundary of* F_0 *and* $R^n(z) \to z_0$ *on* F_0.
(d) *a Siegel disc: the map* $R : F_0 \to F_0$ *is analytically conjugate to a Euclidean rotation of the unit disc onto itself.*
(e) *a Herman ring: the map* $R : F_0 \to F_0$ *is analytically conjugate to a Euclidean rotation of some annulus onto itself.*

Definition 9.7. *A component* U *of the Fatou set* $F(R)$ *is*

(i) *periodic if for some positive integer* n, $R^n(U) = U$
(ii) *eventually periodic if for some positive integer* n, $R^n(U)$ *is periodic*
(iii) *wandering if the sets* $R^n(U)$, $n \geq 0$, *are pairwise disjoint.*

One of the problems which had defied solution for over 60 years is the following theorem, finally solved by Sullivan [30].

Theorem 9.8 (Sullivan). *Every component of the Fatou set of a rational map* R *is eventually periodic. Thus a rational function has no wandering domain.*

In contrast to rational functions, transcendental entire functions may have wandering domain. In our next section we sketch the proof of this important theorem and in the final Section 11, we develop the iteration theory for entire functions and also for meromorphic functions and list a few properties which are different for these functions.

10. Sullivan's no Wandering Domain Theorem

We begin with the following Lemma which will have no significance at the end of the section.

Lemma 10.1. *Suppose* R *has a wandering domain. Then for some component* W *of* $F(R)$, *the components*

$$W, R(W), R^2(W), \ldots, R^n(W)$$

of $F(R)$ *are pairwise disjoint, simply connected and contain no critical points of* R.

Lemma 10.2. *Suppose that* W *is a wandering domain. Then for any compact subset* K *of* W, $diam[R^n(K)] \to 0$ *as* $n \to \infty$.

Proof. Suppose false, then there exists some compact subset K of W and some $\epsilon > 0$ and a sequence $n_j \to \infty$ such that

$$diam[R^{n_j}(K)] \geq \epsilon. \qquad (1)$$

Now $\{R^n\}$ is normal on W and hence given any sequence, in particular the above sequence R^{n_j}, there exists a subsequence of R^{n_j} which converges locally uniformly on W to some analytic function g.

For sake of convenience we relabel this subsequence and so assume R^{n_j} itself has this property.

Now suppose $g(W) = \text{constant} = \alpha$ say, then since $R^{n_j} \to g$ we have $R^{n_j}(W) \to \alpha$. Hence for large n_j, $diam[R^{n_j}(W)] < \epsilon/3$, contradicting (1)

Therefore R^{n_k} should converge to a non-constant function g locally uniformly on W. Now let $\xi \in W$ be such that $g'(\xi) \neq 0$. Then g is one-to-one in a small neighbourhood of ξ and so there exists a disk $D = (|z - \xi| \leq \delta \subset W$ such that $g(z) \neq g(\xi)$ for all $z \in \partial D$.

Thus

$$|g(z) - g(\xi)| > \inf_{w \in \partial D} |g(w) - g(\xi)| > |R^{n_j}(z) - g(z)|$$

for sufficiently large n_j and for all $z \in \partial D$. And hence by Rouche's theorem, $g(z) - g(\xi)$ and $R^{n_j}(z) - g(\xi)$ have the same number of zeros in $|z - \xi| < \delta$.

Since $g(z) - g(\xi)$ has a zero at ξ it follows that $R^{n_j}(D)$ contains $g(\xi)$.

Thus $R^{n_j}(W)$ contains $g(\xi)$, and so W is not a wandering domain. This contradiction proves the Lemma.

Definition 10.3. *Given any function f with continuous partial derivatives in a domain D, we define*

$$\frac{\partial f}{\partial z} = \frac{1}{2i}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \text{ and } \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

Note.

(i) If f is analytic then $\frac{\partial f}{\partial \bar{z}} = 0$ and $\frac{\partial f}{\partial z} = f'(z)$.

(ii) The equation $\frac{\partial f}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z}$ where μ is some suitable complex valued function on D is called *Beltrami equation* and μ is called *dilatation* of f on D.

Note. $\mu : D \to \mathbb{C}$ which we require is to be Lebesgue measurable and to satisfy $\|\mu\|_\infty < 1$ on D, and hence $|\mu| \leq \|\mu\|_\infty$ a.e. on D. Such μ are called *Beltrami coefficient* on D.

Proof of Theorem 9.8. Suppose R has a wandering domain. Then by Lemma 10.1, there exists a component W of $F(R)$ such that the components

$$W, R(W), R^2(W), \ldots, R^n(W), \ldots$$

of $F(R)$, are pairwise disjoint, simply connected and contain no critical points of R.

As W is simply connected (and $\neq \mathbb{C}$, since Julia set is infinite set), there is a conformal equivalence g of Δ onto W.

Next let μ be any Beltrami coefficient on Δ. We transfer μ to ν on W and extend ν from W to $\mathbb{C}$

Solving the Beltrami equation with coefficient ν throughout the sphere, we obtain a ν-conformal map $\phi : \mathbb{C}_\infty \to \mathbb{C}_\infty$ such that $\phi R \phi^{-1}$ is rational of degree equal to deg(R). Further composing with a Möbius map if necessary we suppose that ϕ fixes $0, 1, \infty$.

Thus we have created the composite map

$$\mu \to \nu \to \phi \to \phi R \phi^{-1} \tag{2}$$

from the space of Beltrami coefficient (μ) on Δ onto the space of rational maps $(\phi R \phi^{-1})$ of degree equal to deg(R).

Now roughly speaking keeping (2) in view, the space of Beltrami coefficients μ on Δ is infinite dimensional where as the space of rationals of degree deg(R), is finite dimensional and hence the map $\mu \to \phi R \phi^{-1}$ must map a large subspace of Beltrami coefficient into a single rational function S. These ideas are captured in the following Lemma.

Lemma 10.4. *Suppose $\eta_0 > 0$. For each $t \in [0, 1]$, we can construct a Beltrami coefficient $\mu_t \in \Delta$ such that*

(a) $\|\mu_t\|_\infty < \eta_0$

(b) *the mapping* (1) *above maps each μ_t to the same rational function S.*

Further this construction can be made so that for each z, the map $t \to \phi_t(z)$ is continuous on $[0, 1]$ where the mapping is as follows

$$t \xrightarrow{\mu} \mu_t \xrightarrow{\nu} \nu_t \xrightarrow{\phi} \phi_t \longrightarrow \phi_t R \phi_t^{-1} = R_t, \text{ say.}$$

Since we get the *same* rational function S, we get $\phi_t R \phi_t^{-1} = S = \phi_0 R \phi_0^{-1}$ for every $t \in [0, 1]$.

Put $\Phi_t = \phi_0^{-1} \circ \phi_t$. Then Φ_t has the following property:

For each $t \in [0, 1]$, Φ_t is identity on $J(R)$ and Φ_t maps each component of $F(R)$ onto itself.

So, since Φ_t maps each component of $F(R)$ (and hence W) onto itself we get $\Phi_t(W) = W$, i.e. $\phi_0^{-1} \circ \phi_t(W) = W$. Thus ϕ_t must map W onto the simply connected domain $W_0 (= \phi_0(W))$ where W_0 is clearly independent of t. Thus there is a conformal equivalence h of W_0 onto Δ.

We also note that for any homeomorphism Φ of a simply connected domain Ω onto itself we can define the (hyperbolic) displacement function of Φ by $z \longrightarrow \rho(\Phi(z), z)$ where ρ is the hyperbolic metric of Ω (which exists unless Ω is conformally equivalent to $\mathbb{C}$ or $\mathbb{C}_\infty$.

Next it can be shown that the map $g^{-1}\Phi_t g = g^{-1}\phi_0^{-1}\phi_t g$ of Δ onto itself has bounded displacement function and so extends to the identity on $\partial\Delta$.

Now each of the maps

$$\Delta[\mu_t] \xrightarrow{g} W[\nu_t] \xrightarrow{\phi_t} W_0[o] \xrightarrow{h} \Delta[o]$$

is analytic, (where $\Delta[\mu_t]$ denotes the μ_t-conformal structure on Δ.) Hence the map $h\phi_t g = \psi_t$ say is μ_t-conformal map of Δ onto itself and hence extends to a homeomorphism of the closed disk onto itself. Also on the open disk Δ,

$$\psi_0^{-1}\psi_t = (h\phi_t g)^{-1}(h\phi_0 g) = g^{-1}\Phi_t g.$$

By the previous observation just made above on $\partial\Delta$, $\psi_0^{-1}\psi_t =$ Identity and so $\psi_0 = \psi_t$ on $\partial\Delta$.

Now let Ψ_t be any μ_t conformal map of Δ onto itself. we next make use of the following Theorem:

Let μ be the Beltrami coefficient on a domain D of the complex sphere. Then

(i) *there exists a quasiconformal map with complex dilatation μ on D: and*

(ii) *if* ϕ *and* ψ *are any two such maps then* $\phi\psi^{-1}$ *is analytic.*

Using the above Theorem since ψ_t is already μ_t-conformal, it follows that the function $\psi_t \circ \Psi_t^{-1} = M_t$ say is analytic. Thus $\psi_t = M_t\Psi_t$ is analytic. But $\psi_t = \psi_0$ on $\partial\Delta$ and so $M_t\Psi_t = M_0\Psi_0$ on $\partial\Delta$.

Now in proving Lemma 10.5, we construct Ψ_t so that $\Psi_t = \Psi_0$ on some open arc of $\partial\Delta$. Thus $M_t = M_0$ and so $\Psi_t = \Psi_0$. We show that this is not true and hence a contradiction and hence the theorem gets proved.

11. Dynamics of Entire and Meromorphic Functions

Definition 11.1. *A family $\mathcal{F}$ of meromorphic functions in a domain $D \subset \hat{\mathbb{C}}$ is said to be a normal family if every sequence in $\mathcal{F}$ contains a subsequence which converges locally uniformly in D. Further $\mathcal{F}$ is said to be normal at z_0 if there exists a neighbourhood U of z_0 on which $\mathcal{F}$ is a normal family.*

Theorem 11.2 (Montel). *Let $\mathcal{F}$ be a family of meromorphic functions on a domain D. If there are three fixed values that are omitted by every $f \in \mathcal{F}$, then $\mathcal{F}$ is a normal family in D.*

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a meromorphic function. For f entire f^n is defined for all $z \in \mathbb{C}$. If f is rational then f has a meromorphic extension to $\hat{\mathbb{C}}$. Denoting this extension also by f we see that f^n is defined and meromorphic in $\hat{\mathbb{C}}$. However if f is transcendental meromorphic function, there is no reasonable way to define $f(\infty)$. Thus the iteration in such cases has to be defined to avoid such situation.

Let f be a meromorphic function in $\mathbb{C}$. Let
$F(f) = \{z \in \hat{\mathbb{C}} | \{f^n\}$ is defined and forms a normal family$\}$.
Let $J(f) = \hat{\mathbb{C}} - F(f)$. Then $F(f)$ is called *Fatou set* of f and $J(f)$ is called *Julia set* of f.

If f is transcendental entire, then $\{f^n\}_{n\in\mathbb{N}}$ is defined for all $z \in \mathbb{C}$, and in this case we always take $\infty \in J(f)$. If f is transcendental meromorphic with exactly one pole say z_0 and if this is an omitted value (i.e., $f(z) \neq z_0$ for all $z \in \mathbb{C}$), then clearly $\{f^n\}_{n\in\mathbb{N}}$ is defined for all $z \in \hat{\mathbb{C}} \setminus \{z_0, \infty\}$, and in this case $\{z_0, \infty\} \subset J(f)$. Also in this case $f(z)$ is of the form

$$f(z) = z_0 + \frac{e^{g(z)}}{(z - z_0)^m}$$

for some entire function g and some positive integer m. By suitable translation we may assume that the omitted value is the origin so that we may consider f to be analytic self map of $\mathbb{C} \setminus \{0\}$.

Finally if f is a transcendental meromorphic and has either at least two poles or exactly one pole which is not an omitted value, then there is an interesting way of defining Fatou and Julia sets using the backward orbit and the theorem of Montel.

For $z_0 \in \hat{\mathbb{C}}$, set

$$f^{-n}(z_0) = \{z \in \mathbb{C} : f^n(z) = z_0 \text{ for some } n \in \mathbb{N}\}.$$

Define

$$O^-(z_0) = \cup_{n=1}^{\infty} f^{-n}(z_0).$$

Clearly $\hat{\mathbb{C}} \setminus \overline{O^-(\infty)}$ is the largest open subset of $\hat{\mathbb{C}}$ where all the iterates of f are defined. Also

$$f(\hat{\mathbb{C}} \setminus \overline{O^-(\infty)} \subset \hat{\mathbb{C}} \setminus \overline{O^-(\infty)}$$

and from Picard's theorem, it is easy to see that $\overline{O^-(\infty)}$ is an infinite set and so by Montel's theorem, it follows that $\{f^n\}_{n\in\mathbb{N}}$ is normal in $\hat{\mathbb{C}} \setminus \overline{O^-(\infty)}$. Thus

$$F(f) = \hat{\mathbb{C}} \setminus \overline{O^-(\infty)} \text{and} J(f) = \overline{O^-(\infty)}.$$

In view of these observations the set for all meromorphic functions is divided into four different classes:

R = $\{f : f$ is a rational function$\}$
(We assume the degree to be at least two)
E = $\{f : f$ is transcendental entire$\}$
P = $\{f : f$ is transcendental meromorphic and has exactly one pole and this pole is an omitted value$\}$
M = $\{f : f$ is a transcendental meromorphic and has either at least two poles or exactly one pole which is not an omitted value$\}$.

As is obvious, the reason for this different classification is that many times the properties in these classes differ, and even if the properties are the same the proof of them may differ substantially.

Some of the major differences between the dynamics meromorphic functions of the above classes are:

(i) as mentioned earlier, rational functions do not have wandering domain. In contrast, meromorphic functions may have wandering domains. The first example of a wandering domain was given by Baker [1] who showed that there exists a transcendental entire function which has a wandering domain. Since then several entire and meromorphic functions which have wandering domain have been found. Also there are classes of entire and meromorphic functions which do not have wandering domain. Some of the papers related to the topic on wandering domain are [2, 3, 5, 6, 7, 8, 11, 12, 13, 17, 20, 21, 25, 26, 27, 28].

(ii) A wandering domain of a transcendental entire function may be simply connected or multiply connected and a multiply connected wandering domain must necessarily be bounded. However this need not be true for functions belonging to class **M**. Baker, Kotus, Lu [7] have shown that for $f \in \mathbf{M}$, the Fatou set $F(f)$ can have

  (a) k-connected bounded wandering domain
  (b) k-connected unbounded wandering domain
  (c) bounded wandering domain of infinite connectivity
  (d) unbounded wandering domain of infinite connectivity.

(iii) If f and g are permutable rational functions then it is known that $J(f) = J(g)$. However this is an open question when f and g happen to be transcendental entire. It is pertinent to mention here that several results in affirmation have been obtained under certain restriction on f and g, for instance Bergweiler and Hinkkanen [11] have shown that if f and g are permutable transcendental entire functions having no wandering domain then $J(f) = J(g)$. Some of the papers on permutable transcendental entire are [20, 24, 28] and their references.

(iv) If U is a periodic domain of period p of a meromorphic function f then U is called a *Baker domain* if there exists z_0 on the boundary of U such that $f^{np}(z) \to z_0$ as $n \to \infty$. Rational functions do not have Baker domain, while transcendental entire functions have Baker domain only for $z_0 = \infty$. For $f \in \mathbf{P}$ with pole at the origin, Baker domains are possible with $z_0 \in \{0, \infty\}$.

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RMS/SMF/IMSc Indo-French Conference in Mathematics 2008

The Ramanujan Mathematical Society and the Société Mathématique de France are organising an Indo-French Conference in Mathematics during the period 15th to 19th of December 2008. This conference will be hosted by the Institute of Mathematical Sciences, Chennai.

More details about this meeting will be posted here as they become available. For the moment please send mail the Director, IMSc as per the contact information below for further details.

1 Sessions

The conference will be devoted to the following themes.

- Algebraic Geometry (Géométrie algébrique)
- Complex Geometry (Géométrie complexe)
- Probability Theory (Probabilité)
- Partial Differential Equations/Euclidean Harmonic Analysis (Equations aux dérivées et analyse harmonique euclidienne)
- Number Theory (Théorie des Nombres)
- Lie Groups and Non-Commutative Harmonic Analysis

2 Scientific Committee

- R. Balasubramanian (Institute of Mathematical Sciences, Chennai, India)
- Rajendra Bhatia (Indian Statistical Institute New Delhi, India)
- Aline Bonami (Universite' d'Orle'ans, France)
- Pascal Chossat (Institut non line'aire de Nice, France)
- Jean–Marc Deshouillers (Universite' Bordeaux II, France)
- M. S. Narasimhan (Tata Institute of Fundamental Research/ Indian Institute of Science, Bangalore, India)
- Alladi Sitaram (Indian Statistical Institute, Bangalore, India)
- Michel Waldschmidt (Institut de mathe'matiques de Jussieu, Paris, France)

3 Local Organising Committee

- R. Balasubramanian (Institute of Mathematical Sciences, Chennai, India)
- V. Balaji (Chennai Mathematical Institute, Chennai, India)
- K. N. Raghavan (Institute of Mathematical Sciences, Chennai, India)
- K. H. Paranjape (Institute of Mathematical Sciences, Chennai, India)

4 Speakers

The list of speakers at this conference is given below.

Siva Athreya	Nalini Anantharaman
Arnaud Beauville	Jean Bertoin
Indranil Biswas	Jean–Benoît Bost
Jean–Pierre Demaily	Maria Esteban
Patrick Gérard	Yves Guivarch

Jaya Iyer
Chandrasekhar Khare
Vikram Mehta
Kapil Paranjape
Mihai Paun
C. S. Rajan
Harish Seshadri
R. Sujatha
T. N. Venkataramana
Yves André
Sinnou David
Yogish I. Holla
Parasar Mahanty
Angela Pasquale
Rahul Roy
Emmanuel Trelat
Bruno Kahn
Emmanuel Kowalski
E. K. Narayanan
Etienne Pardoux
Emmanuel Peyre
Swagato Ray
Nimish Shah
S. Thangavelu
Kaushal Verma
Jean–Michel Bismut
Isabelle Gallagher
S. Kesavan
Joseph Oesterlé
Mythily Ramaswamy
V. Srinivas

5 Contact Information

Professor R. Balasubramanian,
Director,
The Institute of Mathematical Sciences,
Chennai 600 113, India

- Email: director@imsc.res.in
  Web: http://www.imsc.res.in/

6 Registration

Those who are interested in attending the conference should register by August 31. We will be able to support a small number of participants in terms of travel and local hospitality. If you are in need of financial assistance, please mention this when you register. There is no registration fee. **To register, please go to website http://www.imsc.res.in/mathweb/rmssmfconf/**

Thematic Year on Analysis and its Applications

August 2008 – July 2009

As a part of its series of annual thematic programs in mathematics, the IISc Mathematics Initiative (IMI) announces the Annual Thematic Year in Analysis and its Applications, beginning August 2008. Its aim is to bring together specialists in mathematical analysis for compact courses and workshops, and to expose younger mathematicians in India to the latest trends in the field. In keeping with the latter goal, compact courses have been planned in the areas of: Complex Analysis in Several Variables, Harmonic Analysis and Operator Theory.

Analysis has had a key role to play in several applied fields. This, plus some of the conceptual trends in analysis today, are intended to be surveyed in the following two workshops:

- Workshop in Harmonic Analysis and PDE (December 05 – December 13, 2008)
- Workshop in Operator Theory (May 18 – 26, 2009)

The year will culminate in the conference:

- Conference in Analysis and its Applications (May 27 – 30, 2009)

Those interested in participating in any of the activities under the annual thematic programme are encouraged to contact one of the organizers at imi@math.iisc.ernet.in

For Further Details and Request for Participation, Please Contact:

IISc Mathematics Initiative (IMI),
Department of Mathematics, Indian Institute of Science,
Bangalore 560 012, India
Ph: +91-80-22933217, 18, 23605390
E-mail: imi@math.iisc.ernet.in
Website: http://math.iisc.ernet.in/~imi/atp-0809.htm

Fourth National Conference on Applicable Mathematics in Wave Mechanics and Vibrations (WMVC-2008)

November 14–16 (Friday–Sunday), 2008

Organised By:

Jaipur College of Engineering & Research Centre,
Jaipur

In Collaboration With:

Von Karman Society for Advanced Study & Research in Mathematical & Social Sciences,
Jalpaiguri, W.B.

Conference Sub-Topics:

- Nonlinear Phenomena in Mathematical and Physical Sciences
- Mathematical and Computational Methods, Astrophysical Problems
- Mechanics of Solids and Structural Mechanics
- Elastic Wave Mechanics, Fluid Mechanics and Plasma Physics
- Vibrations of Beams/Plates/Shells including Thermal and Random Vibrations
- Thermal Stresses, Thermal Buckling and Post-Buckling Analysis
- Mechanical Behaviour of Structures at Elevated and Criogenic Temperatures
- Bio-medical Engineering
- Engineering, Physical and Medical Acoustics, Architectural and Highway Acoustics
- Applications in fields like Music, Speech and Hearing
- Recent Developments in Applied Mathematics and Applications. Including

The Venue: The venue of the Conference is Jaipur College of Engineering and Research Centre (JECRC), Jaipur. Jaipur, Being the capital of Rajasthan it is easily accessible by all means. Daily flights are available to and from all major cities and it is well connected to different parts of the country by roads and trains. Weather of Jaipur in the month of November is very pleasant with average maximum and minimum temperatures of about 28°C and 15°C respectively.

The Institute is situated on Jaipur Kota highway (NH-12) and is about 20 kms from Jaipur railway station and city bus stand, 9 kms from Durgapura railway station and about 5 kms from Jaipur airport.

Abstract:

Two copies of abstract (150–200 words)	: Before July 31, 2008
Notification of acceptance/ invitation	: Before August 18, 2008
One copy of maximum 6-page manuscript of paper with CD typed in MS-Word and with intimation/confirmation for participation and delegate fee	: Before October 31, 2008

Technical Sessions: Technical Sessions are planned for Invited Speakers (Thirty minutes each) and Contributed Papers (Twenty minutes each). **LCD and OHP will be provided for presentation of papers**.

Registration Fees: Nominal Registration Fee is payable by all the Authors, Co-Authors and accompanying Persons as prescribed below:

Sponsored/Instituitional Delegates	– Rs. 1500/-
Author/Co-Author (Per Person)	– Rs. 750/- (Before October 31, 2008)
Accompanying Persons	– Rs. 600/- (Before October 31, 2008)
Late/Spot Registration Fee	– Rs. 1000/-

The above Fees are to be paid by DD (or on-line through SBI having Core-Banking Facility) in favour of "Von Karman Society for Advanced Study and Research in Mathematical and Social Sciences.", A/C No. 10176348340 at State Bank of India, Jalpaiguri Town Branch, Jalpaiguri 735 101 (WB), (SBI Code no. 2070) followed by due intimation of money transfer if made through core-banking facility.

Special Offer: For organisational convenience intending delegates are advised to co-operate by way of sending their early registration fee which will ensure cost-free simple accommodation "on first come first serve basis" untill limited cost-free accommodation is exhausted otherwise it may not be possible to provide accommodation free of charge for late registered delegates.

Conference Publications: Book of Abstracts will be available during the conference. Proceedings will be published after the Conference.

Abstract of Papers Should be Sent to:

Dr. P. Biswas
(Conference Co-Director)
Head, Vibration Research Group and Executive Secretary
Von Karman Society for Advanced Study and Research in Mathematical and Social Sciences
Old Police Line, Jalpaiguri 735 101, West Bengal
Phone: 03561-256894/Mobile: 98231181
E-Mail: biswas_paritosh@yahoo.com

For Further Information Please Write to:

Dr. Abhijit Sarkar
Department of Electronics and Communication Engineering
Jaipur College of Engineering and Research Centre
Shri Ram ki Nangal, Via – Vatika,
Tonk Road, Jaipur 303 905
E-mail: asarkar63@rediffmail.com

For Details Please Visit: www.jecrc.net

ACM-SIAM Symposium on Discrete Algorithms (SODA09)

January 4–6, 2009

Venue: New York Marriott Downtown, New York

Description: This symposium focuses on research topics related to efficient algorithms and data structures for discrete problems. In addition to the design of such methods and structures, the scope also includes their use, performance analysis, and the mathematical problems related to their development or limitations. Performance analyses may be analytical or experimental and may address worst-case or expected-case performance. Studies can be theoretical or based on data sets that have arisen in practice and may address methodological issues involved in performance analysis.

Submission Deadlines:

June 26, 2008 – Pre-Submission Deadline
July 3, 2008 – Final Submission Deadline

Pre-Registration Deadline: December 1, 2008
Hotel Reservation Deadline: December 1, 2008

For Further Details Please Visit:

http://www.siam.org/meetings/da09/

Quantitative and Computational Aspects of Metric Geometry

January 12–16, 2009

Venue: Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, California

Overview: We have witnessed a recent revival of interest in the rich structure and profound properties of metric spaces. Much contemporary research on metric geometry is motivated by the discovery of unexpected connections linking fundamental questions in geometry and analysis with combinatorial optimization, computational complexity, and statistics. This has led to the emergence of an impressive and growing repertoire of common problems and techniques.

Application/Registration: An application/registration form is available at

http://www.ipam.ucla.edu/programs/mg2009/.

Applications received by December 1, 2008, will receive fullest consideration. Encouraging the careers of women and minority mathematicians and scientists is an important component of IPAM's mission and we welcome their applications. You may also simply register and attend without IPAM funding.

For Further Information Contact:
Institute for Pure and Applied Mathematics (IPAM)
Attn: MG2009
460 Portola Plaza
Los Angeles CA 90095-7121
Phone: 310 825-4755
Fax: 310 825-4756
E-mail: mg2009@ipam.ucla.edu
Web site: http://www.ipam.ucla.edu/programs/mg2009/

Group Theory, Combinatorics and Computation

January 5–16, 2009

Venue: The University of Western Australia, Perth, Australia

Description: Special Theme Program on group theory, combinatorics and computation at the University of Western Australia

Topics:

Week 1: An international conference in honor of Professor Praeger's 60th birthday. It will contain invited 1 hour talks and short contributed talks by participants.
Week 2: An informal week of short courses, workshops and problem sessions, especially beneficial to early career researchers and postgraduate students.

For Further Information Contact:

http://sponsored.uwa.edu.au/gcc09/welcome;
E-mail: alice@maths.uwa.edu.au.

Algebraic Geometry

January 12–May 22, 2009

Venue: Mathematical Sciences Research Institute, Berkeley, California

Description: This semester-long "jumbo" program on algebraic geometry will emphasize cross-fertilization between different areas, including classical and complex algebraic geometry, linear series techniques, moduli spaces, enumerative geometry, varieties with group actions, birational geometry, rational curves on algebraic varieties, and classification theory. The full resources of MSRI will be devoted to a comprehensive discussion of these topics. The organizers hope to convey the essential unity of the subject, especially to young researchers and established mathematicians in other fields who use algebraic geometry in their research.

For Further Details Visit:

http://www.msri.org/calendar/programs/programinfo/251/show_program

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Algebraic Lie Theory

January 12–June 26, 2009

Venue: Isaac Newton Institute for Mathematical Sciences, 20 Clarkson Road, Cambridge CB3 OEH, United Kingdom

Description: Lie theory has profound connections to many areas of pure and applied mathematics and mathematical physics. In the 1950s, the original "analytic" theory was extended so that it also makes sense over arbitrary algebraically closed fields, in particular, fields of positive characteristic. Understanding fundamental objects such as Lie algebras, quantum groups, reductive groups over finite or p-adic fields and Hecke algebras of various kinds, as well as their representation theory, are the central themes of "Algebraic Lie Theory". It is anticipated that the activities of the programme will lead to a focalisation and popularisation of the various recent methods, advances and applications of Algebraic Lie Theory.

For Further Details Contact/Visit:

E-mail: s.penton@newton.cam.ac.uk;
http://www.newton.ac.uk/programmes/ALT/index.html.

Pre-ICM International Convention on Mathematical Sciences

December 18–20, 2008

Venue: University of Delhi, Delhi, India

Description: On the initiative of Department of Science and Technology (DST), Government of India, an activity cell to organize "India Mathematics Year 2009 (IMY2009) as Pre-ICM activity" has been set up in the Department of Mathematics, University of Delhi. To launch IMY-2009, we are organizing: Pre-ICM International Convention on Mathematical Sciences (ICMS2008) at the Department of Mathematics, University of Delhi during December 18–20, 2008. The academic programme of the convention will include activities like workshops, symposia, brain-storming sessions, panel discussions, group discussions, seminars, poster sessions, tutorials, compact sessions etc. on various topics of mathematical sciences including interdisciplinary aspects.

For Further Details Contact/Visit:

icmsdu@gmail.com
http://icms2008.du.ac.in.

Algebraic Topology, Braids and Mapping Class Groups

December 8–22, 2008

Venue: Institute for Mathematical Sciences, National University of Singapore, Singapore

Description: The recent progress in topology has shed light on many deep connections between algebraic topology and the theory of braids. A successful program on Braids was organized in May–July, 2007. This present program is going to explore further the connections between algebraic topology and braids, and to establish further research collaborations in algebraic topology in Asia. The present program will consist of a conference on algebraic topology, and a workshop on special topics.

(1) The Second East Asia Conference on Algebraic Topology, December 8–12, 2008.
(2) Workshop on Homotopy, Braids and Mapping Class Groups, December 13–22, 2008.

For Further Information Please Contact:

E-mail: imscec@nus.edu.sg;
http://www.ims.nus.edu.sg/Programs/braids08/index.htm.

4th International Conference on Combinatorial Mathematics and Combinatorial Computing (4ICC)

December 15–19, 2008

Venue: University of Auckland, Auckland, New Zealand

Description: The ICC is held every 10 years. This year it includes the annual ACCMCC meeting of the Combinatorial Society of Australasia, and the New Zealand leg of the map conferences held annually in Slovenia/Slovakia/Arizona-Portugal/New Zealand.

Note: This event is held during summer in New Zealand.

For Further Details Contact:

E-mail: mcw@cs.auckland.ac.nz;
http://www.cs.auckland.ac.nz/research/groups/theory/4ICC/.

Small Ball Inequalities in Analysis, Probability and Irregularities of Distribution

December 8–12, 2008

Venue: American Institute of Mathematics, Palo Alto, California

Description: This workshop, sponsored by AIM and the NSF, will be devoted to a theme common to irregularity of distributions, approximation theory, probability theory and harmonic analysis. In each of these subjects, there are outstanding conjectures in dimensions three and higher that stipulate that functions which satisfy certain conditions on its mixed derivative are necessarily large in sup norm. This workshop will survey these conjectures, seeking both commonalities and differences, describe recent advances, and discuss proof techniques and strategies.

For Further Details Visit:

http://aimath.org/ARCC/workshops/smallballineq.html

The Newton International Fellowship Scheme

http://www.newtonfellowships.org/

The Newton International Fellowship scheme will select the very best early stage post-doctoral researchers from all over the world, and offer support for two years at UK research institutions.The long-term aim of the scheme is to build a global pool of research leaders and encourage long-term international collaboration with the UK.

The Newton International Fellowships scheme is run by The British Academy, The Royal Academy of Engineering and the Royal Society.

The Fellowships cover the broad range of natural and social sciences, engineering and the humanities.

They provide grants of £24,000 per annum to cover subsistence and £8,000 to cover research expenses, plus a one-off relocation allowance of £2,000.

As part of the scheme, all Newton Fellows who remain in research will be granted a 10 year follow-up funding package worth £6,000 per annum.

For more detailed information on the Newton International Fellowships please download the Scheme Notes from

http://www.newtonfellowships.org/PDFs/NIFSchemeNotes.pdf

Call For Papers/Abstracts/Submissions 8th Annual Hawaii International Conference on Statistics, Mathematics and Related Fields

January 13–15, 2009

Submission Deadline: August 29, 2008

E-mail: statistics@hicstatistics.org
Website: http://www.hicstatistics.org

National Workshop on Cryptology – 2008 Department of Computer and Information Sciences, University of Hyderabad

August 8–10, 2008

Website: www.crsindia.com, www.uohyd.ernet.in.

Contact for Further Details:

Mr. Y. V. Subba Rao
Departement of CIS,
University of Hyderabad,
Hyderabad 500 046, India
Email: hcucrypto2008@yahoo.com
hcucrypto2008@gmail.com

The readers may download the Mathematics Newsletter from the RMS website at www.ramanujanmathsociety.org