MATHEMATICS NEWSLETTER

Volume 18 September 2008 No. 2

CONTENTS

Visit: www.ramanujanmathsociety.org

Typeset in LaTeX at Krishtel eMaging Solutions Pvt. Ltd., Chennai - 600 017 Phone: 2434 55 16 and printed at United Bind Graphics, Chennai - 600 004. Phone: 2498 7562, 2466 1807

Mathematica: Introductory Examples Related to Ramanujan, Part 1

Heikki Ruskeepää
Department of Mathematics
University of Turku
FI-20014 Turku, Finland
ruskeepa@utu.fi

Abstract. This two-part article is intended to give an introduction to the *Mathematica* software, developed by Wolfram Research, Inc. Rather than systematically going through the elements of *Mathematica*, we give several examples of applications. Most of the examples are related to results given by Ramanujan, whilst a few at the end of Part 2 relate to the geography, demography, and economy of India. We hope that the examples will provide an overview of the uses of *Mathematica* and an insight into the exciting possibilities that it offers. The second part of the article will appear later.

Part 1

Part 2

Introduction

G. H. Hardy [1] wrote about Srinivasa Ramanujan:

I remember going to see him once when he was lying ill in Putney. I had ridden in taxi-cab No. 1729, and remarked that the number seemed to me rather a dull one, and that I hoped that it was not an unfavorable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as a sum of two cubes in two different ways."

In the words of Wolfram Research, Inc., the developer of the *Mathematica* software, "*Mathematica* has been built from its inception to deliver one vision: the ultimate technical computing environment." *Mathematica* is not only used for symbolic and numerical computation and programming but also for modeling, simulation, visualization, development, documentation, and deployment.

In this article, we introduce the reader to *Mathematica* through several examples. We do not give a systematic account of the elements of *Mathematica*. For such an introduction, we refer the reader to some built-in documentation sources, one of which can be found by choosing **StartupPalette** from the **Help** menu and then clicking the *First Five Minutes with Mathematica* link. Another source is the *Introduction* item in the **Virtual Book** which can also be found from the **Help** menu. There are also books available, for example [4].—This article is based on *Mathematica* 6.

The examples studied in this article relate to the mathematics of Srinivasa Ramanujan (1887–1920). Thus, the examples do not give a balanced overview of *Mathematica* but rather are biased towards the mathematical topics that were of particular interest to Ramanujan. The topics considered include integrals, sums, nested radicals, continued fractions, powers representations, Diophantine equations, Ramanujan tau and tau L functions, primes, divisors, partitions, and composites. We also give examples related to India.

Our main source of material on Ramanujan has been Wolfram *MathWorld*, the on-line mathematics resource at `http://mathworld.wolfram.com/` which has been created and developed by Eric Weisstein. For a CD-ROM about the life and work of Srinivasa Ramanujan, see `http://www/cdac.in/html/nmrc/mathgen.asp`.

In the examples, we mainly use two-dimensional inputs, as they are very easy to read. As an example, we write

$$\int_1^2 \frac{\mathbf{Log[x]^k}}{\mathbf{x-1}}\,\mathbf{dx}$$

instead of

```
Integrate[Log[x]^k/(x-1),{x,1,2}]
```

Inputs can easily be written in the two-dimensional form by using the **BasicMathInput** palette or by using special key combinations. Remember that after writing a command, to get the result one has to press the SHIFT and RETURN keys at the same time.

1. Results Related to Integrals

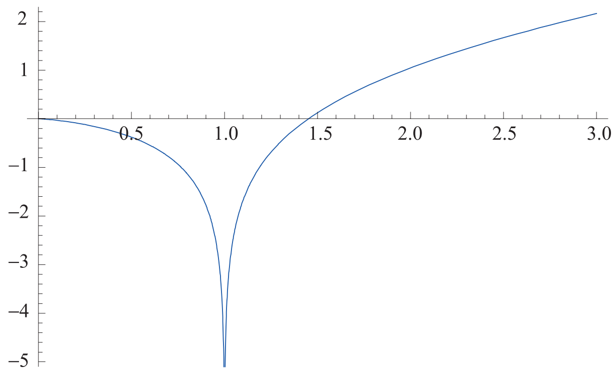
■ Soldner's Constant

The logarithmic integral function li(x) is defined to be $\mathrm{li}(x) = \int_0^x \frac{1}{\ln t}\,dt$, where the integral is calculated as the Cauchy principal value. This function is shown below:

```
f = LogIntegral[x]
```

```
LogIntegral[x]
```

```
Plot[f, {x,0,3}, PlotRange → {-5.1,2.3}]
```



(Note that the arrow → can be written as -> but *Mathematica* automatically replaces these two characters with →.)

Soldner's constant [17] is the root of the li(x) function. The root appears, from the diagram, to be about 1.5. Ramanujan calculated for the root an approximation of 1.45136380. Let us calculate an approximation using *Mathematica*:

```
FindRoot[f, {x,1.5}]
```

```
{x → 1.45137}
```

To obtain more decimal places, one can use a high-precision calculation:

```
sol = FindRoot[f, {x,1.5}, WorkingPrecision → 100]
```

```
{ x →
  1.451369234883381050283968485892027449493032283648015863093004557662425595754517
   835659531357711086829}
```

These are all correct digits. At the calculated zero, the function is, indeed, zero to a very high degree of accuracy:

```
f/. sol
```

$0. \times 10^{-100}$

■ Nielsen–Ramanujan Constants

Consider the integral

$$\texttt{a[k\_]} := \int_1^2 \frac{\texttt{Log[x]}^k}{\texttt{x - 1}}\,dx$$

for various values of k (note that functions in *Mathematica* are defined with := and the arguments in the left-hand side have the underscore _). The values of this integral are called Nielsen–Ramanujan constants [11]. Ramanujan was able to find the value of the constant for $k = 1$ and 2.

Let us see what we can do using *Mathematica*. First we try to calculate the general value of the integral:

```
a[k]
```

$$\int_1^2 \frac{\texttt{Log[x]}^k}{-1 + x}\,dx$$

We can see from the output that *Mathematica* was not able to calculate the general value. However, we can calculate the integral for special values of k:

```
Table[a[k], {k, 4}]
```

$$\left\{\frac{\pi^2}{12}, \frac{\texttt{Zeta[3]}}{4}, \frac{\pi^4}{15} + \frac{1}{4}\pi^2\,\texttt{Log[2]}^2 - \frac{\texttt{Log[2]}^4}{4} - 6\ \texttt{PolyLog}\left[4, \frac{1}{2}\right] - \frac{21}{4}\texttt{Log[2] Zeta[3]},\right.$$

$$\frac{2}{3}\pi^2\texttt{Log[2]}^3 - \frac{4\ \texttt{Log[2]}^5}{5} - \frac{21}{2}\texttt{Log[2]}^2\texttt{Zeta[3]}$$

$$\left. -4\left(\texttt{Log[64] PolyLog}\left[4, \frac{1}{2}\right] + 6\ \texttt{PolyLog}\left[5, \frac{1}{2}\right] - 6\ \texttt{Zeta[5]}\right)\right\}$$

The traditional forms of these expressions are as follows (note that % refers to the previously computed result):

```
% // TraditionalForm
```

$$\left\{\frac{\pi^2}{12}, \frac{\zeta(3)}{4}, \frac{\pi^4}{15}+\frac{1}{4}\pi^2\log^2(2)-\frac{\log^4(2)}{4}-6\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{21}{4}\log(2)\zeta(3), \frac{2}{3}\pi^2\log^3(2) -\frac{4\log^5(2)}{5}-\frac{21}{2}\log^2(2)\zeta(3)-4\left(\log(64)\mathrm{Li}_4\left(\frac{1}{2}\right)+6\mathrm{Li}_5\left(\frac{1}{2}\right)-6\zeta(5)\right)\right\}$$

The general value of the constant has been proved to be

$$\texttt{aa[k\_] := k! Zeta[k+1]} - \frac{\texttt{k Log[2]}^{k+1}}{\texttt{k+1}} - \texttt{k!}\sum_{i=0}^{k-1}\frac{\texttt{PolyLog}\left[\texttt{k+1-i},\frac{1}{2}\right]\texttt{Log[2]}^{i}}{\texttt{i!}}$$

We calculate the first few values:

```
Table[aa[k],{k, 4}] // Simplify
```

$$\left\{\frac{\pi^2}{12}, \frac{\texttt{Zeta[3]}}{4}, \frac{\pi^4}{15}+\frac{1}{4}\pi^2\,\texttt{Log[2]}^2-\frac{\texttt{Log[2]}^4}{4}-6\ \texttt{PolyLog}\left[4,\frac{1}{2}\right]-\frac{21}{4}\texttt{Log[2] Zeta[3]}, \frac{2}{3}\pi^2\texttt{Log[2]}^3-\frac{4\ \texttt{Log[2]}^5}{5}-24\ \texttt{Log[2] PolyLog}\left[4,\frac{1}{2}\right] -\frac{21}{2}\texttt{Log[2]}^2\texttt{Zeta[3]}+24\left(-\texttt{PolyLog}\left[5,\frac{1}{2}\right]+\texttt{Zeta[5]}\right)\right\}$$

These values agree with the values we calculated earlier.

■ Ramanujan's Master Theorem

Suppose that in some neighbourhood of $x = 0$,

$$F(x) = \sum_{k=0}^{\infty}\phi(k)\frac{(-x)^k}{k!}$$

for some function (say, analytic or integrable) $\phi(k)$. Then

$$\int_0^{\infty} x^{s-1}F(x)dx = \Gamma(s)\phi(-s).$$

This is Ramanujan's Master Theorem [6]. As an example, let us choose $\phi(k) = a^k$, where $a > 0$. Then

$$\mathbf{F} = \sum_{k=0}^{\infty}\mathbf{a}^{k}\frac{(-\mathbf{x})^{k}}{\mathbf{k!}}$$

$$\mathbb{E}^{-a\,x}$$

From the theorem we then know that $\int_0^\infty x^{s-1}e^{-a\,x}dx = \Gamma(s)a^{-s}$. We can check this integral:

$$\int_0^{\infty}\mathbf{x}^{s-1}\,\mathbb{E}^{-a\,x}\,\mathbf{dx}$$

```
If[Re[a] > 0 && Re[s] > 0, a^-s Gamma[s],
  Integrate[E^(-a x) x^(-1+s), {x, 0, ∞}, Assumptions → Re[a] ≤ 0 || (Re[a] > 0 && Re[s] ≤ 0)]]
```

Thus, if $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(s) > 0$, then the integral is $\Gamma(s)a^{-s}$; otherwise the integral does not converge. We can define the required assumptions either as

$$\texttt{Assuming}\left[\texttt{Re[a] > 0 \&\& Re[s] > 0, } \int_0^\infty x^{s-1}\, \mathbb{E}^{-a x}\, dx\right]$$

$$a^{-s}\,\texttt{Gamma[s]}$$

or as

$$\texttt{Integrate}\left[x^{s-1}\,\mathbb{E}^{-a x}\texttt{, \{x, 0, }\infty\texttt{\}, Assumptions} \to \texttt{Re[a] > 0 \&\& Re[s] > 0}\right]$$

$$a^{-s}\,\texttt{Gamma[s]}$$

■ Ramanujan's Interpolation Formula

Ramanujan's Interpolation Formula [5] says that if

$$F(x) = \sum_{k=0}^{\infty} \phi(k)(-x)^k,$$

then

$$\int_0^\infty x^{s-1}F(x)dx = \frac{\pi}{\sin(s\pi)}\phi(-s).$$

For example, let us choose $\phi(k) = a^k$ where $a > 0$. Then

$$\mathbf{F} = \sum_{k=0}^{\infty} a^k\, (-x)^k$$

$$\frac{1}{1+a\,x}$$

From the theorem we then know that

$$\int_0^\infty x^{s-1}\frac{1}{1+ax}dx = \frac{\pi}{\sin(s\pi)}a^{-s}.$$

We can check this integral:

$$\int_0^\infty \frac{x^{s-1}}{1+ax}\,dx$$

$$\texttt{If}\Big[0 < \texttt{Re[s]} < 1 \,\&\&\, s\,\texttt{Arg[a]} \le \pi \,\&\&\, (\texttt{Re[a]} \ge 0 \,||\, \texttt{Im[a]} \ne 0),$$

$$a^{-s}\pi\,\texttt{Csc}[\pi s]\texttt{, Integrate}\left[\frac{x^{-1+s}}{1+a\,x}\texttt{, \{x, 0, }\infty\texttt{\}},\right.$$

$$\left.\texttt{Assumptions} \to\, !(0 < \texttt{Re[s]} < 1 \,\&\&\, s\,\texttt{Arg[a]} \le \pi \,\&\&\, (\texttt{Re[a]} \ge 0 \,||\, \texttt{Im[a]} \ne 0))\right]\Big]$$

The assumptions are satisfied, for example, when $a > 0$ and $0 < s < 1$:

$$\texttt{Assuming}\left[\texttt{a > 0 \&\& 0 < s < 1, } \int_0^\infty \frac{x^{s-1}}{1+a\,x}\,dx\right]$$

$$a^{-s}\,\pi\,\texttt{Csc}[\pi\, s]$$

2. Results Related to Sums

■ A Formula for π

Ramanujan presented several formulas for π. One of them [12] states that if

$$\texttt{a[n\_]} := \frac{(-1)^n (1123 + 21460n)\,(2n-1)!!\,(4n-1)!!}{882^{2n+1}\, 32^n\, (n!)^3}$$

then $\sum_{n=0}^{\infty} a_n = \frac{4}{\pi}$. Let us see what value *Mathematica* gives for this sum:

$$\sum_{n=0}^{\infty} \texttt{a[n]}$$

$$\frac{1}{882}\left(1123\ \text{HypergeometricPFQ}\left[\left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\}, \{1,1\}, -\frac{1}{777924}\right] - \frac{5365\ \text{HypergeometricPFQ}\left[\left\{\frac{5}{4}, \frac{3}{2}, \frac{7}{4}\right\}, \{2,2\}, -\frac{1}{777924}\right]}{2074464}\right)$$

We did not get the value of $\frac{4}{\pi}$. However, we can verify that the result is in fact $\frac{4}{\pi}$ by calculating high-precision values for both the value of the sum and for $\frac{4}{\pi}$ and then calculating their difference:

$$\texttt{N[\%, 50] - N}\left[\frac{4}{\pi}, 50\right]$$

$$0. \times 10^{-50}$$

The difference is zero to a very high degree of accuracy, thus verifying that the expressions are the same.

To see how fast the sum converges to $\frac{4}{\pi}$, we calculate the value of the sum when the upper bound is 0, 1, 2, and 3, respectively:

$$\texttt{Table}\left[\sum_{n=0}^{m} \texttt{a[n]}, \{m, 0, 3\}\right]$$

$$\left\{\frac{1123}{882}, \frac{9318469705}{7318708992}, \frac{29456502762912445}{23135083170103296}, \frac{65994970911367394308705 15}{518323289472559769400115 2}\right\}$$

We then calculate the differences:

$$\texttt{N[\%,50] - N}\left[\frac{4}{\pi}, 50\right] \texttt{// N}$$

$$\{3.08565 \times 10^{-6}, -3.17288 \times 10^{-12}, 3.4753 \times 10^{-18}, -3.95304 \times 10^{-24}\}$$

Even the first term is quite a good approximation, and as we add more terms, the series converges very rapidly.

■ Ramanujan's Hypergeometric Identity

Ramanujan's Hypergeometric Identity [16] states that

$$\sum_{n=0}^{\infty} (-1)^n \prod_{k=1}^{n} \left(\frac{2k-1}{2k}\right)^3 = {}_3F_2\left(\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}; \{1, 1\}; -1\right) = {}_2F_1\left(\frac{1}{4}, \frac{1}{4}; 1; -1\right)^2 = \frac{\Gamma\left(\frac{9}{8}\right)^2}{\Gamma\left(\frac{5}{4}\right)^2 \Gamma\left(\frac{7}{8}\right)^2}.$$

To check this formula, we first calculate the sum:

$$\texttt{s} = \sum_{n=0}^{\infty} (-1)^n \prod_{k=1}^{n} \left(\frac{2k-1}{2k}\right)^3$$

$$\frac{\pi}{\sqrt{2}\ \text{Gamma}\left[\frac{5}{8}\right]^2\ \text{Gamma}\left[\frac{7}{8}\right]^2}$$

This looks similar to the expression on the far right in the equation above. Actually, these two expressions are the same:

$$\texttt{s} == \frac{\text{Gamma}\left[\frac{9}{8}\right]^2}{\text{Gamma}\left[\frac{5}{4}\right]^2\ \text{Gamma}\left[\frac{7}{8}\right]^2}\ \texttt{// FullSimplify}$$

```
True
```

(Note here that equations are written with two equal signs == but *Mathematica* automatically replaces them with a special symbol.)
Let us now calculate the values of the hypergeometric function and the generalized hypergeometric function:

$$\text{Hypergeometric2F1}\left[\frac{1}{4},\frac{1}{4},1,-1\right]^2$$

$$\frac{\pi}{\sqrt{2}\ \text{Gamma}\left[\frac{5}{8}\right]^2\ \text{Gamma}\left[\frac{7}{8}\right]^2}$$

$$\text{HypergeometricPFQ}\left[\left\{\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\},\{1,1\},\ -1\right]$$

$$\frac{\pi}{\sqrt{2}\text{Gamma}\left[\frac{5}{8}\right]^2\ \text{Gamma}\left[\frac{7}{8}\right]^2}$$

These values are the same as the value of the sum. Thus, we have checked Ramanujan's Hypergeometric Identity.

■ The Ramanujan ϕ Function

The one-argument Ramanujan phi function [14] is

$$\phi(a) = 1 + 2\sum_{k=1}^{\infty} \frac{1}{(ak)^3 - ak}.$$

This function can be expressed in terms of the polygamma function:

$$1+2\sum_{k=1}^{\infty} \frac{1}{(ak)^3 - ak}\ \texttt{//Simplify}$$

$$-\frac{-a+2\ \text{EulerGamma} + \text{PolyGamma}\left[0,1+\frac{1}{a}\right] + \text{PolyGamma}\left[0,\frac{-1\ +\ a}{a}\right]}{a}$$

Another representation is in terms of harmonic numbers:

```
phi[a_] = % // FullSimplify
```

$$-\frac{-a+\ \text{HarmonicNumber}\left[-\frac{1}{a}\right] +\ \text{HarmonicNumber}\left[\frac{1}{a}\right]}{a}$$

Some special values of the phi function are as follows:

```
Table[phi[a],{a, 2, 6}]
```

$$\left\{\frac{1}{2}\left(2-\text{HarmonicNumber}\left[-\frac{1}{2}\right]-\text{HarmonicNumber}\left[\frac{1}{2}\right]\right),\right.$$
$$\frac{1}{3}\left(3-\text{HarmonicNumber}\left[-\frac{1}{3}\right]-\text{HarmonicNumber}\left[\frac{1}{3}\right]\right),$$
$$\frac{1}{4}\left(4-\text{HarmonicNumber}\left[-\frac{1}{4}\right]-\text{HarmonicNumber}\left[\frac{1}{4}\right]\right),$$
$$\frac{1}{5}\left(5-\text{HarmonicNumber}\left[-\frac{1}{5}\right]-\text{HarmonicNumber}\left[\frac{1}{5}\right]\right),$$
$$\left.\frac{1}{6}\left(6-\text{HarmonicNumber}\left[-\frac{1}{6}\right]-\text{HarmonicNumber}\left[\frac{1}{6}\right]\right)\right\}$$

```
% // FunctionExpand
```

$$\left\{\text{Log}[4],\frac{1}{3}\left(2\left(-\text{Log}[2]+\frac{\text{Log}[3]}{2}\right)+2\,\text{Log}[6]\right),\right.$$
$$\frac{\text{Log}[8]}{2},\frac{1}{5}\left(2\,\text{Log}[10]+\frac{1}{2}\text{Log}\left[\frac{5}{8}-\frac{\sqrt{5}}{8}\right]-\frac{1}{2}\sqrt{5}\,\text{Log}\left[\frac{5}{8}-\frac{\sqrt{5}}{8}\right]\right.$$
$$\left.\left.+\frac{1}{2}\text{Log}\left[\frac{5}{8}+\frac{\sqrt{5}}{8}\right]+\frac{1}{2}\sqrt{5}\text{Log}\left[\frac{5}{8}+\frac{\sqrt{5}}{8}\right]\right),\frac{1}{6}(4\,\text{Log}[2]+3\,\text{Log}[3])\right\}$$

```
% // FullSimplify
```

$$\left\{\text{Log}[4],\ \text{Log}[3],\frac{\text{Log}[8]}{2},\frac{\text{ArcCoth}\left[\sqrt{5}\right]}{\sqrt{5}}+\frac{\text{Log}[5]}{2},\frac{\text{Log}[432]}{6}\right\}$$

The value of the phi function at $a = 5$ can also be expressed in terms of the golden ratio (note that with [[ ]] we can pick out parts of expressions):

$$\%[[4]]==\left(\frac{1}{5}\sqrt{5}\text{Log}[\text{GoldenRatio}]+\frac{1}{2}\text{Log}[5]\right)\text{// FullSimplify}$$

```
True
```

■ The Ramanujan Theta Function

Let us define

$$\text{F}[\text{a\_},\text{b\_},\text{s\_}]:=\sum_{n=-s}^{s}a^{\frac{n(n+1)}{2}}b^{\frac{n(n-1)}{2}}$$

The Ramanujan theta function $f(a, b)$ is then given by $F(a, b, \infty)$ [15]. The one-argument form of this function is $f(-q) = f(-q, -q^2)$, so let us define

$$\text{F}[\text{-q\_},\text{s\_}]:=\text{F}[-q,\ -q^2,s]$$

The first few terms of $f(-q)$ are then

```
F[-q,8]
```

$1-q-q^2+q^5+q^7-q^{12}-q^{15}+q^{22}+q^{26}-q^{35}-q^{40}+q^{51}+q^{57}-q^{70}-q^{77}+q^{92}+q^{100}$

It can be shown that $f(-q)=\prod_{k=1}^{\infty}(1-q^k)$. If we form the product of the first 100 terms, we get the same coefficients as above:

$$\text{Take}\left[\prod_{k=1}^{100}(1-q^k)\,//\,\text{Expand},\ 17\right]$$

$1-q-q^2+q^5+q^7-q^{12}-q^{15}+q^{22}+q^{26}-q^{35}-q^{40}+q^{51}+q^{57}-q^{70}-q^{77}+q^{92}+q^{100}$

■ The Ramanujan φ Function

Ramanujan's phi function $\varphi(q)$ [15] is defined to be $f(q,q)$, where the latter function is the Ramanujan theta function considered above. Clearly, $\varphi(q)=\sum_{n=-\infty}^{\infty}q^{n^2}$. Further, $\varphi(q)=\vartheta_3(0,q)$, where $\vartheta_3(0,q)$ is the Jacobi theta function. Indeed,

$$\sum_{n=-\infty}^{\infty} q^{n^2}$$

```
EllipticTheta [3,0,q]
```

```
% // TraditionalForm
```

$\vartheta_3(0,q)$

It can be shown that $\varphi(e^{-\pi})=\frac{\pi^{1/4}}{\Gamma(\frac{3}{4})}$. We verify this property:

$$\text{EllipticTheta}\ [3,0,\mathbb{E}^{-\pi}]-\frac{\pi^{\frac{1}{4}}}{\text{Gamma}\left[\frac{3}{4}\right]}\ //\ \text{N}$$

```
0.
```

■ The Ramanujan Mock Theta Function

Let us define

$$\text{f}[\text{q}\_,\ \text{s}\_]\ :=\sum_{n=0}^{s}\frac{q^{n^2}}{\prod_{k=1}^{n}\left(1+q^k\right)^2}$$

Ramanujan's mock theta function $f(q)$ is then the infinite sum $f(q,\infty)$ [9]. Here are the first few terms:

```
f[q,3]
```

$$1+\frac{q}{(1+q)^2}+\frac{q^4}{(1+q)^2(1+q^2)^2}+\frac{q^9}{(1+q)^2(1+q^2)^2(1+q^3)^2}$$

We are interested in the series expansion of the mock theta function. It turns out that the first eleven terms of the series expansion of $f(q,3)$ equal those of $f(q)$:

```
Series[f[q,3],{q,0,10}]
```

$1+q-2q^2+3q^3-3q^4+3q^5-5q^6+7q^7-6q^8+6q^9-10q^{10}+O[q]^{11}$

The coefficients are

```
CoefficientList[%,q]
```

{1,1,-2,3,-3,3,-5,7,-6,6,-10}

We calculate the coefficients of the series expansion of the mock theta function using an alternative method. Let us define

$$\texttt{g[q\_, s\_] := 1 + 4}\sum_{k=1}^{s} \texttt{(-1)}^{k} \frac{q^{\frac{1}{2}k(3k+1)}}{1+q^{k}}$$

Then $g(q, \infty)$ is the generating function of Dragonette's gamma sequence. Let us calculate the first 12 coefficients of this generating function (it suffices to take $s = 2$):

```
gamma = CoefficientList[Series[g[q,2],{q,0,11}],q]
```

{1,0,-4,4,-4,4,-4,8,-4,0,-4,8}

The coefficient $A(n)$ of the nth term of the series expansion of the mock theta function is then $A(n) = \sum_{r=0}^{n-1} P(r)\gamma(n-r)$, where $P(r)$ is the partition function (giving the number of ways of writing the integer r as a sum of positive integers). Thus, the first few coefficients of the series expansion of the mock theta function are:

```
Table[Sum[PartitionsP[r]gamma[[n-r]],{r, 0,n-1}], {n, 11}]
```

{1,1,-2,3,-3,3,-5,7,-6,6,-10}

These coefficients are the same as the ones we obtained earlier.

3. Nested Radicals and Continued Fractions

■ Some Examples of Nested Radicals

Nested radicals appear, for example, in some values of trigonometric functions:

$$\texttt{Table}\left[\texttt{Sin}\left[\frac{\pi}{2^n}\right], \{n,5\}\right] \texttt{ // FunctionExpand}$$

$$\left\{1, \frac{1}{\sqrt{2}}, \frac{\sqrt{2-\sqrt{2}}}{2}, \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}, \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}\right\}$$

The golden ratio can be expressed as a nested radical:

$$\phi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}$$

An approximation can be calculated as follows:

$$\sqrt{\texttt{Nest}\left[\texttt{1+}\sqrt{\texttt{\#}}\texttt{\&,1,10}\right]}$$

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{2}}}}}}}}}}$$

We can compare the decimal values of the golden ratio and the above approximation:

```
{GoldenRatio,%}// N

{1.61803,1.61803}
```

■ A Nested Radical by Ramanujan

Ramanujan discovered the following nested radical [10]:

$$a+n+x=\sqrt{\left((a+n)^2+ax+x\sqrt{(a+n)^2+a(n+x)+(n+x)\sqrt{(a+n)^2+a(2n+x)+(2n+x)\sqrt{\ldots}}}\right)}$$

The following code calculates the left-hand side and an m-term approximation to the right-hand side:

```
Rnr[x_,a_,n_,m_]:=
  {x + n + a, Sqrt[Fold[a (x + #2 n) + (n+a)^2+ (x + #2 n) √#1&,0,Range[m-1,0,-1]]]}
```

For example, a 3-term approximation is as follows:

```
Rnr[x,a,n,3]
```

$$\left\{a+n+x,\sqrt{\left((a+n)^2+ax+x\sqrt{(a+n)^2+a(n+x)+(n+x)\sqrt{(a+n)^2+a(2\ n+x)}}\right)}\right\}$$

Here are some special cases:

```
Rnr[x,0,1,6]
```

$$\left\{1+x,\sqrt{1+x\sqrt{1+(1+x)\sqrt{1+(2+x)\sqrt{1+(3+x)\sqrt{5+x}}}}}\right\}$$

```
Rnr[2,0,1,8]
```

$$\left\{3,\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+6\sqrt{22}}}}}}\right\}$$

From the following decimal values we can see how the nested radicals converge to 3:

```
Table[Rnr[2,0,1,m][[2]] //N,{m,20}]

{1.,1.73205,2.23607,2.55983,2.75505,2.8671,
  2.92917,2.96272,2.98055,2.98992,2.9948,2.99733,2.99863,
  2.9993,2.99964,2.99982,2.99991,2.99995,2.99998,2.99999}
```

■ A Continued Fraction by Ramanujan

Consider the following constant:

```
(cc=E^(2π/5) (⁴√5 √GoldenRatio - GoldenRatio)) // TraditionalForm
```

$$\mathbb{E}^{2\pi/5}\left(\sqrt[4]{5}\sqrt{\phi}-\phi\right)$$

```
cc // N
```

```
0.998136
```

A continued fraction approximation of this constant is

```
cf = ContinuedFraction[cc,6]
```

```
{0,1,535,2,38,10}
```

In other words, the constant is approximated by a value expressed in the form

$$0+\cfrac{1}{1+\cfrac{1}{535+\cfrac{1}{2+\cfrac{1}{38+\cfrac{1}{10}}}}}$$

$$\frac{413401}{414173}$$

```
% // N
```

```
0.998136
```

Mathematica also has a command to construct a number directly from the list representing the continued fraction:

```
FromContinuedFraction[cf]
```

$$\frac{413401}{414173}$$

Ramanujan [13] presented a completely different continued fraction for the constant considered above:

$$\frac{1}{1+}\,\frac{e^{-2\pi}}{1+}\,\frac{e^{-4\pi}}{1+}\,\frac{e^{-6\pi}}{1+}\cdots=e^{2\pi/5}\left(\sqrt[4]{5}\sqrt{\phi}-\phi\right).$$

One way to construct a continued fraction with, say, the first four terms, is the following:

```
c=0; Table[c=E^(-2nπ)/(1+c), {n,4,0,-1}]
```

$$\left\{\mathbb{E}^{-8\pi},\ \frac{\mathbb{E}^{-6\pi}}{1+\mathbb{E}^{-8\pi}},\ \frac{\mathbb{E}^{-4\pi}}{1+\frac{\mathbb{E}^{-6\pi}}{1+e^{-8\pi}}},\ \frac{\mathbb{E}^{-2\pi}}{1+\frac{\mathbb{E}^{-4\pi}}{1+\frac{\mathbb{E}^{-6\pi}}{1+\mathbb{E}^{-8\pi}}}},\ \frac{1}{1+\frac{\mathbb{E}^{-2\pi}}{1+\frac{\mathbb{E}^{-4\pi}}{1+\frac{\mathbb{E}^{-6\pi}}{1+\mathbb{E}^{-8\pi}}}}}\right\}$$

Here, the last term is the continued fraction. Another way is as follows:

```
FoldList[E^(-2 #2 π)/(1+ #1)&,0,Range[4,0,-1]]
```

$$\left\{0, \mathbb{E}^{-8\pi}, \frac{\mathbb{E}^{-6\pi}}{1+\mathbb{E}^{-8\pi}}, \frac{\mathbb{E}^{-4\pi}}{1+\frac{\mathbb{E}^{-6\pi}}{1+\mathbb{E}^{-8\pi}}}, \frac{\mathbb{E}^{-2\pi}}{1+\frac{\mathbb{E}^{-4\pi}}{1+\frac{\mathbb{E}^{-6\pi}}{1+\mathbb{E}^{-8\pi}}}}, \frac{1}{1+\frac{\mathbb{E}^{-2\pi}}{1+\frac{\mathbb{E}^{-4\pi}}{1+\frac{\mathbb{E}^{-6\pi}}{1+\mathbb{E}^{-8\pi}}}}}\right\}$$

Let us construct the first four continued fractions:

```
cf2=Table[Fold[E^(-2 #2 π)/(1+ #1)&,0,Range[n,0,-1]],{n,4}]
```

$$\left\{\frac{1}{1+\mathbb{E}^{-2\pi}}, \frac{1}{1+\frac{\mathbb{E}^{-2\pi}}{1+\mathbb{E}^{-4\pi}}}, \frac{1}{1+\frac{\mathbb{E}^{-2\pi}}{1+\frac{\mathbb{E}^{-4\pi}}{1+\mathbb{E}^{-6\pi}}}}, \frac{1}{1+\frac{\mathbb{E}^{-2\pi}}{1+\frac{\mathbb{E}^{-4\pi}}{1+\frac{\mathbb{E}^{-6\pi}}{1+\mathbb{E}^{-8\pi}}}}}\right\}$$

We can check how accurate these continued fractions are:

```
N[cf2 - cc, 50] // N
```

$\{-6.48813 \times 10^{-9}, 4.22533 \times 10^{-17}, -5.13865 \times 10^{-28}, 1.16704 \times 10^{-41}\}$

The convergence is fast.

■ The Rogers–Ramanujan Continued Fraction

The Rogers–Ramanujan continued fraction [3] is defined by

$$R(q) = \frac{q^{1/5}}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\cdots$$

To calculate an approximation to this function, we define

```
R[q_,n_]:=q^(1/5) Fold[q^#2/(1+ #1)&,0, Range[n,0,-1]]
```

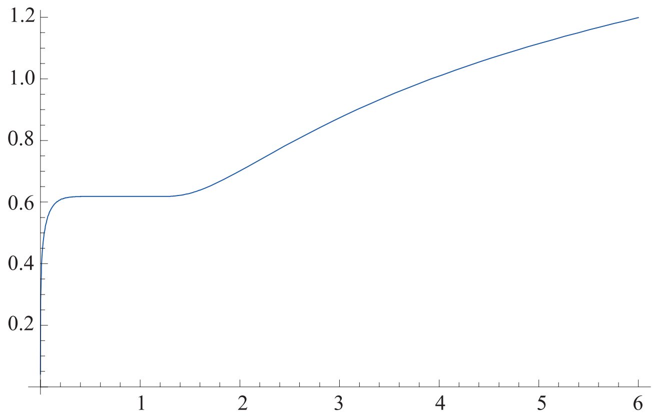
For example,

```
R[q,4]
```

$$\frac{q^{1/5}}{1+\frac{q}{1+\frac{q^2}{1+\frac{q^3}{1+q^4}}}}$$

We take 50 terms and plot the function:

```
Plot[R[q,50], {q,0,6}]
```



It can be shown that $r_n = R(e^{-n\pi})$ is an algebraic number for $n = 0, 1, 2, \ldots$. The values of $r_0 = R(1)$, $r_1 = R(e^{-\pi})$, and $r_2 = R(e^{-2\pi})$ are as follows:

```
r0 = GoldenRatio-1;
```

$$r1=\frac{1}{8}\left(3+\sqrt{5}\right)\left(\sqrt[4]{5}-1\right)\left(\sqrt{10+2\sqrt{5}}-\left(3+\sqrt[4]{5}\right)\left(\sqrt[4]{5}-1\right)\right);$$

$$r2=-GoldenRatio+\sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)};$$

(Note that we used the semicolon ; at the end of these commands. The semicolon suppresses the display of the result. Thus, the results are calculated and stored in the memory but not shown on the screen.) Recall that the value of the golden ratio is

```
RootReduce[GoldenRatio]
```

$$\frac{1}{2}\left(1+\sqrt{5}\right)$$

The decimal values of r_0, r_1, and r_2 are

```
{r0, r1, r2}// N
{0.618034,0.511428,0.284079}
```

Below we check these three values:

$$\{R[1,50]-r0, R[\mathbb{E}^{-\pi},50]-r1, R[\mathbb{E}^{-2\pi},50]-r2\}//\ N$$

$$\{0.,\ 0.,\ 2.22045\times 10^{-16}\}$$

Construct the minimal polynomials for the three values:

```
mp0 = MinimalPolynomial[r0]
```

$$-1+\#1+\#1^2\&$$

```
mp1= MinimalPolynomial[r1]
```

$1-14\#1+22\#1^2-22\#1^3+30\#1^4+22\#1^5+22\#1^6+14\#1^7+\#1^8\&$

```
mp2 = MinimalPolynomial[r2]
```

$1-2\#1-6\#1^2+2\#1^3+\#1^4\&$

The three values are certain zeros of these polynomials:

```
NSolve[mp0[x]==0,x][[2]]
```

{x → 0.618034}

```
NSolve[mp1[x]==0,x][[8]]
```

{x → 0.511428}

```
NSolve[mp2[x]==0,x][[3]]
```

{x → 0.284079}

For another example of an interesting continued fraction due to Ramanujan, see [7].

4. Powers Representations

■ Finding Powers Representations

Let us investigate the mathematical question in the story about Ramanujan at the beginning of the Introduction. Are there numbers n_1 and n_2 such that $1729 = n_1^3 + n_2^3$? *Mathematica* has a special command to solve this kind of problem:

```
?PowersRepresentations
```

PowersRepresentations[n, k, p] gives the distinct representations of the integer n as a sum of k non-negative p^{th} integer powers.≫

Thus, we write the following command:

```
pr = PowersRepresentations[1729,2,3]
```

{{1,12},{9,10}}

The result means that 1729 is equal to 1^3+12^3 and 9^3+10^3 and thus 1729 really does have two representations as sum of two cubes. Later, we demonstrate that 1729 is the smallest integer expressible as the sum of two cubes in two different ways, as Ramanujan asserted. However, first we verify in several ways that the above result is correct. In doing these verifications, we simultaneously learn about some important features of *Mathematica*.

■ Simple Verifications

It is straightforward to verify the powers representations:

$\{1^3+12^3,\ 9^3+10^3\}$

{1729, 1729}

To study *Mathematica* further, let us verify the result using an alternative method. We first calculate the third powers of the numbers that constitute the powers representation:

```
pr^3
{{1,1728},{729,1000}}
```

Note that *Mathematica* does the required calculation for each element of a list automatically, and so one does not need do the calculation for each individual element of the list separately. The above result is a (2×2) matrix with rows (1, 1728) and (729, 1000). With **`Total`** we can add the rows element-wise, that is, calculate the column sums:

```
Total[pr^3]
{730, 2728}
```

Thus, by first transposing the matrix, we get the row sums:

```
Total[(pr^3)^T]
{1729,1729}
```

This again verifies the powers representation. The transposition symbol $^{\mathsf{T}}$ can be written as ESC tr ESC (just press these four keys in turn). We could also use **`Transpose`**:

```
Total[Transpose[pr^3]]
{1729,1729}
```

■ Yet Another Verification

Let us verify the powers representation in yet another way; this allows us to introduce so-called pure functions and the important **`Map`** command. To verify the result, we need the following sums:

```
{Total[{1,1728}], Total[{729,1000}]}
{1729,1729}
```

A simple way to do this calculation is to use the **`Map`** command:

```
Map[Total[#]&, {{1,1728}, {729,1000}}]
{1729,1729}
```

Here, the function **`Total[#]&`** is applied to each element of the second argument of **`Map`**; the elements are, in this example, the rows {1, 1728} and {729, 1000} of the matrix in question. Thus, we get the sums of the rows. The function to be applied, **`Total[#]&`**, is a *pure function*. The argument of such a function is written as **#** and at the end of the function we have the ampersand, **&**.

We can also apply **`Map`** in a shorter way, with the aid of the symbol sequence **`/@`**:

```
Total[#]&/@{{1,1728}, {729,1000}}
{1729,1729}
```

Actually, in a simple pure function such as the one above, we can even omit the argument **#** and the ampersand **&**:

```
Total/@{{1,1728},{729,1000}}
```

```
{1729,1729}
```

Thus, we get the following way to verify the powers representation:

```
Total/@pr^3
```

```
{1729,1729}
```

■ Finding Powers Representations for Several Numbers

Next, we will find the powers representations of the numbers 1725, . . . , 1730 as a sum of two cubes:

```
Table[PowersRepresentations[n,2,3],{n,1725,1730}]
```

```
{{},{},{},{{0,12}},{{1,12},{9,10}},{}}
```

This means that the numbers 1725, 1726, 1727 and 1730 do not have a representation as the sum of two cubes; note that { } is an empty list. The number 1728 has the representation $1728 = 0^3 + 12^3$. The number 1729 has the two representations that we have already studied: $1729 = 1^3 + 12^3$ and $1729 = 9^3 + 10^3$.

To do the same calculation in another way, note that with **Range** we get lists of consecutive integers:

```
Range[10]
```

```
{1,2,3,4,5,6,7,8,9,10}
```

```
Range[1725,1730]
```

```
{1725,1726,1727,1728,1729,1730}
```

Thus, we can also use **Map**:

```
pr = PowersRepresentations[#,2,3]&/@Range[1725,1730]
```

```
{{},{},{},{{0,12}},{{1,12},{9,10}},{}}
```

Let us then select the results that have at least two elements in order to find the numbers that can be represented as a sum of two cubes in at least two ways. Use **Select**:

```
Select[pr, Length[#] ≥ 2&]
```

```
{{{1,12},{9,10}}}
```

The criterion used by **Select** is expressed as a pure function. In our example, we are interested in all of the elements whose length was at least 2. Here are some examples of lengths:

```
{Length[{}],Length[{{0,12}}],Length[{{1,12},{9,10}}]}
```

```
{0,1,2}
```

■ A Demanding Computation

Now, we will find which integers from 1 to 21000 can be represented as the sum of two cubes in at least two ways. First we calculate the powers representations:

```
Timing[pr = PowersRepresentations[#,2,3] &/@ Range[21000];]
```

```
{76.4082, Null}
```

Here, we have used **Timing** to show the number of seconds used in the calculation; the time was about one minute on my not very fast computer. Then we find the representations having a length of at least two:

```
pr2 = Select[pr, Length[#] ≥ 2&]
```

```
{{{1,12},{9,10}},{{2,16},{9,15}},{{2,24},{18,20}},{{10,27},{19,24}}}
```

We found four numbers that can be represented as the sum of two cubes in two ways. Next, let us calculate these numbers.

■ Was Ramanujan Right 1

What are the four numbers mentioned above? A simple way to find them is the following:

$$\{\{1^3+12^3,\ 9^3+10^3\},\ \{2^3+16^3,9^3+15^3\},\{2^3+24^3,18^3+20^3\},\{10^3+27^3,19^3+24^3\}\}$$

```
{{1729,1729},{4104,4104},{13832,13832},{20683,20683}}
```

Thus, 1729, 4104, 13832, and 20683 are the only integers between 1 and 21000 that can be represented as a sum of two cubes in at least two ways. Now we see that the number 1729 is the smallest. Ramanujan was right!

■ Was Ramanujan Right 2

Another way to find the four numbers is the following. First calculate the third powers:

$pr2^3$

```
{{{1,1728},{729,1000}},{{8,4096},{729,3375}},
  {{8,13824},{5832,8000}},{{1000,19683},{6859,13824}}}
```

Then calculate, with **Map**, the transposes of these four matrices:

$Transpose /@ pr2^3$

```
{{{1,729},{1728,1000}},{{8,729},{4096,3375}},
  {{8,5832},{13824,8000}},{{1000,6859},{19683,13824}}}
```

Lastly, we calculate the column sums of these matrices, again using **Map**:

```
Total /@ %
```

```
{{1729,1729},{4104,4104},{13832,13832},{20683,20683}}
```

The above steps can be collected together:

```
Total /@ (Transpose /@ pr2^3)
{{1729,1729},{4104,4104},{13832,13832},{20683,20683}}
```

■ Was Ramanujan Right 3

We can also map the **Total** function at the second *level* of $\mathbf{pr2}^3$:

```
Map[Total, pr2^3,{2}]
{{1729,1729},{4104,4104},{13832,13832},{20683, 20683}}
```

The default level in **Map** is the first level, that is, the level where we have the elements of the list in question. At the first level of $\mathbf{pr2}^3$, we have the four matrices but at the second level we have the rows of the matrices. Mapping **Total** to the rows, we get the row sums.

■ Further Taxicab Numbers

The nth taxicab number Ta(n) is the smallest number representable in n ways as the sum of two positive cubes [18]. We have seen that Ta(2) $= 1729$. The next three taxicab numbers are introduced below, where we only show that the numbers in question can be written as the sum of two cubes in 3, 4, and 5 ways, respectively:

```
PowersRepresentations[87539319,2,3]
{{167,436},{228,423},{255,414}}
PowersRepresentations[6963472309248,2,3]
{{2421,19083},{5436,18948},{10200,18072},{13322,16630}}
PowersRepresentations[48988659276962496,2,3]
{{38787,365757},{107839,362753},{205292,342952},{221424,336588},{231518,331954}}
```

The sixth taxicab number is strongly believed to be the number shown in the next command. However, this remains to be proved.

```
PowersRepresentations[24153319581254312065344,2,3]
{{582162,28906206},{3064173,28894803},{8519281,28657487},
  {16218068,27093208},{17492496,26590452},{18289922,26224366}}
```

5. The Landau–Ramanujan Constant

■ Introduction

Let $S(x)$ be the number of positive integers not exceeding x which can be expressed as the sum of two squares. For example: $1 = 0^2 + 1^2$; $2 = 1^2 + 1^2$; 3 cannot be expressed as the sum of two squares; $4 = 0^2 + 2^2$; $5 = 1^2 + 2^2$; 6 and 7 cannot be expressed as the sum of two squares; $8 = 2^2 + 2^2$. Accordingly, $S(1) = 1$, $S(2) = 2$, $S(3) = 2$, $S(4) = 3$, $S(5) = 4$, $S(6) = 4$, $S(7) = 4$, and $S(8) = 5$.

We can use *Mathematica* to find the number of ways in which integers can be written as the sum of two squares:

```
Table[Length[PowersRepresentations[n,2,2]],{n,25}]
```

```
{1,1,0,1,1,0,0,1,1,1,0,0,1,0,0,1,1,1,0,1,0,0,0,0,2}
```

Taking the sign function gives 1 if such a representation exists and 0 if it does not exist:

```
% // Sign
```

```
{1,1,0,1,1,0,0,1,1,1,0,0,1,0,0,1,1,1,0,1,0,0,0,0,1}
```

Then we can calculate the cumulative sums:

```
% // Accumulate
```

```
{1,2,2,3,4,4,4,5,6,7,7,7,8,8,8,9,10,11,11,12,12,12,12,12,13}
```

These are the first few values of the $S(x)$ function at integer points.

■ A limit

Landau has shown that

$$\lim_{x\to\infty} \frac{\sqrt{\ln(x)}}{x} S(x) = K = 0.764223653\ldots.$$

Here, $S(x)$ is the function introduced above. Ramanujan presented a slightly different result. The constant K is the Landau–Ramanujan constant [8]. Let us see how fast the convergence is. Consider the first 10,000 integers:

```
(ss = Accumulate[
    Sign[Table[Length[PowersRepresentations[n,2,2]],{n,10000}]]];) // Timing
```

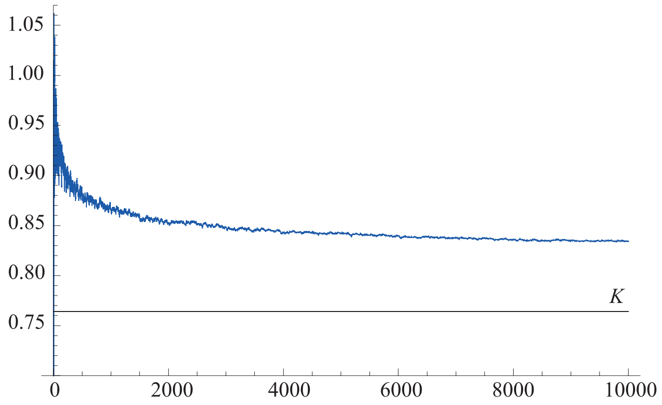
```
{13.7525, Null}
```

We calculate the coefficients:

```
tt = Table[Sqrt[Log[n]]/n, {n, 1., 10000}];
```

Then we plot the values of $\frac{\sqrt{\ln(x)}}{x} S(x)$ at $x = 1, 2, \ldots, 10,000$:

```
ListLinePlot[tt ss,
  Epilog → Line[{{0,0.764224}, {10000,0.764224}}], Text[K,{9800,0.778}]},
  PlotRange → {0.7,1.07}]
```



After 10,000 terms, we are still quite far away from the limit, and so the convergence is very slow.

■ Another Limit

Landau showed that the constant K can also be represented as

$$K = \frac{1}{\sqrt{2}} \prod_{p \text{ prime} \equiv 3 (\text{mod} 4)} \left(1 - \frac{1}{p^2}\right)^{-\frac{1}{2}}.$$

Let us try this formula, considering only the first 2000 primes:

```
pp = Select[Prime[Range[2000]], Mod[#,4]== 3&];
```

This list has about one thousand elements:

```
pp // Length
```

1013

We then calculate an approximation to K:

$$\frac{1}{\sqrt{2}} \text{Product}\left[\left(1.-\frac{1}{p^2}\right)^{-\frac{1}{2}}, \{p, pp\}\right] \text{// InputForm}$$

```
0.7642226322911497
```

This result is correct to five decimal places, and thus this second formula gives a rapidly converging sequence for K.

■ Yet Another Limit

Yet another representation is given below:

$$K = \frac{1}{\sqrt{2}} \prod_{n=1}^{\infty} \left[\left(1 - \frac{1}{2^{2^n}}\right) \frac{\zeta(2^n)}{\beta(2^n)}\right]^{\frac{1}{2^{n+1}}}.$$

Here, $\zeta(x)$ is the Riemann zeta function and $\beta(x)$ is the Dirichlet beta function:

$$\beta[x\_] = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^x}$$

$$4^{-x}\left(\text{Zeta}\left[x, \frac{1}{4}\right] - \text{Zeta}\left[x, \frac{3}{4}\right]\right)$$

Only three terms of the product are required to give an accuracy of eight decimal places:

$$\frac{1.}{\sqrt{2}} \prod_{n=1}^{3} \left(\left(1 - \frac{1}{2^{2^n}}\right) \frac{\text{Zeta}[2^n]}{\beta[2^n]}\right)^{\frac{1}{2^{n+1}}} \text{ // InputForm}$$

```
0.7642236524796345
```

6. The Diophantine Equation—Fourth Powers

■ A Theorem by Ramanujan

Ramanujan derived several theorems that provide infinite families of solutions for sums of equal powers [2]. One of these theorems is as follows. Let $a_1, \ldots, a_6$ be

$$a[1] = 8s^2 + 40s\,t - 24t^2;$$

$$a[2] = 6s^2 - 44s\,t - 18t^2;$$

$$a[3] = 14s^2 - 4s\,t - 42t^2;$$

$$a[4] = 9s^2 + 27t^2;$$

$$a[5] = 4s^2 + 12t^2;$$

$$a[6] = 15s^2 + 45t^2;$$

Then $\sum_{i=1}^{5} a_i{}^4 = a_6{}^4$. We can check that this result holds true:

$$\sum_{i=1}^{5} a[i]^4 == a[6]^4 \text{ // Expand}$$

```
True
```

■ Searching Solutions

To get examples of positive integers a_i that satisfy $\sum_{i=1}^{5} a_i^4 = a_6^4$, we can calculate the values of $a_1, \ldots, a_6$ defined above for several integer values of s and t and then pick the values of s and t that give positive values for the a_i.

First, we collect the a_i expressions into one list:

```
aa = Array[a,6]
```

$$\{8s^2+40s\,t-24t^2,\ 6s^2-44s\,t-18\,t^2,\ 14s^2-4s\,t-42\,t^2,\ 9s^2+27t^2,\ 4s^2+12\,t^2, 15\,s^2+45t^2\}$$

As an example, we try the values $s = 0, 1, 2$ and $t = 0, 1, 2$:

```
(aas = Flatten[Table[{s,t,aa}, {s, 0, 2}, {t,0,2}], 1]) // Column
{0,0,{0,0,0,0,0,0}}
{0,1,{-24,-18,-42,27,12,45}}
{0,2,{-96,-72,-168,108,48,180}}
{1,0,{8,6,14,9,4,15}}
{1,1,{24,-56,-32,36,16,60}}
{1,2,{-8,-154,-162,117,52,195}}
{2,0,{32,24,56,36,16,60}}
{2,1,{88,-82,6,63,28,105}}
{2,2,{96,-224,-128,144,64,240}}
```

The a_i are positive for $(s, t) = (1, 0)$ and $(s, t) = (2, 0)$. Thus, $8^4+6^4+14^4+9^4+4^4 = 15^4$ and $32^4+24^4+56^4+36^4+16^4 = 60^4$.

■ Searching Positive Solutions

Note that we can test whether given numbers in a list are positive using the **Positive** command:

```
pos = Positive[{-24,-18,-42,27,12,45}]
{False, False, False, True, True, True}
```

We can then apply the logical AND operation using **Apply**. Write either

```
Apply [And, pos]
False
```

or

```
And @@ pos
False
```

On the other hand, if all the elements are positive, then the result is True:

```
Positive[{8,6,14,9,4,15}]
{True, True, True, True, True, True}
And @@%
True
```

So, we can continue as follows to select the lists having positive elements:

```
aas = Flatten[Table[aa,{s,0,2},{t,0,2}],1]
```

```
{{0,0,0,0,0,0},{-24,-18,-42,27,12,45},{-96,-72,-168,108,48,180},
  {8,6,14,9,4,15},{24,-56,-32,36,16,60},{-8,-154,-162,117,52,195},
  {32,24,56,36,16,60},{88,-82,6,63,28,105},{96,-224,-128,144,64,240}}
```

```
sol = Select[aas, And @@ Positive[#]&]
```

```
{{8,6,14,9,4,15},{32,24,56,36,16,60}}
```

To show the representations in the form of $\sum_{i=1}^{5} a_i{}^4 = a_6{}^4$, we proceed as follows:

```
Map[Superscript[#, 4]&, sol, {2}]
```

$\{\{8^4, 6^4, 14^4, 9^4, 4^4, 15^4\}, \{32^4, 24^4, 56^4, 36^4, 16^4, 60^4\}\}$

```
Map[Total[Most [#]]==Last[#]&, %]
```

$\{4^4+6^4+8^4+9^4+14^4==15^4,\ 16^4+24^4+32^4+36^4+56^4==60^4\}$

■ More Solutions

Next we do a somewhat more extensive search:

```
sol = Select[Flatten[Table[aa, {s,0,10}, {t,0,10}], 1], And @@ Positive[#]&];
Map[Total[Most[#]] == Last[#]&, Map[Superscript[#, 4]&, sol, {2}]] // Column
```

$4^4+6^4+8^4+9^4+14^4==15^4$

$16^4+24^4+32^4+36^4+56^4==60^4$

$36^4+54^4+72^4+81^4+126^4==135^4$

$64^4+96^4+128^4+144^4+224^4==240^4$

$100^4+150^4+200^4+225^4+350^4==375^4$

$144^4+216^4+288^4+324^4+504^4==540^4$

$196^4+294^4+392^4+441^4+686^4==735^4$

$256^4+384^4+512^4+576^4+896^4==960^4$

$14^4+268^4+603^4+808^4+822^4==1005^4$

$324^4+486^4+648^4+729^4+1134^4==1215^4$

$72^4+336^4+756^4+984^4+1056^4==1260^4$

$400^4+600^4+800^4+900^4+1400^4==1500^4$

$142^4+412^4+927^4+1176^4+1318^4==1545^4$

References

[1] G. H. Hardy, *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work*, AMS Chelsea Publishing, 1999, p. 12.

[2] T. Piezas III and E. W. Weisstein, Diophantine Equation—4th Powers, from *MathWorld*—A Wolfram Web Resource, http://mathworld.wolfram.com/DiophantineEquation4thPowers.html.

[3] T. Piezas III and E. W. Weisstein, Rogers–Ramanujan Continued Fraction, from *MathWorld*—A Wolfram Web Resource, http://mathworld.wolfram.com/Rogers-RamanujanContinuedFraction.html.

[4] H. Ruskeepää, *Mathematica Navigator: Mathematics, Statistics, and Graphics,* Third Edition, Elsevier Academic Press, 2009, to appear. Also to appear in Elsevier Indian Edition.

[5] J. Sondow and E. W. Weisstein, Ramanujan's Interpolation Formula, from *MathWorld*—A Wolfram Web Resource, http://mathworld.wolfram.com/RamanujansInterpolationFormula.html.

[6] J. Sondow and E. W. Weisstein, Ramanujan's Master Theorem, from *MathWorld*—A Wolfram Web Resource, http://mathworld.wolfram.com/RamanujansMasterTheorem.html.

[7] J. L. Varona, A Remarkable Formula of Ramanujan, from The Wolfram Demonstrations Project, http://demonstrations.wolfram.com/ARemarkableFormulaOfRamanujan/.

[8] E. W. Weisstein, Landau-Ramanujan Constant, from *MathWorld*—A Wolfram Web Resource, http://mathworld.wolfram.com/Landau-RamanujanConstant.html.

[9] E. W. Weisstein, Mock Theta Function, from *MathWorld*—A Wolfram Web Resource, http://mathworld.wolfram.com/MockThetaFunction.html.

[10] E. W. Weisstein, Nested Radical, from *MathWorld*—A Wolfram Web Resource, http://mathworld.wolfram.com/NestedRadical.html.

[11] E. W. Weisstein, Nielsen–Ramanujan Constants, from *MathWorld*—A Wolfram Web Resource, http://mathworld.wolfram.com/Nielsen-RamanujanConstants.html.

[12] E. W. Weisstein, Pi Formulas, from *MathWorld*—A Wolfram Web Resource, http://mathworld.wolfram.com/PiFormulas.html.

[13] E. W. Weisstein, Ramanujan Continued Fractions, from *MathWorld*—A Wolfram Web Resource, http://mathworld.wolfram.com/RamanujanContinuedFractions.html.

[14] E. W. Weisstein, Ramanujan phi-Function, from *MathWorld*—A Wolfram Web Resource, http://mathworld.wolfram.com/RamanujanPhi-Function.html.

[15] E. W. Weisstein, Ramanujan Theta Functions, from *MathWorld*—A Wolfram Web Resource, http://mathworld.wolfram.com/RamanujanThetaFunctions.html.

[16] E. W. Weisstein, Ramanujan's Hypergeometric Identity, from *MathWorld*—A Wolfram Web Resource, http://mathworld.wolfram.com/RamanujansHypergeometricIdentity.html.

[17] E. W. Weisstein, Soldner's Constant, from *MathWorld*—A Wolfram Web Resource, http://mathworld.wolfram.com/SoldnersConstant.html.

[18] E. W. Weisstein, Taxicab Number, from *MathWorld*—A Wolfram Web Resource, http://mathworld.wolfram.com/TaxicabNumber.html.

A ‘pearl’ of Number Theory: the theorem of van der Waerden

Sukumar Das Adhikari
Harish-Chandra Research Institute
Chhatnag Road, Jhusi, Allahabad 211 019, India
e-mail: adhikari@mri.ernet.in

1. Introduction

The theorem of van der Waerden, about which we are going to discuss here, is one among the ‘pearls’ that Khinchin presented [17] in his ‘Three pearls of Number Theory’. As we shall see in this small expository article, this result has led to many interesting developments in Combinatorics and Number Theory.

Let $\mathbf{Z}^+$, be the set of positive integers.

If one considers the partition $\mathbf{Z}^+ = X \cup Y$, where

$$X = \left\{n \in \mathbf{Z}^+ : n \in [2^r, 2^{r+1}) \text{ for some even integer } r\right\}$$

and

$$Y = \left\{n \in \mathbf{Z}^+ : n \in [2^r, 2^{r+1}) \text{ for some odd integer } r\right\},$$

then it is clear that neither X nor Y will contain an infinite arithmetic progression.

However, we have the following [29]:

Theorem. **(van der Waerden).** *Given positive integers k and r, there exists a positive integer $W(k, r)$ such that for any r-colouring of $\{1, 2, \dots W(k, r)\}$, there is a monochromatic arithmetic progression (A.P.) of k terms.*

Here an *r-colouring* of a set S is a map $\chi : S \to \{c_1, \dots, c_r\}$. Writing $S = \chi^{-1}(c_1) \cup \chi^{-1}(c_2) \cup \dots \cup \chi^{-1}(c_r)$, an r-colouring of a set S is nothing but a partition of S into r parts where elements belonging to the same part receive the same colour. A subset A of S is called *monochromatic* if $A \subset \chi^{-1}(c_i)$ for some $i \in \{1, 2, \dots, r\}$. Thus given an r-colouring $\chi : \mathbf{Z}^+ \to \{c_1, \dots, c_r\}$, an A.P. $a, a + b, \dots, a + kb$ will be monochromatic if $\chi(a) = \chi(a + b) = \cdots = \chi(a + kb)$.

The above theorem implies that for any finite partition $\mathbf{Z}^+ = X_1 \cup X_2 \cup \cdots \cup X_r$ of $\mathbf{Z}^+$, at least one X_i will contain arithmetic progression of any given length.

We should remark that van der Waerden’s Theorem is a Ramsey-type theorem. Ramsey Theory can be characterized as the subject dealing with results that talk about the phenomena of ‘large’ substructures of certain structures retaining certain regularities. Most often, we come across results saying that if a large structure is divided into finitely many parts, at least one of the parts will retain certain regularity properties of the original structure.

To our readers we recommend van der Waerden’s personal account [30] of finding its proof. It contains the formulation of the problem with the valuable suggestions due to Emil Artin and Otto Schreier and depicts how the sequence of basic ideas occurred as an elaboration of the psychology of invention. It should also be mentioned that the result was originally (see [12] for instance) conjectured by Schur; since van der Waerden came to know it through Baudet, he calls it Baudet’s conjecture.

At this point, it will be appropriate to mention one of the early Ramsey-type results due to Schur [24]:

Theorem. **(Schur).** *For any r-colouring of $\mathbf{Z}^+$, $\exists$ a monochromatic subset $\{x, y, z\}$ of $\mathbf{Z}^+$ such that $x + y = z$. (The situation is described by saying that the equation $x + y = z$ has a monochromatic solution.)*

One observes that a three term arithmetic progression $x < y < z$ is a solution of the equation $x + z = 2y$ and by van der Waerden’s theorem, for any finite colouring of $\mathbf{Z}^+$, this equation has a monochromatic solution. We remark that, in this direction, successful investigations of Rado ([20], [21], [22]) provided necessary and sufficient conditions for a system of homogeneous linear equations over $\mathbf{Z}$ to possess monochromatic solutions for finite colouring of $\mathbf{Z}^+$. One may look into [12] for Rado’s theorem and some related results.

The following result [2] (see also [18], [1]) generalizes van der Waerden’s Theorem to higher dimensions.

Theorem. **(Grünwald).** *Let $d, r \in \mathbf{Z}^+$, the set of positive integers. Then given any finite set $S \subset (\mathbf{Z}^+)^d$, and an r-colouring of $(\mathbf{Z}^+)^d$, there exists a positive integer 'a' and a point 'v' in $(\mathbf{Z}^+)^d$ such that the set $aS + v$ is monochromatic.*

Remark. We note that when $d = 1$, one derives van der Waerden's Theorem by taking $S = \{1, \ldots , k\}$, in the above theorem.

In the next section we give an account of further developments along this theme. In the final section, we shall give a proof of Grünwald's Theorem.

2. Further Developments

First, we discuss the theorem of Hales and Jewett [15] which reveals the combinatorial nature of van der Waerden's theorem. As has been said in [12]:

"the Hales-Jewett theorem strips van der Waerden's theorem of its unessential elements and reveals the heart of Ramsey theory".

So, this 'pearl of number theory' belongs to the ancient shore of Combinatorics! Indeed, van der Waerden's theorem was a prelude to a very important theme where interplay of several areas of Mathematics would be seen. Development of this theme, saw the results of Roth and Szemerédi and a number of different proofs of these results including the ergodic proof of Szemeredi's theorem due to Furstenberg. And, in the recent years, we have the results of Gowers and the Green-Tao Theorem.

We need some definitions before we can state Hales–Jewett Theorem.

Write

$$C_t^n = \{x_1x_2 \ldots x_n : x_i \in \{1, 2, \ldots t\}\}.$$

In other words, C_t^n is the collection of words of length n over the alphabet of t- symbols $1, 2, \ldots t$.

Then, by a *combinatorial line* in C_t^n we mean a set of t points in C_t^n ordered as $X_1, X_2, \ldots , X_t$ where $X_i = x_{i1}x_{i2} \ldots x_{in}$ such that for j belonging to a nonempty subset I of $\{1, \ldots n\}$ we have $x_{sj} = s$ for $1 \le s \le t$ and $x_{1j} = \cdots = x_{tj} = c_j$ for some $c_j \in \{1, \ldots t\}$ for j belonging to the complement (possibly empty) of I in $\{1, \ldots n\}$.

For example, for $t = 3$ and $n = 5$, the following is a combinatorial line in C_3^5:

11122
21222
31322

Theorem. **(Hales-Jewett).** *Given any two positive integers r and t, there exists a positive integer $n = HJ(r, t)$ such that if C_t^n is r-coloured then there exists a monochromatic combinatorial line.*

Observing the above example of the combinatorial line in C_3^5, and identifying the collection of words in C_3^5 with the set of integers in their usual expression in decimal system, it is easy to see that the above combinatorial line corresponds to a three term arithmetic progression with common difference 10100.

Thus, if we consider an r-colouring of C_t^n (with suitably large t), induced from a given r-colouring of the integers which have base d representation with $d \ge t$, Hales-Jewett Theorem would imply the existence of a monochromatic arithmetic progression of t terms.

It was felt that, in the case of van der Waerden's theorem, while considering finite partition of $\mathbf{Z}^+$, only the 'size' of the part matters. Indeed, Erdős and Turan [6] conjectured that any subset of $\mathbf{Z}^+$ with positive upper natural density contains arithmetic progressions of arbitrary length, where, for $A \in \mathbf{Z}^+$, *the upper natural density*, $\bar{d}(A)$ of A is defined by

$$\bar{d}(A) = \limsup_{N\to\infty} \frac{|A \cap [N]|}{N},$$

where N denotes the set $\{1, 2, \ldots , N\}$.

We remark that in connection with Schur's theorem, the situation is quite different; though the set of even integers and the set of odd integers have the same upper natural density 1/2 in $\mathbf{Z}^+$, there is no solution of $x + y = z$ in the subset of odd positive integers.

Towards the above mentioned conjecture of Erdős and Turan, in 1953 Roth [23] proved that any subset A of the set $\mathbf{Z}^+$ of positive integers with positive upper natural density will always contain a three-term arithmetic progression. Later, Szemerédi first improved [25] Roth's result to that of A possessing a four-term arithmetic progression and finally in 1974, in a famous paper [26] proved the general Erdős-Turan conjecture by a sophisticated combinatorial argument. Later, Furstenberg [7] gave an ergodic theoretic proof of Szemerédi's theorem which opened up the subject of Ergodic Ramsey Theory (see [8], [3] and [19]). There have been other important proofs of Szemerédi's theorem since then.

The area of Ergodic Ramsey Theory has many important developments which include the density version of the Hales-Jewett theorem by Furstenberg and Katznelson [9] and the polynomial extension of Hales-Jewett theorem by Bergelson and Leibman [4]. An expository account of the Bergelson-Leibman result is available in [3].

Defining $r_k(n)$ to be the smallest integer such that whenever $A \subset [n]$ satisfies $|A| > r_k(n)$, A contains an arithmetic progression of k terms, Szemerédi's result [26] implies that

$$r_k(n) = o(n).$$

For the case $k = 3$, Roth's proof [23] gave $r_3(n) = O\left(\frac{n}{\log\log n}\right)$; successive improvements in this direction were obtained by Heath–Brown [16], Szemerédi [27] and Bourgain [5], the result of Bourgain being

$$r_3(n) = O\left(n \cdot \sqrt{\frac{\log\log n}{\log n}}\right).$$

Regarding estimates of $r_k(n)$ for $k > 3$, Gowers [10] has made a remarkable breakthrough while establishing

$$r_4(n) < \frac{n}{(\log\log n)^d} \text{ for some absolute constant } d > 0,$$

where the method seems to go through for $r_k(n)$ for $k \geq 4$.

Apart from the original paper [10], we recommend the two beautiful articles [11] and [5] for getting an idea as well as the background of the proof.

The following conjecture of Erdős is still open.

Conjecture. **(Erdős).** If $A \subset \mathbf{Z}^+$ satisfies

$$\sum_{a \in A} \frac{1}{a} = \infty,$$

then A contains arithmetic progressions of arbitrary length.

Recently, in the Green-Tao Theorem [14] we have seen a major breakthrough. Green and Tao showed that the set of primes contains arithmetic progressions of arbitrary length.

We refer to [28] and [13] for an accessible account of the recent developments in the area of Additive Combinatorics; the reader will find the extensive bibliography provided in those volumes very helpful.

3. Proof of Grünwald's Theorem

We work with fixed d. Now, the theorem will be proved if we prove the following statement for all finite sets $S \subset (\mathbf{Z}^+)^d$.

$A(S)$: For each $k \in \mathbf{Z}^+$, $\exists\ n = n(k)$ such that for every k-colouring of $B_n \stackrel{def}{=} \{(a_1, \ldots, a_d) : a_i \in \mathbf{Z}^+, 1 \leq a_i \leq n\}$, B_n contains a monochromatic subset of the form $aS + v$ for some $a \in \mathbf{Z}^+$ and $v \in B_n$.

We remark that the above statement not only proves the theorem, given the number of colours used and the given set S, it tells us (as in our statement of van der Waerden's theorem) about the size of the finite cube where the monochromatic subset of the form $aS + v$ can be found. In fact, by the 'Compactness Principle' (see [12], for instance), the above statement is equivalent to the statement in the theorem; we shall not go into this.

Since $A(S)$ is obviously true if $|S| = 1$, it is enough to show that $A(S) \Rightarrow A(S \cup \{s\})$ for any $s \in (\mathbf{Z}^+)^d$.

For the induction procedure, once $A(S)$ is established for a given S, we prove the the following intermediate statement $C(p)$ corresponding to a positive integer p. Once $C(p)$ is established for any positive integer p, it will lead to the statement $A(S \cup \{s\})$ and we shall be through.

$C(p)$: Let $S \subset (\mathbf{Z}^+)^d$ be fixed for which $A(S)$ is true. Then for given $k \in \mathbf{Z}^+$ and $s \in (\mathbf{Z}^+)^d$, $\exists n = n(p, k, s) \in \mathbf{Z}^+$ such that for each k-colouring of B_n, there are positive integers $a_0, a_1, \ldots, a_p$ and a point $u \in (\mathbf{Z}^+)^d$ such that the each of the $(p + 1)$ sets

$$T_q \stackrel{\text{def}}{=} u + \sum_{0 \leq i < q} a_i S + \left(\sum_{q \leq i \leq p} a_i\right) s, \quad 0 \leq q \leq p,$$

are monochromatic subsets of B_n.

$C(0)$ holds trivially and we have to show that $C(p) \Rightarrow C(p + 1)$.

Let $n = n(p, k, s)$ be the integer specified for $C(p)$. Now, given a k-colouring of $(\mathbf{Z}^+)^d$, we define the *associated colouring* of $(\mathbf{Z}^+)^d$ such that two points u and v will have the same colour in this new colouring iff the lattice points in the cubes $u + B_n$ and $v + B_n$ are identically coloured in the original k-colouring of $(\mathbf{Z}^+)^d$.

Clearly, this associated colouring of $(\mathbf{Z}^+)^d$ is a k'- colouring where $k' \stackrel{\text{def}}{=} k^{n^d}$.

Now, from $A(S)$, it follows that $\exists$ an integer $n' = n'(k')$ such that for every k'-colouring of $B_{n'}$, $B_{n'}$ contains a monochromatic subset of the form $a'S + v'$ for some $a'(\neq 0) \in \mathbf{Z}^+$ and $v' \in B_{n'}$.

Let $N = n + n' + 1$. Let a k-colouring of B_N be given. In an arbitrary way we extend this to a k-colouring of $(\mathbf{Z}^+)^d$. Now, corresponding to the associated k'- colouring of $(\mathbf{Z}^+)^d$, $B_{n'}$ contains a monochromatic subset of the form $a'S + v'$. This means that the $|S|$-cubes $B_n + a't + v'$, $t \in S$ are coloured identically in the original k- colouring. We observe that all these cubes lie in B_N. By $C(p)$, for any $t \in S$, the cube $B_n + a't + v'$ contains monochromatic sets

$$T_q(t) = a't + v' + u + \sum_{0\le i<q} a_i S + \left(\sum_{q\le i\le p} a_i\right) s, \;\; 0 \le q \le p.$$

Setting $b_0 = a'$ and $b_i = a_{i-1}$, $1 \le i \le p+1$, we claim that the sets

$$T'_q = (v' + u) + \sum_{0\le i<q} b_i S + \left(\sum_{q\le i\le p+1} b_i\right) s, \;\; 0 \le q \le p+1$$

are monochromatic.

For $q = 0$,

$$T'_0 {=} (v' + u) + \left(\sum_{0\le i\le p+1} b_i\right) s$$

is a singleton and the claim is established.

For $q \ge 1$, $T'_q = \cup_{t\in S} T_{q-1}(t)$. Since $B_n + a't + v'$ are identically coloured for different t's belonging to S, it follows that the monochromatic sets $T_{q-1}(t)$ are of the same colour and hence $T'_q = \cup_{t\in S} T_{q-1}(t)$ is a monochromatic subset of B_N.

Thus $C(p+1)$ holds with $n(p+1, k, s) = N$.

Now that $C(p)$ is established for all integers $p \ge 0$, the particular case $p = k$ gives us an integer $n = n(k, k, s)$ such that given any k-colouring of B_n, $\exists (k+1)$ monochromatic sets $T_0, \dots, T_k$ in B_n. By pigeonhole principal, two of these sets, say T_r, T_q with $r < q$ are of the same colour.

Writing

$$T_r = u + \sum_{0\le i<r} a_i S + \left(\sum_{r\le i<q} a_i\right) s + \left(\sum_{q\le i<k+1} a_i\right) s$$

and

$$T_q = u + \sum_{0\le i<r} a_i S + \sum_{r\le i<q} a_i S + \left(\sum_{q\le i<k+1} a_i\right) s,$$

and choosing $s_0 \in S$ (S being nonempty), it follows that the set

$$T = u + \left(\sum_{0\le i<r} a_i\right) s_0 + \left(\sum_{r\le i<q} a_i\right) (S \cup \{s\}) + \left(\sum_{q\le i<k+1} a_i\right) s$$

is contained in B_N and is monochromatic.

Setting $a = \sum_{r\le i<q} a_i$ and $v = u + \left(\sum_{0\le i<r} a_i\right) s_0 + \left(\sum_{q\le i<k+1} a_i\right) s$, $T = a(S \cup \{s\}) + v$ and this establishes $A(S \cup \{s\})$.

References

[1] S. D. Adhikari, *Aspects of combinatorics and combinatorial number theory*, Narosa, New Delhi, 2002.

[2] P. G. Anderson, *A generalization of Baudet's conjecture (van der Waerden's Theorem)*, Amer. Math. Monthly, **83**, 359–361 (1976).

[3] V. Bergelson, *Ergodic Ramsey Theory – an Update*, London Mathematical Society Lecture Note Series 228, Cambridge University Press, 1996.

[4] V. Bergelson and A. Leibman, *Set-polynomials and polynomial extension of the Hales-Jewett theorem*, Ann. of Math. (2) **150**, No. 1, 33–75 (1999).

[5] J. Bourgain, *On triples in arithmetic progression*, Geom. Funct. Anal., **9**, No. 5, 968–984 (1999).

[6] P. Erdős and P. Turan, *On some sequences of integers*, J. London Math. Soc. **11**, 261–264 (1936).

[7] H. Furstenberg, *Ergodic behaviour of diagonal measures and a theorem of Szemerédi on arithmetic progressions*, J. d'Analyse Math. **31**, 204–256 (1977).

[8] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, 1983.

[9] H. Furstenberg and Y. Katznelson, *A density version of the Hales-Jewett theorem*, J. Analyse Math., **57**, 64–119 (1991).

[10] W. T. Gowers, *A new proof of Szemerédi's theorem for arithmetic progressions of length four*, GAFA, **8**, 529–551 (1998).

[11] W. T. Gowers, *Fourier analysis and Szemerédi's theorem*, Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998). Doc. Math., Extra Vol. I, 617–629 (1998).

[12] R. L. Graham, B. L. Rothschild and J. H. Spencer, *Ramsey Theory*, John Wiley & Sons, 1980.

[13] A. Granville, M. B. Nathanson and J. Solymosi (Eds.), *Additive Combinatorics*, CRM Proceedings and Lecture Notes, Volume 43, American Mathematical Society, 2007.

[14] B. Green and T. Tao, *The primes contain arbitrarily long arithmetic progressions*, Annals of Math., to appear.

[15] A. W. Hales and R. I. Jewett, *Regularity and positional games*, Trans. Amer. Math. Soc. **106**, 222–229 (1963).

[16] D. R. Heath-Brown, *Integer sets containing no arithmetic progressions*, J. London Math. Soc. (2) **35**, 385–394 (1987).

[17] A. Y. Khinchin, *Three Pearls of Number Theory*, Graylock Press, 1952.

[18] M. Lothaire, *Combinatorics on Wards*, Addison – Wesley, 1983.

[19] R. McCutcheon, *Elemental methods in ergodic Ramsey theory*, Springer, Lect. Notes Math., **Vol. 1722**, 1999.

[20] R. Rado, *Verallgemeinerung eines Satzes von van der Waerden mit Anwendungen auf ein problem der Zalentheorie, Sonderausgabe aus den Sitzungsbericten der Preuss*, Akad. der Wiss. Phys. -Math. klasse **17**, 1–10 (1933).

[21] R. Rado, *Studien zur Kombinatorik*, Math. Z. **36**, 424–480 (1933).

[22] R. Rado, *Some recent results in combinatorial analysis*, Congrès International des Mathematiciens, Oslo, 1936.

[23] K. F. Roth, *On certain sets of integers*, J. London Math. Soc. **28**, 104–109 (1953).

[24] I. Schur, *Über die Kongruenz* $x^m + y^m \equiv z^m \pmod{p}$, Jber. Deutsch. Math. Verein. **25**, 114–117 (1916).

[25] E. Szemerédi, *On sets of integers containing no four elements in arithmetic progression*, Acta. Math. Acad. Sci. Hungar. bf 20, 89–104 (1969).

[26] E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression*, Acta. Arith. bf 27, 199–245 (1975).

[27] E. Szemerédi, *Integer sets containing no arithmetic progressions*, Acta. Math. Hungar.,**56**, 155–158 (1990).

[28] T. Tao and V. H. Vu, *Additive Combinatorics*, Cambridge University Press, 2006.

[29] B. L. van der Waerden, *Beweis einer Baudetschen Vermutang,* Nieuw Arch. Wisk. **15**, 212–216 (1927).

[30] B. L. van der Waerden, *How the proof of Baudet's conjecture was found*, Studies in Pure Mathematics (Edited by L. Mirsky), Academic Press, 251–260 (1971).

Some Topics in Prime Number Theory

T. N. Shorey*
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road, Mumbai 400 005, India
E-mail: shorey@math.tifr.res.in

Dedicated to Professor K. Ramachandra on his 75th birthday

Definition. An integer p is called prime if

(i) $p > 1$
(ii) p has no divisor other than 1 and itself

Examples. 2, 3, 5, 7, 13, 19, 23, $\cdots$ are primes.
We denote by P the set of all primes and we write the elements of P in the increasing order as

$$p_1 < p_2 < p_3 < \cdots$$

Now consider

$$\triangle = p_1 \cdots p_n + 1.$$

Observe that $\triangle$ is an integer > 1. Therefore there exists

$$q \in P \text{ with } q \mid \triangle.$$

Then

$$q \neq p_i \text{ for } 1 \leq i \leq n$$

since $q \nmid \triangle - 1$. Thus

$$q > p_n.$$

We started with primes $p_1, \cdots, p_n$ and we found a prime $q > p_n$. Therefore we have proved

Theorem 1. (Euclid) *P is infinite.*

Let us have a closer look at the proof of Euclid. For this, we introduce the following important function for counting

*This article is based on the text of my talk in VSRP (2005) at School of Mathematics, TIFR, Mumbai.

primes:

$$\pi(x) = \sum_{p \le x} 1,$$

the number of primes $\le x$. Thus

$$\pi(4) = 2, \pi(10) = 4.$$

Let $n > 1$. We have shown that

$$p_{n+1} \le q \le p_1 \cdots p_n + 1.$$

Therefore

$$p_{n+1} \le p_n^n.$$

Thus we have bounded $(n+1)$-th prime in terms of n-th prime. Using this inequality, we show that

$$p_n \le 2^{2^{2^n}}.$$

The proof is by induction on n. It is fine when $n = 1$ and assume for n. Then

$$p_{n+1} \le p_n^n \le (2^{2^{2^n}})^n \le 2^{2^{2^{n+1}}}.$$

Now

$$n = \pi(p_n) \le \pi(2^{2^{2^n}}).$$

Let $x \ge x_0$ where x_0 is sufficiently large absolute constant. Then

$$2^{2^{2^n}} \le x < 2^{2^{2^{n+1}}}$$

for some n. Now

$$\pi(x) \ge \pi(2^{2^{2^n}}) \ge n \ge \log\log\log x.$$

Thus Euclid's proof not only shows that primes are infinitely many in numbers but it also gives a lower bound for $\pi(x)$ and the lower bound tends to infinity with x. We have a definite result here:

Prime Number Theorem.

$$\lim_{x\to\infty} \frac{\pi(x)}{(x/\log\ x)} = 1.$$

Thus for $\epsilon > 0$,

$$\pi(x) \ge (1-\epsilon)\frac{x}{\log\ x} \text{ for } x \ge x_1(\epsilon) \qquad (1)$$

and

$$\pi(x) \le (1+\epsilon)\frac{x}{\log\ x} \text{ for } x \ge x_2(\epsilon). \qquad (2)$$

Let $x = p_N$. Then

$$\lim_{N\to\infty} \frac{\pi(p_N)}{(p_N \mid \log\ p_N)} = 1$$

i.e.,

$$\lim_{N\to\infty} (\frac{N}{p_N \mid \log p_N}) = 1.$$

i.e.,

$$\lim_{N\to\infty} (\frac{p_N}{N \log N}) = 1. \qquad (3)$$

Now we use Euclid's proof to show

Theorem 1′. *There are infinitely many primes of the form* $4n+3$ *with* $n > 0$.

Proof. By contradiction. Suppose that $q_1, \cdots, q_m$ are all the primes $\equiv 3 (\text{mod } 4)$. Consider

$$\triangle' = 4q_1 \cdots q_m - 1.$$

There exists $q \in P, q \mid \triangle'$ and $q \equiv 3$ (mod 4) since $\triangle' \equiv 3$ (mod 4). Then

$$q = q_i \text{ with } 1 \le i \le m$$

implying $q \mid (\triangle' + 1)$ which is a contradiction.

The above proof can not be used to show that there are infinitely many primes of the form $4n + 1$ with $n > 0$. But we have a general result.

Theorem 2. (Dirichlet) *Let* $a > 0$ *and* $b > 0$ *be integers with* $gcd\ (a, b) = 1$. *Then there are infinitely many primes of the form* $an + b$ *with* $n > 0$.

Put

$$f(X) = aX + b.$$

Then

$$f(X) \in \mathbb{Q}[X]$$

satisfying

(i) $f(X)$ is irreducible over $\mathbb{Q}$
(ii) the leading coefficient of f is positive

(iii) Let p be a prime. Then

$$f(X) \not\equiv 0 (\text{mod } p).$$

We have the following conjecture:

Conjecture. **(Schinzel)** *Let $f(X) \in \mathbb{Q}[X]$ satisfying* (i), (ii) *and* (iii). *Then there are infinitely many positive integers m such that $f(m)$ is prime.*

It is easy to see that the assumptions (i), (ii) and (iii) for f are necessary. Further Schinzel formulated a more general conjecture valid for an arbitrary number of polynomials.

A powerful tool for studying primes is the Riemann Zeta function. To begin with, it is defined in the half plane

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, s = \sigma + it, \sigma > 1.$$

Its connection with primes is given by Euler Identity

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, s = \sigma + it, \sigma > 1.$$

$\zeta(s)$ can be continued analytically in the whole plane except at $s = 1$ where it has simple pole. We call the extended function also $\zeta(s)$. Now

$$\zeta(s) = 0 \text{ for } s = -2 - 4, \cdots$$

But these are not all the zeros of $\zeta(s)$. We have

Theorem 3. (Hardy) *There are infinitely many zeros of $\zeta(s)$ on $s = \sigma + it$ with $\sigma = \frac{1}{2}$.*

Further we have the famous conjecture:

Riemann Hypothesis. *Apart from $s = -2, -4, \cdots$, all the zeros of $\zeta(s)$ lie on the line $s = \sigma + it$ with $\sigma = \frac{1}{2}$.*

For $n \geq 1$, let

$$h(N) = p_{N+1} - p_n.$$

Consider the interval

$$(p_N, 2p_N].$$

It is well-known that there is $q \in P$ with $q \in (p_N, 2p_N]$. Thus

$$q \geq p_{N+1}$$

and

$$h(N) = p_{N+1} - p_N \leq q - p_N < p_N.$$

This can be improved by using Prime Number Theorem as follows: Let $0 < \theta < 1$. Then

$$h(N) < \theta N \text{ with } N \geq N_1(\theta).$$

For this we should consider the interval

$$(p_N, p_N + \theta p_N]$$

and show that it has a prime. This is equivalent to showing

$$\pi(p_N + \theta p_N) - \pi(p_N) > 0.$$

We use the lower bound (1) for the first term and upper bound (2) for the second term for deriving the above inequality. There is a

Conjecture. **(Cramer)** *There exists an absolute constant N_2 such that*

$$h(N) \leq (\log p_N)^2 \text{ for } N \geq N_2.$$

This is a very difficult conjecture. Even Riemann Hypothesis implies

$$h(N) \leq P_N^{\frac{1}{2}+\epsilon}, N \geq N_3(\epsilon).$$

It is known due to Baker, Harman and Pintz [1] that

$$h(N) \leq p_N^{\frac{1}{2}+\frac{1}{40}+\epsilon} \quad \text{for } N \geq N_4(\epsilon).$$

Let

$$E(N) = \frac{p_{N+1} - p_N}{\log p_N}$$

and we put

$$E_1^* = \lim sup_{N\to\infty} E(N),$$
$$E_2^* = \lim inf_{N\to\infty} E(N).$$

By Prime Number Theorem

$$E_1^* \geq 1 \text{ and } E_2^* \leq 1.$$

For deriving this, we observe that

$$p_{2M} - p_M = \sum_{i=1}^{M} (p_{M+i} - p_{M+i-1})$$

and use Prime Number Theorem for getting

$$(1-\epsilon)M \log M \leq p_{2M} - p_M \leq (1+\epsilon)M \log M \text{ for } M \geq M_0(\epsilon).$$

Schönhage [9] showed that $E_1^* = \infty$ and we refer to his paper for more precise formulation of his result. There are primes of the type 3,5 or 5,7 or 11,13 These are called twin primes. It has been conjectured that there are infinitely many twin primes. Then

$$E_2^* = 0.$$

Erdős [4] was the first to show that

$$E_2^* < 1$$

and Bombieri, Friedlander, Iwaniec [2] showed that

$$E_2^* \leq \frac{6}{7}.$$

For integers $n > 0$ and $k \geq 2$, we write

$$\triangle_0 = \triangle_0(n,k) = n(n+1)\cdots(n+k-1).$$

Further we denote by $P(\triangle_0)$ and $\omega(\triangle_0)$ the greatest prime factor and the number of distinct prime divisors of $\triangle_0$, respectively. As already stated, there is a prime between X and $2X$. This is a particular case $n = k+1$ of the following result dating back to 1892.

Theorem 4. (Sylvester [12]) *We have*

$$P(\triangle_0) > k \text{ if } n > k.$$

Thus a product of k consecutive positive integers each exceeding k is divisible by a prime greater than k. By applying Theorem 4 with $n = k+1$, we have

$$P(\triangle_0(k+1,k)) > k.$$

Therefore there is an integer between $k+1$ and $2k$ divisible by a prime exceeding k and this integer has to be a prime. The assumption $n > k$ in Theorem 4 is necessary since

$$P(\triangle_0(1,k)) = P(1.2\cdots k) \leq k.$$

We have several more such instances. Let $n > 1$ and $k = n!+1$. We write

$$\triangle_0(n,k) = n(n+1)\cdots(n!+1)(n!+2)\cdots(n!+n)$$

and we observe that

$$P(\triangle_0) \leq n!+1 = k,$$

since $n!+2, \cdots, n!+n$ are all composites. Thus there are infinitely many pairs (n,k) for which

$$P(\triangle(n,k)) \leq k.$$

This is special about consecutive integers. Let $d > 1$, gcd $(n,d) = 1$ and $k \geq 3$. Then Shorey and Tijdeman [11] showed that

$$P(n(n+d)\cdots(n+(k-1)d)) > k$$

unless $(n,d,k) = (2,7,3)$. We observe that $P(2.9.16) = 3$ and therefore it is necessary to exclude the tuple (2,7,3). Also the assumption $k \geq 3$ is necessary since

$$P(1(1+2^r-1)) = P(1.2^r) = 2 \text{ for } r = 1,2,\ldots.$$

We know that

$$k! \mid \triangle_0(n,k).$$

Thus

$$\omega(\triangle_0(n,k)) \geq \pi(k).$$

Theorem 4 can be re-formulated as

$$\omega(\triangle_0) > \pi(k) \text{ if } n > k. \tag{4}$$

Let us see how far we can go. We observe that

$$\triangle_0(k+1,k) = (k+1)\cdots(2k)$$

and

$$\omega(\triangle_0(k+1,k)) = \pi(k) + \pi(2k) - \pi(k) = \pi(2k)$$

In fact, we can say a little more. We consider

$$\triangle_0(k+2,k) = (k+1)(k+2)\cdots(2k)\frac{(2k+1)}{(k+1)}$$

and

$$\omega(\triangle_0(k+2,k)) = \pi(k) + \pi(2k) - \pi(k) - 1 = \pi(2k) - 1$$

if $k+1$ is prime and $2k+1$ is composite. There are infinitely many such k. We have already seen that there are infinitely many primes $p \equiv 2 (\text{mod } 3)$. Let $k = p-1$. Then $k+1 = p$ is prime and

$$2k+1 = 2(k+1) - 1 \equiv 0 (\text{mod } 3)$$

implying the assertion. There are examples when $\omega(\triangle_0(n,k)) < \pi(2k) - 1$. For example

$$\omega(\triangle_0(74, 57)) = \pi(2k) - 2, \omega(\triangle_0(3936, 3879)) = \pi(2k) - 3,$$
$$\omega(\triangle_0(1304, 1239)) = \pi(2k) - 4, \omega(\triangle_0(3932, 3880)) = \pi(2k) - 5$$

but we do not know whether there are infinitely many such pairs. But this is the case under Schinzel's Conjecture as observed by Balasubramanian, Laishram, Thangadurai and myself. Sylvester's theorem can not be sharpened to

$$\omega(\triangle_0(n,k)) \geq \pi(2k) \text{ for } n > k.$$

On the other hand, Laishram and Shorey [6] showed that

$$\omega(\triangle_0(n,k)) \geq \min(\pi(k) + \left[\frac{3}{4}\pi(k)\right] - 1, \pi(2k) - 1) \text{ for } n > k.$$

If the interval $[n, n+k)$ is contained in an interval (p_N, p_{N+1}), then the estimate (4) may be sharpened considerably. Infact Grimm [5] conjectured that

$$\omega(\triangle_0) \geq k$$

if $n, n+1, \cdots, n+k-1$ are all composites. This conjecture, according to Erdős, implies that there exist absolute constants $\alpha > 0$ and N_5 such that

$$p_{N+1} - p_N < p_N^{\frac{1}{2}-\alpha} \text{ for } N \geq N_5.$$

The conjecture has been confirmed by Ramachandra, Shorey and Tijdeman [8] when $\log k \leq C_1(\log n)^{1/2}$ for $n \geq N_6$ where C_1 and N_6 are absolute constants.

Now we point out a relation between Theorem 4 and Diophantine equations. Let k be fixed and $P(\triangle_0) \leq k$. Then the terms of $\triangle_0$ are composed of fixed primes. For any three distinct terms $n+i_1, n+i_2$ and $n+i_3$ of $\triangle_0$ with $i_1 < i_2 < i_3$, we have Siegel's identity

$$(i_3 - i_2)(n + i_1) + (i_2 - i_1)(n + i_3) = (i_3 - i_1)(n + i_2).$$

Therefore we have an equation of the form

$$X_1 + X_2 = X_3$$

where X_1, X_2 and X_3 are composed of primes from a given set. This leads us to a fundamental and central problem in Diophantine equations:

a b c Conjecture. Let *a*, *b and c be positive integers such that* gcd $(a, b, c) = 1$ *and*

$$a + b = c.$$

Let $\epsilon > 0$. *Then there exists* $K = K(\epsilon)$ *such that*

$$c \leq K\left(\prod p\right)^{1+\epsilon}$$

where the product is taken over all prime divisors of abc.

This conjecture has several consequences. For example, it can be applied to Fermat equation (5) to obtain the following result.

Theorem 5. *Let $p \geq 3$ be prime and x, y, z be positive integers such that gcd $(x, y, z) = 1$ and*

$$x^p + y^p = z^p. \quad (5)$$

Then

$$\max(p, x, y, z) \leq C_2$$

where C_2 is an absolute constant.

Now we show that *a b c* conjecture implies Theorem 5.

Proof. Assume (5). Then $p > 3$ by Euler. We apply *a b c* conjecture with

$$a = x^p, b = y^p, c = z^p, \epsilon = \frac{1}{6}.$$

Then gcd $(a, b, c) = 1$ and $a + b = c$. We obtain

$$z^p \leq K\left(\prod_{p|xyz} p\right)^{7/6} \leq K(xyz)^{7/6} \leq Kz^{7/2}$$

where K is an absolute constant. Thus $2^{p-\frac{7}{2}} \leq z^{p-\frac{7}{2}} \leq K$. Hence p and z are bounded since $p > 3$. Consequently x and y are bounded.

Wiles [13] has proved that Fermat's equation does not hold. Now we state a result proved recently coming out of the ideas of Wiles and others on Fermat's equation and the theory of linear forms in logarithms. We define the Fibonacci sequence

$$F_0 = 0, F_1 = 1$$

and

$$F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2.$$

We write the members of the sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \cdots.$$

Thus $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_6 = 8$, $F_{12} = 144$ are powers. It has been proved recently by Bugeaud, Mignotte and Siksek [3] that these are the only powers in the Fibonacci sequence.

Fibonacci sequences are binary recursive sequences. Let $u_0, u_1 \in \mathbb{Q}$ and $r, s \in \mathbb{Q}$ with $s \neq 0, r^2 + 4s \neq 0$. Then we consider the binary recursive sequence $\{u_m\}$ given by

$$u_m = ru_{m-1} + su_{m-2} \text{ for } m \geq 2.$$

Let α and β be roots of $X^2 - rX - s$. Then $\alpha\beta \neq 0$, $\alpha \neq \beta$ and $\alpha + \beta = r$, $\alpha\beta = -s$. Now we show by induction on m that

$$u_m = a\alpha^m + b\beta^m \text{ for } m \geq 0$$

where

$$a = \frac{u_0\beta - u_1}{\beta - \alpha}, b = \frac{u_1 - u_0\alpha}{\beta - \alpha}.$$

Then $\{u_m\}$ is called non-degenerate if $ab \neq 0$ and α/β is not a root of unity. It has been proved by Pethő [7] and Shorey and Stewart [10], independently, that there are only finitely many powers in a non-degenerate binary recursive sequence and the proof depends on the theory of linear forms in logarithms.

References

[1] R. C. Baker, G. Harman and J. Pintz, *The difference between consecutive primes* II, Proc. London Math. Soc. **83** (2001), 532–562.

[2] E. Bombieri, J. B. Friedlander and H. Iwaniec, *Primes in arithmetic progressions to large moduli,* Acta Math. **156** (1986), 203–251.

[3] Y. Bugeaud, M. Mignotte and S. Siksek, *Classical and modular approaches to exponential diophantine equations* I, Fibonacci and Lucas perfect powers, Annals of Math., to appear.

[4] P. Erdős, *The difference of consecutive primes*, Duke Math. Journal **6** (1940), 438–441.

[5] C. A. Grimm, *A conjecture on consecutive composite numbers*, American Math. Monthly 76 (1969), 1126–1128.

[6] Shanta Laishram and T. N. Shorey, *Number of prime divisors in a product of consecutive integers*, Acta Arith. **113** (2004), 327–341.

[7] A. Pethő, *Perfect powers in second order linear recurrences*, Journal Number Theory **15** (1982), 5–13.

[8] K. Ramachandra, T. N. Shorey and R. Tijdeman, *On Grimm's problem relating to factorisation of a block of consecutive integers* II, Journal Reine Angew Math. **288** (1976), 192–201.

[9] Arnold Schönhage, *Eine Bemerkung zur Konstruktion grosser Primzahllücken*, Arch. Math. **14** (1963), 29–30.

[10] T. N. Shorey and C. L. Stewart, *On the equation* $ax^2 + bx^ty + cy^2 = d$ *and pure powers in recurrence sequences*, Math. Scand. **52** (1983), 24–36.

[11] T. N. Shorey and R. Tijdeman, *On the greatest prime factor of an arithmetical progression*, A tribute to Paul Erdős, ed. A. Baker, B. Bollobas and A. Hajnal, Cambridge University Press (1990), 385–389.

[12] J. Sylvester, *On arithmetical series*, Messenger Math. **21** (1892), 1–19, 87–120.

[13] A. Wiles, *Modular elliptic curves and Fermat's Last Theorem*, Annals of Math. **141** (1995), 443–551.

Refresher Course in Real Analysis

Ramanujan Institute for Advanced Study in Mathematics
University of Madras
Sponsored by UGC-Academic Staff College
2008–2009

There will be a refresher course (batch No. XXIV), **for college teachers**, in Real Analysis, from 05-11-2008 to 25-11-2008, at Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chepauk, Chennai 600 005. In this course, Real Analysis will be developed from the basics. Applications of Real Analysis in other areas will be discussed after developing the basic theory. Experts in Analysis from leading institutions will deliver lectures for this refresher course.

For Applications and Other Information, Please Contact:

Dr. K. A. Chandrasekar (Director),
UGC – Academic Staff College
University of Madras, Chennai 600 005

Phone: 25368778 Ext: 269, 343
Email: ugcascuomyahoo.com

If You Have Further Questions, You May Also Contact:

Prof. E. Thandapani (Coordinator)
RIASM, University of Madras, Chennai 600 005
Phone: 25360357 Ext: 25
Email: ethandapaniyahoo.co.in

TATA Institute of Fundamental Research

TIFR Centre For Applicable Mathematics Advanced Instructional School On Partial Diffrential Equations (AIS-PDE)

December 15, 2008–January 6, 2009

For Further Details Visit/Contact:

http://math.tifrbng.res.in/aispde

Annual Thematic Programs

at

Indian Institute of Science Bangalore
August 2008–July 2009

For Further Details Visit/Contact:

http://math.iisc.ernet.in/ imi/atp-0809.htm
Email: imi@math.iisc.ernet.in

For Details About Gate-2009

Visit: http://gate.iitm.ac.in/

For Details About PH.D.

Programme at IIT Madras
Visit: http://www.iitm.ac.in

24th Annual Conference of the Mathematical Society-Banaras Hindu University, Varanasi (ACMS-BHU-2008)

December 30–31, 2008

For Details Contact:

1. Prof. S. D. Singh by Mobile: 09335665529
2. Dr. B. S. Bhadauria by Mobile: 09453641182

The Mathematical Society-BHU, Department of Mathematics, Faculty of Science, Banaras Hindu University, Varanasi 221 005, U.P.
Emails: acmsbhu2008@gmail.com, acmsbhu@yahoo.in

National Conference on History of Mathematics

Its Role in Science and Society, Imphal

December 19–21, 2008

For Further Details Contact:

1. B. S. Yadav, TU-67, Vishakha Enclave, Pitampura, Delhi 110 088.
   Tel: 011-27343878, Mob: +919891002425,
   Email: bsyadav@indianshm.com
2. M. Ranjit Singh, Dept. of Mathematics, Manipur University, Canchipur 795 003, Imphal, Manipur.
   Email: mranjitmu@rediffmail.com

Indo-German Workshop-Cum-Lecture Series on Computational Models and Methods Driven by Industrial Problems

November 2008–February 2009

Interested Participants May Contact/Visit:

Prof. S. Sundar, Department of Mathematics, IIT Madras, Chennai 600 036, India
Email: slnt@iitm.ac.in; http:/cmmdip0809.iitm.ac.in/

The readers may download the Mathematics Newsletter from the RMS website at www.ramanujanmathsociety.org