MATHEMATICS NEWSLETTER

Volume 19 December 2010 No. 2

CONTENTS

Visit: `www.ramanujanmathsociety.org`

Typeset in LaTeX at Krishtel eMaging Solutions Pvt. Ltd., Chennai - 600 017 Phone: 2434 55 16 and printed at United Bind Graphics, Chennai - 600 004. Phone: 2498 7562, 2466 1807

Wiener's Theorem, Infinite Matrices and Banach Algebras

S. H. Kulkarni
Department of Mathematics, Indian Institute of Technology – Madras, Chennai 600 036
Email: shk@iitm.ac.in

1. Introduction

This article is based on an invited talk given by me at the National Symposium on Mathematical Methods and Applications (NSMAA 2009) organized by the the Department of Mathematics, Indian Institute of Technology Madras on December 22, 2009. A preliminary version of this article appeared in the Proceedings of the National Symposium on Mathematical Methods and Applications 2009 - Invited Talks, Editors: P. V. Subrahmanyam and S. Sundar. The Department has been organizing such a symposium every year in honour of the celebrated Indian Mathematician Srinivasa Ramanujan. I thank the organizers for inviting me to give a talk in this symposium. It is an honour to be associated with the illustrious memory of Ramanujan.

An objective of this talk is to highlight the connections between some apparently unrelated theorems and the role of Banach algebras in these theorems. The first of these theorems is the following well known theorem due to the famous mathematician Norbert Wiener. Wiener's proof can be found in his book [11].

Theorem 1.1 (Wiener's Theorem). *Let f be a periodic function on $[-\pi, \pi]$. Suppose f has an absolutely convergent Fourier series and $f(t) \neq 0$ for all $t \in [-\pi, \pi]$. Then $1/f$ also has absolutely convergent Fourier series.*

Gelfand gave an elegant proof of this theorem using the techniques from Banach Algebras in his celebrated paper [2]. This was the first paper in which the theory of Banach algebras was developed systematically. Gelfand's proof is much shorter than the original proof of Wiener and it attracted the attention of Mathematicians to the theory of Banach algebras.

The second theorem is due to Jaffard [5] and deals with the decay of off-diagonal entries of doubly infinite matrices. Note that such matrices can be regarded as the operators on $\ell^2(\mathbb{Z})$, the space of all square summable doubly infinite sequences of complex numbers.

Theorem 1.2 (Jaffard's theorem). *If a matrix A with entries $A(k,l), k,l \in \mathbb{Z}$ is invertible on $\ell^2(\mathbb{Z})$ and there are constants $C > 0$, $r > 1$ such that*

$$|A(k,l)| \leq C(1 + |k-l|)^{-r} \quad \textit{for all} \quad k,l \in \mathbb{Z},$$

then,

$$|A^{-1}(k,l)| \leq C(1 + |k-l|)^{-r} \quad \textit{for all} \quad k,l \in \mathbb{Z}.$$

In order to discuss next theorem in our list, we need a definition.

Definition 1.3 (Band dominated operators). *We shall regard an infinite matrix $A = [A(k,l)]$, $k,l \in \mathbb{Z}$ as an operator on $\ell^2(\mathbb{Z})$. Such a matrix is called a band matrix or band operator if there exists $n \in \mathbb{N}$ such that $A(k,l) = 0$ for $|k-l| > n$. A band dominated matrix (or operator) is a limit (in the operator norm) of a sequence of band operators.*

Then the theorem can be simply stated as follows: If a band dominated operator is invertible, then its inverse is also band dominated. (Note that this not true for band operators.)

A more detailed statement is given below.

Theorem 1.4. *For an infinite matrix $A = [A(k,l)]$, $k,l \in \mathbb{Z}$, let A_n denote the nth band approximation of A given by $A_n(k,l) = A(k,l)$ for $|k-l| \leq n$ and $A_n(k,l) = 0$ otherwise. If A is invertible on $\ell^2(\mathbb{Z})$ and there exist positive constants r, C such that*

$$\|A - A_n\| \leq Cn^{-r} \quad \textit{for all} \quad n \in \mathbb{N},$$

then there exists a sequence $\{B_n\}$ of band matrices such that

$$\|A^{-1} - B_n\| \leq Cn^{-r} \quad \textit{for all} \quad n \in \mathbb{N},$$

This brings us to the main question of the talk.

What is the Connection Between these Theorems?

In other words, is there any common theme that leads to each of these theorems as a special case? The answer is yes and this connection/common theme is provided by the theory of Banach algebras. In the next section, we review some basic concepts from the theory of Banach algebras that are needed to understand this connection. More information about above theorems and related issues can be found in [3], [4] and [9]. The last section contains the details about this common theme.

2. Banach algebras

Our objects of interest are spectra of elements in a Banach algebra. We begin with the definition of a complex algebra.

Definition 2.1 (Complex Algebras). *A complex algebra A is a ring that is also a complex vector space such that*

$$(\alpha a)b = \alpha(ab) = a(\alpha b) \quad \text{for all} \quad a, b \in A,\ \alpha \in \mathbb{C}$$

A is called commutative if $ab = ba$ for all $a, b \in A$.

We shall assume that A has a unit element 1 satisfying $1a = a = a1$ for all $a \in A$.

Definition 2.2 (Banach algebras). *Let A be a complex algebra. An algebra norm on A is a function $\|.\| : A \to \mathbb{R}$ satisfying:*

(1) $\|a\| \geq 0$ *for all* $a \in A$ *and* $\|a\| = 0$ *if and only if* $a = 0$.
(2) $\|\alpha a\| = |\alpha|\|a\|$ *for all* $a \in A$ *and* $\alpha \in \mathbb{R}$
(3) $\|a + b\| \leq \|a\| + \|b\|$ *for all* $a, b \in A$.
(4) $\|ab\| \leq \|a\|\|b\|$ *for all* $a, b \in A$.

A complex normed algebra is a complex algebra A with an algebra norm defined on it. A Banach algebra is a complete normed algebra.

We shall assume that A is *unital*, that is A has unit 1 with $\|1\| = 1$.

Next we recall some standard examples of Banach algebras.

Example 2.3 (Algebras of functions). Let X be a compact Hausdorff space, and let $C(X)$ denote the set of all complex-valued continuous functions on X. Then $C(X)$ is a commutative Banach algebra under pointwise operations and the sup norm given by

$$\|f\| := \sup\{|f(x)| : x \in X\}, \quad f \in C(X)$$

Example 2.4 (Algebras of operators). Let H be a complex Hilbert space and let $BL(H)$ denote the set of all bounded (continuous) linear operators on H. Then $BL(H)$ is a Banach algebra under the usual operations and the operator norm given by

$$\|T\| := \sup\{\|T(x)\| : x \in H, \|x\| \leq 1\}, \quad T \in BL(H)$$

When H is of dimension n, $BL(H)$ can be identified with $\mathbb{C}^{n\times n}$, the algebra of all matrices of order $n \times n$ with complex entries.

More examples and basic theory of Banach algebras can be found in the following books: [1] and [7].

We now define our main objects of interest.

Definition 2.5 (Spectrum). *Let A be a complex Banach algebra with unit 1 and let $a \in A$. The spectrum $\sigma_A(a)$ of a is defined to be the set of all complex numbers λ such that $\lambda 1 - a$ is not invertible in A.*

The Spectral Radius $r(a)$ of a is defined by

$$r_A(a) := \sup\{|\lambda| : \lambda \in \sigma_A(a)\}$$

The subscript A will be dropped when the algebra under consideration is fixed and no confusion is likely.

Thus when $A = C(X)$ and $f \in A$, $\sigma(f)$ coincides with the range of f.

Similarly when $A = \mathbb{C}^{n\times n}$ and $M \in A$, $\sigma(M)$ is the set of all eigenvalues of A.

Properties of Spectrum:

We now recall a few well known properties of the spectrum. Let A be a complex Banach algebra with unit 1 and let $a \in A$. Then,

- $\sigma(a)$ is a nonempty compact subset of $\mathbb{C}$.
- The Spectral Radius Formula:

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$$

- The map $a \to \sigma(a)$ is upper semicontinuous. This means that given any open subset U of $\mathbb{C}$ containing $\sigma(a)$, there exists $\delta > 0$ such that for every $b \in A$ with $\|a - b\| < \delta$, $\sigma(b) \subseteq U$.

3. Inverse-closed subalgebras

We are now in a position to say something about the main question posed in the Introduction. We may note that each of the theorems mentioned there deals with some elements in some Banach algebra. In particular, each theorem says that if an element in a Banach algebra has a a particular property, then its inverse, if exists, also has the same property. This observation naturally leads to the following definition.

Definition 3.1 (Inverse-closedness). *Let A be a complex Banach algebra with unit 1 and let B be a subalgera of A containing 1. Then B is called inverse-closed in A if*
$a \in B$ and $a^{-1} \in A$ implies $a^{-1} \in B$.

Now the main question posed in the Introduction can be reformulated as follows:

When is *B* Inverse-Closed in *A*?

Before discussing this question further, we may note that this concept has been given some other names also in the literature.

Let A and B be as above.

- B is inverse-closed in A.
- (B, A) is a Wiener pair. (Naimark)
- B is a spectral subalgebra of A. (Palmer)
- B is a local subalgebra of A.
- Spectral invariance, spectral permanece (Arveson)

The justification for the last of these names is due to the following characterization of inverse-closedness in terms of the spectrum.

Theorem 3.2. *Let A be a complex Banach algebra with unit 1 and let B be a subalgebra of A containing 1. Then B is inverse-closed in A if and only if for every $x \in B$,*
$\sigma_B(x) = \sigma_A(x)$.

The following theorem gives a condition for the two spectra to coincide. More information can be found in [4].

Theorem 3.3. *Let A be a complex Banach algebra with unit 1. Let $Inv(A)$ denote the set of all invertible elements in A. Then, for each $a \in A$, $\sigma_A(a) = \{\lambda \in \mathbb{C} : \lambda 1 - a \notin Inv(A)\}$ Next, let B be a closed subalgebra of A containing 1. Then ,*

(1) *$Inv(B)$ is a union of components of $B \cap Inv(A)$.*
(2) *For $x \in B$, $\sigma_B(x)$ is the union of $\sigma_A(x)$ and a (possibly empty) collection of bounded components of the complement of $\sigma_A(x)$.*

In particular, if the the complement of $\sigma_A(x)$ is connected, then $\sigma_B(x) = \sigma_A(x)$.

A proof of the above theorem can be found in [8]. The situation is better when the Banach algebra under consideration has some additional structure, namely involution. In particular, if it is a B^*-algebra, we can give a very simple condition for a subalgebra to be inverse-closed. We recall some relevant definitions.

Definition 3.4 (Involutions). *An involution on a complex algebra A is a mapping $a \to a^*$ of A into A that satisfies the following axioms:*

(1) *$(a + b)^* = a^* + b^*$ for all a and b in A;*
(2) *$(\alpha a)^* = \overline{\alpha} a^*$ for all a in A and α in $\mathbb{C}$;*
(3) *$(ab)^* = b^* a^*$ for all a and b in A;*
(4) *$(a^*)^* = a$ for all a in A.*

Definition 3.5 (B^*-algebras). *A Banach algebra A with an involution $x \to x^*$ is called a B^*-algebra if*

$$\|x^* x\| = \|x\|^2$$

for every $x \in A$.

These algebras are also known as C^*-algebra.

Known fact: If A is a B^*-algebra, then $\sigma_A(xx^*) \subseteq [0, \infty)$ for every $x \in A$. (See [8])

We now give the simple condition mentioned above.

Theorem 3.6 (See [8]). *Suppose A is a B^*-algebra with unit 1, B is a closed subalgebra of A, $1 \in B$ and $x^* \in B$ for every $x \in B$. Then $\sigma_B(x) = \sigma_A(x)$ for every $x \in B$.*

In other words, B is inverse-closed in A.

Proof. Suppose $x \in B$ has inverse in A. Then x^* and hence xx^* also have inverses in A. Hence $\sigma_A(xx^*) \subseteq (0, \infty)$. Thus the complement of $\sigma_A(xx^*)$ is connected. This implies $\sigma_B(xx^*) = \sigma_A(xx^*) \subseteq (0, \infty)$. Hence $(xx^*)^{-1} \in B$ and consequently $x^{-1} = x^*(xx^*)^{-1} \in B$.

Remark 3.7. We are now in a position to explain the connection between the three theorems stated in the introduction. In fact, as explained below each of these theorems is a special case of Theorem 3.6.

(1) Let $A = C(\Gamma)$, where Γ denotes the unit circle and B be the set of continuous functions in A with absolutely convergent Fourier series. We have already seen in Example 2.3 that A is a Banach algebra. It is routine to

check that B is a closed subalgebra of A. The map $f \to \overline{f}$, where $\overline{f}$ denotes the complex conjugate of f, is an involution on A making it a B^*-algebra. Also, it is easy to prove that if the Fourier series of f converges absolutely, then so does that of $\overline{f}$. In other words, B satisfies the hypotheses of Theorem 3.6 and is hence an inverse-closed subalgebra of A. This is Wiener's theorem (Theorem 1.1).

(2) Let $A = BL(\ell^2(\mathbb{Z}))$ and B be the set of all matrices satisfying the off diagonal decay conditions given in Jaffard's theorem (Theorem 1.2). Then A is a Banach algebra, as seen in Example 2.4. It requires some work to check that B is a closed subalgebra of A. Next, for $T \in A$, let T^* denote the adjoint of T. Then the map $T \to T^*$ is an involution on A and A is B^*-algebra with respect to this involution. This is well known and can be found in many books, for example, [7], [8]. Further, it is also well known that if $\alpha_{i,j}$ is the (i, j)th entry of the matrix of T, then the (i, j)th entry of the matrix of T^* is $\overline{\alpha_{j,i}}$. Hence if T satisfies the off diagonal decay conditions, then so does T^*. Thus B satisfies the hypotheses of Theorem 3.6 and is hence an inverse-closed subalgebra of A. This is precisely the statement of Jaffard's theorem (Theorem 1.2).

(3) Let $A = BL(\ell^2(\mathbb{Z}))$ and B be the set of all band dominated matrices. Then, as above, A is a B^*-algebra and B is a closed subalgebra of A. Also, if $T \in A$ is band dominated, then so is T^*. Thus again by Theorem 3.6, it follows that B is an inverse-closed subalgebra of A. This implies Theorem 1.4.

References

[1] Frank F. Bonsall and John Duncan, *Complete normed algebras*, Springer-Verlag, New York, 1973. MR 0423029 (54 #11013).

[2] I. Gelfand, Normierte Ringe, Rec. Math. [Mat. Sbornik] N. S. **9 (51)** (1941), 3–24. MR0004726 (3,51f).

[3] K. Grochening and A. Klotz, Noncommutative approximation: Inverse closed subalgebras and off-diagonal decay of matrices. Preprint

[4] K. Grochening. Wiener's lemma: Theme and variations. An introduction to spectral invariance. Applied and Numerical Harmonic Analysis. Birkhauser, Boston, 2009.

[5] S. Jaffard, Propriétés des matrices "bien localisées" près de leur diagonale et quelques applications, Ann. Inst. H. Poincaré Anal. Non Linéaire **7** (1990), no. 5, 461–476. MR1138533 (93h:47035).

[6] B. V. Limaye, *Functional analysis*, second edition, New Age, New Delhi, 1996. MR1427262 (97k:46001).

[7] T. W. Palmer, *Banach algebras and the general theory of *-algebras. Vol. I*, Encyclopedia of Mathematics and its Applications, 49, Cambridge Univ. Press, Cambridge, 1994. MR1270014 (95c:46002).

[8] W. Rudin, *Functional analysis*, Second edition, McGraw-Hill, New York, 1991. MR1157815 (92k:46001).

[9] T. Strohmer, Pseudodifferential operators and Banach algebras in mobile communications, Appl. Comput. Harmon. Anal. **20** (2006), no. 2, 237–249. MR2207837 (2007k:42100).

[10] A. E. Taylor and D. C. Lay, *Introduction to functional analysis*, second edition, John Wiley & Sons, New York, 1980. MR0564653 (81b:46001).

[11] Wiener, Norbert. The Fourier integral and certain of its applications. Reprint of the 1933 edition. With a foreword by Jean-Pierre Kahane. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1988. xviii+201 pp. ISBN: 0-521-35884-1 MR0983891 (90b:42022).

On a Theorem of Euler

Joseph Oesterlé
Institut de Mathématiques de Jussieu, Université Pierre et Marie Curie, 4 place Jussieu 75005 - Paris, France
Email: oesterle@math.jussieu.fr

Leonhard Euler (1707–1783) stated and proved in 1738 the following theorem ([1]): *Nullus cubus, ne quidem numeris fractis exceptis, unitate auctus quadratum efficere potest praeter unicum casum, quo cubus est 8.*

Euler's paper is written in Latin. A free translation of the aforementionned theorem is : *No cube, not even of a fractional number, can yield a square when augmented by a unit, except in the single case where the cube is 8.*

In this statement, a fractional number means for Euler a positive rational number. If one wants to take care of all rational numbers, the theorem should be modified as follows:

Theorem. *Let x be a rational number. If x^3+1 is the square of a rational number, then x is equal to -1, 0 or 2.*

This theorem allows us to determine all points with rational coordinates of the curve defined by the equation $y^2 = x^3+1$. This curve (completed by a point at infinity) is one of the simplest examples of what we now call an elliptic curve over $\mathbf{Q}$. Euler's proof is particularly interesting, since it provides a nice example of a method called 2-descent to determine the set of rational points on such a curve.

We shall prove the theorem by following essentially Euler's original argument. Let us write $x = t-1$. If $t \neq 0$, x^3+1 can be written as $t^2\left(t-3+\frac{3}{t}\right)$. We have to prove that if $t-3+\frac{3}{t}$ is a square, t is either equal to 1 or to 3. Since the expression $t-3+\frac{3}{t}$ is unchanged when t is replaced by $\frac{3}{t}$, we may restrict our attention to the case when the numerator of t is not divisible by 3. The theorem may therefore be restated as follows:

Proposition. *Let t be a non-zero rational number whose numerator is not divisible by 3. If $t-3+\frac{3}{t}$ is the square of a rational number, then $t=1$.*

Since $t-3+\frac{3}{t}$ is a square, t cannot be negative. Let us write $t=\frac{a}{b}$, with a and b natural numbers, and a coprime to $3b$. Then $t-3+\frac{3}{t}$ takes the form $\frac{a^2-3ab+3b^2}{ab}$. Moreover a, b and $a^2-3ab+3b^2$ are pairwise coprime, hence each of these numbers is a square.

The two roots of the second degree equation $(a-bu)^2 = a^2-3ab+3b^2$ in u are therefore rational numbers. Since their sum is $\frac{2a}{b}$, one of them has a numerator not divisible by 3. Let u be such a root. We have $a^2-2abu+b^2u^2 = a^2-3ab+3b^2$, hence $-2tu+u^2 = -3t+3$. This implies $2u \neq 3$ and $t = \frac{u^2-3}{2u-3}$.

We can write u in irreducible form as $\frac{m}{n}$ where $n \geq 1$ and m is an integer coprime to $3n$. We then get $\frac{a}{b} = \frac{m^2-3n^2}{n(2m-3n)}$. The assumption that m is coprime to $3n$ implies that m^2-3n^2 is coprime to $n(2m-3n)$. We therefore must have $a = \pm(m^2-3n^2)$. But a is a square and $3n^2-m^2$ cannot be a square, since it is congruent to 2 or 3 mod 4. Hence the only possiblity left is $a = m^2-3n^2$ and $b = n(2m-3n)$.

Since a is a square, the two roots of the second degree equation $(m-nv)^2 = m^2-3n^2$ in v are rational numbers. Their sum is $\frac{2m}{n}$, and m is coprime to 3, hence one of these roots has a numerator not divisible by 3. Let v denote such a root. We have $m^2-2mnv+n^2v^2 = m^2-3n^2$, hence $-2uv+v^2 = -3$ and $u = \frac{v^2+3}{2v}$.

Since b is a square, $\frac{b}{n^2} = 2u-3$ is a square, and therefore $v-3+\frac{3}{v}$ is a square.

Surprise: v is a rational number satisfying the same conditions as t!

The proposition therefore follows from the following observation, and from the fact that no infinite sequence of natural numbers can be strictly decreasing.

Lemma. *If $t \neq 1$, then $v \neq 1$ and the denominator of v is strictly smaller than the denominator of t.*

If $v=1$, then $u = \frac{v^2+3}{2v} = 2$ and $t = \frac{u^2-3}{2u-3} = 1$. Hence $t \neq 1$ implies $v \neq 1$. Let us write v in irreducible form as $\frac{p}{q}$ where $q \geq 1$ and p is an integer coprime to $3q$. Since $v-3+\frac{3}{v}$ is a square, v is positive and we have $p \geq 1$. The relation $u = \frac{v^2+3}{2v}$ may be written as $\frac{m}{n} = \frac{p^2+3q^2}{2pq}$. But pq is coprime to p^2+3q^2, hence (m,n) is either equal to $\left(\frac{p^2+3q^2}{2}, pq\right)$ or to its double, and $n(2m-3n)$ is either $pq(p^2-3pq+3q^2)$ or 4 times this number. This proves that $b > q$ except when p and $p^2-3pq+3q^2$ are both equal to 1. But in that case, we have $p=q=1$, hence $v=1$ and $t=1$. This completes the proof.

References

[1] Leonhardi Euleri: Theorematum quorundam arithmeticorum demonstrationes, in Commentationes Arithmeticae, Vol. I, Ed. Ferdinand Rudio, Lipsiae et Berolini, B.G. Teubner, MCMXV.

Extensions of Modules

H. Guhan Venkat

Dedicated to the memory of M. HARIKUMAR

Abstract. We construct extensions of finitely generated torsion modules over Euclidean Domains using *relations matrix* and also obtain an equivalence between them for isomorphic and equivalent extensions when considered as $\mathbb{Z}$ modules i.e. for finite abelian groups.

1. Introduction

Let R be a Euclidean Domain (hence naturally a PID) and let M be a finitely generated torsion R-module. From the Structure Theorem, we know that

$$M \cong R/(p_1^{\alpha_1}) \oplus R/(p_2^{\alpha_2}) \oplus \cdots \oplus R/(p_m^{\alpha_m})$$

where $p_1^{\alpha_1}, \dots, p_m^{\alpha_m}$ are positive powers of (not necessarily distinct) primes in R. Also

$$M \cong R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_n)$$

where $a_1|a_2|\dots|a_n$ are non-zero elements in R which are also not units.

Let $x_1, \dots, x_n \in M$ be a generating set. We will call an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_mx_m = 0$$

with $a_i \in R$, a *relation* in R. Note that the multiplication is the action of R on M. Further, a *relations matrix* in M is a set of relations written as a matrix, i.e.

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & a_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 0.$$

A *relations matrix* is called complete if every relation in M can be expressed as an R-linear combination of rows of the matrix. Note that by performing suitable elementary row operations on a complete relations matrix A in M, the Smith Normal form of A is

$$A_M = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}.$$

To compute the Smith Normal Form, we require R to be Euclidean and this justifies our assumption eventhough the results are valid for PIDs.

2. Extensions

Let L, N, M be R-modules such that

$$0 \to L \xrightarrow{f} N \xrightarrow{g} M \to 0$$

is a Short Exact Sequence. We know that $N = L \oplus M$ if the annihilators of L and M are relatively prime (i.e. no primes in common occur in the primary decomposition of the two modules). Thus we consider the extension problem for p-primary components of torsion module.

Fix a prime p in R. From now on, we will consider only p-primary torsion finitely generated modules over R. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_l$ be an increasing sequence of positive integers. Let the R-module L_p be of type λ where $\lambda = (\lambda_1, \ldots, \lambda_l)$, i.e.

$$L_p \cong R/(p^{\lambda_1}) \oplus R/(p^{\lambda_2}) \oplus \cdots \oplus R/(p^{\lambda_l}).$$

Let $y_1, \ldots, y_l \in L_p$ be a generating set for L_p. Similarly, let M_p be an R-module of type μ where $\mu = (\mu_1, \ldots, \mu_m)$ with $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_m$ an increasing sequence of positive integers. Let $x_1, \ldots, x_m$ be a generating set for M_p. Let N_p be any extension of of M_p by L_p

$$0 \to L_p \xrightarrow{i} N_p \xrightarrow{\pi} M_p \to 0.$$

For each $x \in M_p$, let n_x be an element in N_p mapping onto x by π (these are called a section of π)

$$\begin{aligned} p^{\mu_j} x_j &= 0 \forall j \in [1, m] \\ \Rightarrow p^{\mu_j} \pi(n_{x_j}) &= 0 \\ \Rightarrow \pi(p^{\mu_j} n_{x_j}) &= 0. \end{aligned}$$

Now $p^{\mu_j} n_{x_j} \in Ker(\pi) = L_p$ and since every element in L_p is an R-linear combination of $y_1, \ldots, y_l$, we can write

$$\begin{aligned} p^{\mu_1} n_{x_1} &= a_{11} y_1 + a_{12} y_2 + \cdots + a_{1l} y_l \\ p^{\mu_2} n_{x_2} &= a_{21} y_1 + a_{22} y_2 + \cdots + a_{2l} y_l \\ \vdots \quad & \qquad \vdots \\ p^{\mu_m} n_{x_m} &= a_{m1} y_1 + a_{m2} y_2 + \cdots + a_{ml} y_l. \end{aligned}$$

This gives the complete *relations matrix* (actually a system of relations) for N_p

$$\begin{pmatrix} p^{\lambda_1} & & & & & & \\ & p^{\lambda_2} & & & & & \\ & & \ddots & & & & \\ a_{11} & a_{12} & \ldots & a_{1l} & p^{\mu_1} & & \\ a_{21} & a_{22} & \ldots & a_{2l} & & p^{\mu_2} & \\ \vdots & \vdots & \ddots & \vdots & & & \ddots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & & & & p^{\mu_m} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_l \\ n_{x_1} \\ n_{x_2} \\ \vdots \\ n_{x_m} \end{pmatrix} = 0.$$

For brevity, we write them as blocks $\left(\begin{smallmatrix} p^{\lambda} & \\ A & p^{\mu} \end{smallmatrix}\right)\left(\begin{smallmatrix} y \\ n_x \end{smallmatrix}\right) = 0$.

We will call the matrix A as the *matrix of extension*. Note that two such matrices $A_1 \equiv A_2 \text{mod}(p^{\lambda})$ define the same extension and hence we can in particular choose our matrix of extension to be modulo p^{λ}.

Corollary 1. *The extension above is split if* $A = 0$.

3. Extensions of finite abelian p-groups

In this section, we fix $R = \mathbb{Z}$, i.e. all modules are finite abelian p-groups for p prime in $\mathbb{Z}$. Using the results obtained above, we will consider isomorphic and equivalent extensions and relate it to well known results from Homological Algebra. Let G_λ and G_μ be finite abelian p-groups of type $(\lambda_1, \ldots, \lambda_l)$ and $(\mu_1, \ldots, \mu_m)$ respectively. Consider all extensions, E of G_μ by G_λ, i.e.

$$0 \to G_\lambda \xrightarrow{i} E \xrightarrow{\pi} G_\mu \to 0$$

is a short exact sequence. The image of G_λ is identified in E as an inclusion. A *presentation* for E is then

$$\begin{pmatrix} p^{\lambda} & \\ A & p^{\mu} \end{pmatrix} \begin{pmatrix} y \\ e_x \end{pmatrix} = 0$$

where $y = (y_1, \ldots, y_l)$ is a generating set in G_λ and $e_x = (e_{x_1}, \ldots, e_{x_m})$ is a section of a generating set of G_μ.

3.1 Isomorphic and Equivalent Extensions

Suppose we have the following commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_\lambda & \xrightarrow{i_1} & E & \xrightarrow{\pi_1} & G_\mu & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow h & & \downarrow g & & \\ 0 & \longrightarrow & G'_\lambda & \xrightarrow{i_2} & E' & \xrightarrow{\pi_2} & G'_\mu & \longrightarrow & 0. \end{array}$$

Theorem 1 (Short Five Lemma). *When f and g are isomorphisms, then h is automatically an isomorphism.*

Proof. The proof can be found in any standard book on Algebra such as [1]. □

Such extensions are called isomorphic extensions. Let G'_λ be generated by $y' = (y'_1, \ldots, y'_l)$, the image of y under f and G'_μ be generated by $x' = (x'_1, \ldots, x'_m)$, the image of x under g. It follows that the *section* of x', denoted by $e'_{x'}$, is the image of e_x under h. A presentation for E' is then

$$\begin{pmatrix} p^{\lambda} & \\ A' & p^{\mu} \end{pmatrix} \begin{pmatrix} y' \\ e'_{x'} \end{pmatrix} = 0.$$

The isomorphisms f, g and h can be regarded as matrices F, G and H as done in [2]. F, G and H are $l \times l$, $m \times m$ and $(l+m) \times (l+m)$ matrices respectively. We are interested in a relation between the matrices of extension, A and A' for isomorphic extensions considered above.

Theorem 2. *Two extensions E and E' are isomorphic if and only if $A = GA'F$ where A and A' are matrices of extensions E and E' respectively and F and G are matrices corresponding to isomorphisms of G_λ and G_μ respectively.*

Proof. If E and E' were isomorphic extensions, then by the *Short Five Lemma*, we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_\lambda & \xrightarrow{i_1} & E & \xrightarrow{\pi_1} & G_\mu & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow h & & \downarrow g & & \\ 0 & \longrightarrow & G'_\lambda & \xrightarrow{i_2} & E' & \xrightarrow{\pi_2} & G'_\mu & \longrightarrow & 0 \end{array}$$

where f, g and h are isomorphisms with corresponding matrices denoted by F, G and H. We can write $H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$ where h_{11} and h_{22} are $l \times l$ and $m \times m$ blocks.

From the commutativity of the diagram, it follows that

$$i_2(f(k)) = h(i_1(k)) \qquad \forall k \in G_\lambda$$

$$\Rightarrow \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} F \\ 0 \end{bmatrix} = \begin{bmatrix} h_{11} \\ h_{21} \end{bmatrix}$$

$$g(\pi_1(j)) = \pi_2(h(j)) \qquad \forall j \in E$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix} \begin{bmatrix} p^\lambda & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} p^\lambda & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} F & h_{12} \\ 0 & h_{22} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix} = \begin{bmatrix} p^\lambda F & p^\lambda h_{12} \\ 0 & h_{22} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} H \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & G \end{bmatrix}$$

Note that the block h_{12} could be arbitrary in the automorphism $[H]$. However, the $p^\lambda h_{12}$ term is zero due to the p^λ factor which is the annihilator of G_λ.

Since h is an isomorphims $h(0_E) = 0_{E'}$. Now applying h to the relation matrix in E, we have

$$\begin{pmatrix} p^\lambda & \\ A & p^\mu \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} y \\ e_x \end{pmatrix} = 0. \tag{1}$$

But $\begin{pmatrix} p^\lambda & \\ A' & p^\mu \end{pmatrix}$ gives a complete set of relations in E' and hence the relations in (1) can be obtained by an elementary row operation.

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} p^\lambda & \\ A' & p^\mu \end{pmatrix} = \begin{pmatrix} p^\lambda & \\ A & p^\mu \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & G \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} \alpha p^\lambda + \beta A' & \beta p^\mu \\ \gamma p^\lambda + \delta A' & \delta p^\mu \end{pmatrix} = \begin{pmatrix} p^\lambda F & * \\ AF & p^\mu G \end{pmatrix}. \tag{3}$$

Note that we consider the *relations matrix* modulo p^λ and therefore some terms vanish and we have

$$AF = \delta A' = p^\mu G p^{-\mu} A' \tag{4}$$

$$\Rightarrow A = GA'F \tag{5}$$

and the Theorem follows. Note that by abuse of notation, we still call the isomorphisms F^{-1} and $p^\mu G p^{-\mu}$ as F and G respectively. □

Two extensions are called equivalent if there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_\lambda & \xrightarrow{i_1} & E & \xrightarrow{\pi_1} & G_\mu & \longrightarrow & 0 \\ & & \| & & \downarrow h & & \| & & \\ 0 & \longrightarrow & G_\lambda & \xrightarrow{i_2} & E' & \xrightarrow{\pi_2} & G_\mu & \longrightarrow & 0\,. \end{array}$$

Corollary 2. *Two extensions E and E' are equivalent if and only if their matrices of extension are equal modulo p^λ.*

Proof. In this case, $F = G = I$ and the result follows. □

4. Homological Algebra

An introduction to Homological Algebra can be found in [1] or any other standard book. We state many standard results without proofs here. In this section, all the modules are considered as $\mathbb{Z}$-modules unless stated explicitly. Associated to each pair of modules L, N are a sequence of cohomology groups denoted by $\mathrm{Ext}^n_{\mathbb{Z}}(L, N)$. Let L, N be $\mathbb{Z}$ modules. We have the following well known theorem:

Theorem 3. *Equivalent extensions of L by N correspond bijectively to classes in $Ext^1_{\mathbb{Z}}(L, N)$. The split extension corresponds to the trivial class.*

This gives us a relevant corollary

Corollary 3. *Classes in $Ext^1_{\mathbb{Z}}(L, N)$ correspond bijectively to matrices of extensions of according size. The trivial class corresponds to the zero matrix.*

Proof. The proof follows from Theorem 2 and Corollary 2 above. □

Now we will illustrate two examples. Consider the extensions of $\mathbb{Z}/p\mathbb{Z}$ by itself. It is known that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$. The isomorphisms F and $G \in (\mathbb{Z}/p\mathbb{Z})^{\times}$. The matrix of extension $A \in [0, p-1]$.

When $A = 0 \Rightarrow A_E = \left(\begin{smallmatrix} p & \\ 0 & p \end{smallmatrix}\right)$ which gives the split extension $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$.

$A \neq 0 \Rightarrow A_E = \left(\begin{smallmatrix} p & \\ A & p \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} 1 & \\ & p^2 \end{smallmatrix}\right)$ which gives the cyclic extension $\mathbb{Z}/p^2\mathbb{Z}$. There are in all $p-1$ non equivalent but isomorphic cyclic extensions.

The bijection defined in Corollary 2 takes each A to the p-modulo class of A in $\mathbb{Z}/p\mathbb{Z}$. Also note that $A_1 = FA_2G \;\; \forall A_1, A_2 \in [1, p-1]$ for some $F, G \in (\mathbb{Z}/p\mathbb{Z})^{\times}$ which, by Theorem 2, implies that all non-split extensions are cyclic and agrees with the well known result.

Now, consider the extension of $\mathbb{Z}/p\mathbb{Z}$ by D - any finite abelian p-group of rank n. We know that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, D) \cong D/pD$ which is an elementary abelian group of same rank n.

From the Main Theorem, we know that equivalent extensions are in bijection with $1 \times n$ mod p matrices. The explicit bijection is given as follows

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \longrightarrow (a_1, a_2, \ldots, a_n) \in (\mathbb{Z}/p\mathbb{Z})^n \cong D/pD$$

and similarly the inverse.

Acknowledgements

I am extremely grateful to Dr. Amritanshu Prasad for his guidance.

References

[1] Dummit, David S. and Foote, Richard M. *Abstract Algera* Third Edition John Wiley and Sons

[2] Hillar, Christopher J. and Rhea, Darren. *Automorphisms of finite Abelian Groups* arXiv:math/0605185v1 [math.GR]

Weighted Composition Operators and Non-Autonomous Abstract Cauchy Problems

R. K. Singh and V. K. Pandey

Department of Mathematics and Astronomy, Lucknow University, U.P. India

E-mail: rajkishor@hotmail.com, vikashpandeylko@rediffmail.com

Abstract. A characterization of well-posedness of non-autonomous linear abstract Cauchy problems is studied with the help of a c_0-semigroup of weighted composition operators on some Banach spaces of continuous functions.

Keywords. Weighted spaces, weighted composition operators, c_0-semigroup, evolution family of operators, evolution semigroups, abstract Cauchy problems.

2000 Mathematics Subject Classification. Primary 47B37, 47D06; Secondary 47E15, 37B55.

1. Introduction

Let X be Banach space, let Y be a completely regular Hausdorff space, and let V be a system of weights on Y. Then the weighted space $CV_b(Y, X)$ consists of functions $f \in C(Y, X)$ such that vf is a bounded function for every $v \in V$. The weighted space $CV_0(Y, X)$ consists of continuous functions f from Y to X such that vf vanishes at infinity. These spaces are locally convex linear spaces with a family of semi-norms associated with $v \in V$ and sometimes they are Banach spaces. Linear operations on these spaces are defined point-wise. If $\varphi \in C(Y, Y)$ and $w \in C(Y, B(X))$ (or $w \in C(Y, \mathbb{C})$), then the mapping W_φ^w taking f to $w \cdot f \circ \varphi$ is a linear transformation on these weighted spaces and it is called the weighted composition operator induced by the pair (w, φ). Sometimes it is a continuous operator depending on inducing functions w and φ. If $\{A(t) : t \in \mathbb{R}\}$ is a family of linear operators on X, then the initial-value problem $\frac{dx}{dt} = A(t)x(t), x(s) = x_s, s \in \mathbb{R}$ is known as abstract linear non-autonomous Cauchy problem. In case $A(t) = A$ for every $t \in \mathbb{R}$, we call it autonomous abstract Cauchy problem. With some conditions on A the theory of c_0-semigroups of bounded operators provides complete solution of the problem. Stability of solutions or stability of the dynamical system induced by autonomous problem is characterized in term of spectrum of the generator of the evolution semigroup which turns out to be a semigroup of weighted composition operators on $C_0(\mathbb{R}, X)$ [2]. Non-autonomous problem is very difficult to handle, but a semigroup of weighted composition operators known as evolution semigroup of a two-parameter family of bounded operators on X is employed to characterize the well-posedness of abstract Cauchy problems in non-autonomous case. We have presented some preliminary results on weighted spaces of continuous functions and weighted composition operators on them. Every c_0-semigroup on X gives rise to a dynamical system and thus every linear autonomous differential equation gives rise to a dynamical system in light of Hille-Yosida theorem under certain conditions on A. Such type of theorem is not known in non-autonomous case which makes problem very difficult. However, an evolution family of operators gives rise to a c_0-semigroup of weighted composition operators which is used in characterization of well-posedness of non-autonomous differential equations. In this more or less expository article we present interaction between concrete operators of multiplication and composition, dynamical systems and spaces of continuous functions to study some abstract Cauchy problems.

2. Preliminaries

Let Y be a locally compact space. Then a set V of positive real valued upper-semi continuous functions is called a system of weights on Y whenever

(1) For $v_1, v_2 \in V$ and $\alpha > 0$, there exists $v \in V$ such that $\alpha v_i(y) \leq v(y)$ for $y \in Y$ and $i = 1, 2$.
(2) For every $y \in Y$, there exists $v \in V$ such that $v(y) \neq 0$.

If V is the set of all positive functions or V is the set of all positive constant functions or V is the set of characterstic functions of all compact subsets of Y, then in each case V is a system of weights on Y. Let X be a Banach space and let $C(Y, X)$ denote the vector space of all continuous X-valued functions on Y with pointwise vector operations. Let $CV_b(Y, X)$ and $CV_0(Y, X)$ be defined as $CV_b(Y, X) = \{f \in C(Y, X) : vf \text{ is bounded for every } v \in V\}$ and $CV_0(Y, X) = \{f \in C(Y, X) : vf \text{ vanishes at infinity for every } v \in V\}$. Then $CV_b(Y, X)$ and $CV_0(Y, X)$ are linear spaces and $CV_0(Y, X)$ is contained in $CV_b(Y, X)$. If $v \in V$ and $f \in CV_b(Y, X)$, then define semi-norm $\|f\|_v$ as $\|f\|_v = \sup\{v(y)\|f(y)\| : y \in Y\}$. The family $\{\|\cdot\|_v : v \in V\}$ of semi-norms gives a locally convex topology on $CV_b(Y, X)$ and $CV_0(Y, X)$. These spaces are known as the weighted spaces of continuous functions and include a large class of spaces of continuous functions. For example, if V is the system of positive constant functions, then $CV_b(Y, X) = C_b(Y, X)$ and $CV_0(Y, X) = C_0(Y, X)$. In case Y is compact $CV_b(Y, X) = C(Y, X) = C_0(Y, X)$. If $V = \{\lambda v : \lambda \geq 0\}$, where v is a strictly positive continuous function on Y, then $C(Y, X)$ is a Banach space. The weighted space of continuous functions were first introduced by Nachbin [9], and later they were studied by Bierstedt [1], Proalla [13] and Summers [23], where they took X as a locally convex linear topological space and locally convex topology on weighted spaces was determined by V and a family of continuous semi-norms on X. For further details on these spaces we refer to [13,15,16]. These spaces make contact with many areas of mathematics and mathematical sciences like topological dynamics, ergodic theory, abstract Cauchy problems.

By a continuous dynamical system on a Banach space X, we mean a continuous map $\pi : \mathbb{R} \times X \to X$ such that

(1) $\pi(0, x) = x$ for $x \in X$
(2) $\pi(s + t, x) = \pi(s, \pi(t, x))$ for $s, t \in \mathbb{R}$ and $x \in X$.

Here π is called a flow or a motion or an action on X and X is called a phase space or a state space. Usually, the dynamical system is denoted by the triple $(\pi, \mathbb{R}, X)$. For every $t \in \mathbb{R}$, the map $\pi^t : X \to X$ defined by $\pi^t(x) = \pi(t, x)$ is a homeomorphism. If π^t is linear for every $t \in \mathbb{R}$, then $(\pi, \mathbb{R}, X)$ is called a linear dynamical system. In case $\mathbb{R} = \mathbb{R}^+$, we get a continuous semi-dynamical system. If $\mathbb{R}$ is replaced by $\mathbb{Z}$ or $\mathbb{Z}^+$, then $(\pi, \mathbb{Z}, X)$ is called a discrete dynamical system or $(\pi, \mathbb{Z}^+, X)$ is called discrete semi dynamical system. Iterations of a map on X gives rise to discrete dynamical system. By $B(X)$ we denote the Banach algebra of all bounded linear operators on X. By a semigroup of operators on X we mean a continuous map $T : \mathbb{R}^+ \to B(X)$ such that $T(0) = I, T(s + t) = T(s)T(t)$. If T is continuous with respect to strong operator topology on $B(X)$, then it is called a c_0-semigroup of operators. In case T is continuous with respect to norm topology , it is called uniformly continuous semigroup. If $A \in B(X)$, then $T(t) = e^{tA}$ is well known uniformly continuous semigroup and every uniformly continuous semigroup on X is of this type. Let $v(t) = e^{-|t|}$ for $t \in \mathbb{R}$ and let $V = \{\lambda v : \lambda \geq 0\}$. Then V is a system of weights on $\mathbb{R}$ and $CV_0(\mathbb{R}, X)$ is a Banach space. For $t \in \mathbb{R}$, define $T(t) : CV_0(\mathbb{R}, X) \to CV_0(\mathbb{R}, X)$ as $(T(t)f)(s) = f(t + s)$. Then T is a c_0-semigroup of operators on $CV_0(\mathbb{R}, X)$ and plays an important role in study of motion on $CV_0(\mathbb{R}, X)$. Every c_0-semigroup T of operators gives rise to a semidynamical system π_T on X given by $\pi_T(t, x) = T(t)x$ for $t \in \mathbb{R}^+$ and $x \in X$. If T is a c_0-semigroup of operators on X, and if $D = \{x \in X : \lim_{t\to 0} \frac{T(t)x-x}{t}$ exists$\}$, then define $Ax = \lim_{t\to 0^+} \frac{T(t)x-x}{t}, x \in D$. A is called the infinitesimal generator of T and D is called domain of A denoted as $D(A)$. A is densely defined closed linear operator and plays significant role in study of dynamics induced by semigroups or linear autonomous abstract Cauchy problem given by $\dot{x} = Ax, x(0) = x_0$. For details we refer to [2,3,10].

3. Weighted Composition Operators on Weighted Spaces of Continuous Functions

Let Y_1 be a subset of a locally compact Hausdroff space Y, let $\varphi : Y_1 \to Y$ be a continuous map and let $w : Y_1 \to B(X)$ (or $\mathbb{C}$) be a continuous map. For $f \in CV_b(Y, X)$, define the function $w \cdot f \circ \varphi$ on Y as $w \cdot f \circ \varphi(y) = w(y)f(\varphi(y))$ if $y \in Y_1$ and $w \cdot f \circ \varphi(y) = 0$ for if $y \in Y \backslash Y_1$. The map w is called operator-valued (or scalar-valued) weight function on Y_1. It turns out that the map $f \to w \cdot f \circ \varphi$ is a linear transformation on $CV_b(Y, X)$. It is called generalized weighted composition operator on $CV_b(Y, X)$ induced by w and φ and we denote it by W_φ^w. In case $Y_1 = Y$, W_φ^w is called weighted composition operator on $CV_b(Y, X)$. If $w(y) = I$, the identity operator for $y \in Y$, then $W_\varphi^w f = f \circ \varphi = C_\varphi f$, where C_φ is the composition operator on $CV_b(Y, X)$ defined as $C_\varphi f = f \circ \varphi$. If $\varphi(y) = y$, for $y \in Y$, then $W_\varphi^w f = w \cdot f = M_w f$, where M_w is the multiplication operator defined as $M_w f = w \cdot f$. Thus the class of the weighted composition operators on $CV_b(Y, X)$ (or any function space on Y) includes the class of composition operators and the class of multiplication operators. Two simple operations of composition and multiplication of functions give rise to these classes of operators on function spaces. They have played significant roles in characterization of isometries, lattice homomorphisms, disjointness preserving maps [15,17]. Recently they have been used to study dynamical properties of systems given by some differential equations making contact with abstract Cauchy problems. Some evolution semigroups turn out to be semigroups of weighted composition operators [2,8]. The celebrated Banach-Stone theorem says that if S is a surjective linear isometry of $C(Y)$, then $S = W_\varphi^w$, where φ is a homeomorphism on the compact Hausdroff space Y and w is a unimodular scalar-valued function on Y. The vector-valued anologue of Banach-Stone theorem on $C(Y, X)$ was presented by Jerison in 1950. It says that every linear onto isometry on $C(Y, X)$ is a weighted composition operator induced by a continuous map φ and an operator valued map w. Characterization of linear isometries on $CV_b(Y, X)$ and $CV_0(Y, X)$ in terms of weighted composition operators is still not completely resolved. However, in case X is a strictly convex Banach space, then into isometries on $C_0(Y, X)$ are generalized weighted composition operators. From this it follows that every isometry of $C(Y)$, Y a compact Hausdroff space, is a generalized weighted composition operator. For details we refer to [17]. An operator S on $CV_b(Y, X)$ is said to be disjointness preserving if $f \perp g$ implies that $Sf \perp Sg$, where $f \perp g$ means $\{y \in Y : f(y) \neq 0\} \cap \{y \in Y : g(y) \neq 0\} = \phi$. It has been shown that every disjointness preserving linear operator on $CV_0(Y, X)$ is a weighted composition operator [15,21]. This we record in the following theorem

Theorem 3.1. *An operator S on $CV_0(Y, X)$ is disjointness preserving iff $S = W_\varphi^w$, for some continuous maps $\varphi : Y \to Y$ and $w : Y \to B_s(X)$, where $B_s(X)$ denotes $B(X)$ with strong operator topology.*

The lattice homomorphisms are also studied in terms of weighted composition operators. The semigroups of weighted composition operators have also been studied on function spaces of analytic functions and integrable functions making contact with differentiable dynamics and measurable dynamics [6,22]. In this article our interest is only in weighted composition operators on weighted spaces of continuous functions so that we can employ them in study of semigroups induced by evolutionary processes dealing with autonomous and non-autonomous differential equations. If V is a system of weights on Y, and $\varphi : Y \to Y$ is a continuous map, then $V \circ \varphi$ defined as $V \circ \varphi = \{v \circ \varphi : v \in V\}$ is also a system of weights on Y. Let V_1 and V_2 be two systems of weights on Y. Then we say that $V_2 \geq V_1$ if for every $v \in V_1$, there exists $u \in V_2$ such that $v(y) \leq u(y)$ for every $y \in Y$. If $w : Y \to \mathbb{C}$ is a continuous map, then $V \cdot |w| = \{v \cdot |w| : v \in V\}$ is also a system of weights on Y. The following theorem characterizes continuous composition operators and continuous multiplication operators on weighted spaces.

Theorem 3.2. *Let Y be a locally compact space, let $\varphi \in C(Y, Y)$ and let $w \in C(Y, \mathbb{C})$. Let X be Banach space. Then*

(1) *C_φ is a continuous operator on $CV_b(Y, X)$ iff $V \leq V \circ \varphi$.*
(2) *M_w is a continuous multiplication operator on $CV_b(Y, X)$ (or $CV_0(Y, X)$) iff $V \cdot |w| \leq V$.*
(3) *If $w : Y \to B(X)$ is a continuous map, then M_w is a continuous operator on $CV_b(Y, X)$ iff for every $v \in V$ there exists $u \in V$ such that $v(y)\|w(y)x\| \leq u(y)\|x\|$ for every $y \in Y$ and every $x \in X$.*

For proof of these results we refer to [16,19]. In the following theorem we present the characterization of continuous weighted composition operators induced by scalar-valued weight functions [15].

Theorem 3.3.

(1) *Let $\varphi \in C(Y, Y)$ and let $w \in C(Y, \mathbb{C})$. Then the weighted composition operator W_φ^w is continuous on $CV_b(X, \mathbb{C})$ iff $V \cdot |w| \leq V \circ \varphi$.*
(2) *W_φ^w is continuous on $CV_b(Y, X)$ iff $V \cdot |w| \leq V \circ \varphi$.*
(3) *$W_\varphi^w : CV_0(Y, X) \to CV_0(Y, X)$ is continuous iff $V \cdot |w| \leq V \circ \varphi$ and for every $\varepsilon > 0$, $v \in V$ and compact subset K of Y, $\varphi^{-1}(K) \cap \{y \in Y : v \cdot |w|(x) \geq \varepsilon\}$ is compact.*

Let w be an operator-valued weight which is continuous with respect to strong operator topology on $B(X)$. In the following theorem we have results on continuity of weighted composition operators induced by continuous operator-valued weight function.

Theorem 3.4.

(1) *Let $\varphi \in C(Y, Y)$ and let $w \in C(Y, B(X))$. If W_φ^w is continuous from $CV_0(Y, X)$ to $CV_0(Y, X)$, then for every $v \in V$ there exists $u \in V$ such that $v(y)\|w(y)x\| \leq u(\varphi(y))\|x\|$ for $y \in Y$ and every $x \in X$.*
(2) *W_φ^w is continuous on $CV_b(Y, X)$ iff for $v \in V$ there exists $u \in V$ such that $v(y)\|w(y)x\| \leq u(\varphi(y))\|x\|$ for $y \in Y$ and $x \in X$.*

The proof can be seen in [20].

Remark 3.5.

(i) If $w \in C(Y, B(X))$ induces a continuous multiplication operator M_w on $CV_b(Y, X)$ and $\varphi \in C(Y, Y)$ induces a continuous composition operator C_φ on $CV_b(Y, X)$, then the pair (w, φ) induces a continuous weighted composition operator W_φ^w on $CV_b(Y, X)(CV_0(Y, X))$.
(ii) If w is a bounded map and φ induces continuous composition operator, then the pair (w, φ) induces continuous weighted composition operator W_φ^w on $CV_b(Y, X)(CV_0(Y, X))$.
(iii) Let χ_K denote the characterestic functions of set K and let $V = \{\alpha\chi_K : \alpha \geq 0, K \text{ a compact subset of } Y\}$. Then V is a system of weights on Y and $CV_b(Y, X)$ is $C(Y, X)$ with compact open topology. It turns out that every pair (w, φ), where $w \in C(Y, B(X))$ and $\varphi \in C(Y, Y)$

induces a continuous weighted composition operator on $CV_b(Y, X)$. In particular, if Y has discrete topology, then every pair (w, φ) induces a continuous weighted composition operator on $CV_b(Y, X)$, where $V = \{\alpha\chi_K : K$ is finite and $\alpha \geq 0\}$.

Let $\pi : \mathbb{R}^+ \times Y \to Y$ be a semi-dynamical system on Y or a flow on Y. Then for every $t \in \mathbb{R}^+$, the map $\pi_t : Y \to Y$ defined by $\pi_t(y) = \pi(t, y)$ is a continuous map. Under some conditions, the flow π can be lifted to a linear flow $\bar{\pi}$ on weighted space $CV_b(Y, X)$ (or $CV_b(Y, X)$) given by $\bar{\pi}(t, f) = C_{\pi_t} f$, where C_{π_t} is the composition operator induced by π_t. For example if $V \leq V \circ \pi_t$ for every $t \in \mathbb{R}$, then C_{π_t} is a continuous operator and flow $\bar{\pi}$ is well defined [15]. By a scalar-valued cocycle over a flow π, we mean a continuous map $w : \mathbb{R}^+ \times Y \to \mathbb{C}$ such that $w(t+s, y) = w(t, \pi_s(y))w(s, y)$ for $s, t \in \mathbb{R}^+$, $y \in Y$ and $w(o, y) = 1$, for every $y \in Y$. We denote $w(t, y)$ by $w_t(y)$. Thus $W = (w_t : t \geq 0)$ is a family of weights on Y induced by the cocycle w. A cocycle w on Y over π is called a coboundary if $w_t(y) = g(\pi_t(y))/g(y)$ for some continuous functions g on Y such that $g(y) \neq 0$ for every $y \in Y$. A cocycle w is said to be exponentially bounded if there exist $M > 1$ and $\alpha \in \mathbb{R}$ such that $|w_t(y)| \leq Me^{\alpha t}$ for every $t \in \mathbb{R}^+$ and $y \in Y$. If w is a cocycle over a dynamical system π on Y, the pair (w, π) gives rise to a family $\{W_{\pi_t}^{w_t} : t \geq 0\}$ of weighted composition operators on weighted spaces $CV_b(Y, X)$ (or $CV_0(Y, X)$) given by $W_{\pi_t}^{w_t} f = w_t \cdot f \circ \pi_t$. Under some conditions on w and π, $W_{\pi_t}^{w_t}$ is a continuous weighted composition operator on $CV_b(Y, X)$ (or $CV_0(Y, X)$). Define the mapping $E : \mathbb{R}^+ \to B(CV_b(Y, X))$ as $E(t) = W_{\pi_t}^{w_t}$. It turns out that E is a semigroup of weighted composition operators induced by the pair (w, π). If w is a coboundary, then $E(t)$ is similar to C_{π_t}. For details we refer to [5]. If $w : \mathbb{R}^+ \times Y \to B(X)$ is a strongly continuous operator-valued cocycle over flow π on Y such that each π_t defines a continuous composition operator on $CV_b(Y, X)$ (or $CV_0(Y, X)$), then w generates a c_0-semigroup E in case each $w_t : Y \to B(X)$ defines a multiplication operator on weighted spaces. In case X is a Banach algebra, we can get semigroups of weighted composition operators induced by vector-valued cocycle $w : \mathbb{R}^+ \times Y \to X$ over the flow π. Thus every flow on Y can be lifted to several flows on the weighted spaces of continuous functions on Y. Some of these weighted spaces are Banach spaces of continuous functions and lifted flows are dynamical systems induced by c_0-semigroups. Some asymptotic properties in abstract Cauchy problems can be studied with the help of spectral properties of these semigroups. In the next section we shall study some special weighted composition operators on $CV_0(Y, X)$, when V is the system weights on $\mathbb{R}$.

4. Evolution Families and Evolution Semigroups of Weighted Composition Operators

As we know, a semigroup of operators on a Banach space X is a one-parameter family of operators with some conditions and plays significant roles in study of linear autonomous abstract Cauchy problems. In case of linear non-autonomous Cauchy problems, the theory of semigroups is not adequate to find solutions of the problems. In order to attempt a solution of non-autonomous problems a two parameter family is needed. These two parameter families of operators on X, known as evolution families will be studied in this section. These families create semigroups of operators on weighted spaces of continuous functions known as evolution semigroups.

Definition 4.1. *Let X be a Banach space and let $\mathbb{R}^2_{\geq} = \{(s, t) : s, t \in \mathbb{R}$ and $s \geq t\}$. Then $\mathbb{R}^2_{\geq}$ is a semigroup under pointwise addition. A function $U : \mathbb{R}^2_{\geq} \to B(X)$ is an (two-parameter) evolution family of operators on X if*

(i) *$U(s, s) = I$ for every $s \in \mathbb{R}$.*
(ii) *$U(s, r)U(r, t) = U(s, t)$ for $s \geq r \geq t$.*
(iii) *U is strongly continuous.*

U is said to be exponentially bounded if there exist $M \geq 1$ and $\alpha \in \mathbb{R}$ such that $\|U(s, t)\| \leq Me^{\alpha(s-t)}$ for every $(s, t) \in \mathbb{R}^2_{\geq}$. If $T : \mathbb{R}^+ \to B(X)$ is a c_0-semigroup, then define $U_T : \mathbb{R}^2_{\geq} \to B(X)$ by $U_T(s, t) = T(s-t)$. It is evident that U_T is an evolution family. Thus every c_0-semigroup of operators gives rise to an evolution family which is exponentially bounded. If $\lambda \in \mathbb{C}$, and if U is an evolution family of operators on X, then $U_\lambda : \mathbb{R}^2_{\geq} \to B(X)$ defined as $U_\lambda(s, t) = e^{-\lambda(s-t)}U(s, t)$ is also an evolution family.

Definition 4.2 (Evolution semigroups on $CV_0(Y, X)$). *Let $\pi : \mathbb{R} \times Y \to Y$ be a dynamical system and $w : \mathbb{R} \times Y \to B(X)$ be operator-valued strongly continuous cocycle over π such that the pair (w_t, π_t) induces weighted composition* *operator on $CV_0(Y, X)$ for every $t \in \mathbb{R}$. Define the map $E : \mathbb{R}^+ \to B(CV_0(Y, X))$ as*

$$\begin{aligned}(E(t)f)(y) &= w_t(\pi_t^{-1}(y)(f \circ \pi_t^{-1}(y)) \\ &= w_t(\pi_{-t}(y))(f \circ \pi_{-t}(y))\end{aligned}$$

Then E is a semigroup of continuous operators on $CV_0(Y, X)$. E is called Mather evolution semigroup induced by the pair (w, π).

We are interested in the case when $CV_0(Y, X)$ is a Banach space. For examples, if V is the system of all constant weights, then $CV_0(Y, X)$ is the Banach space $C_0(Y, X)$ of all X-valued continuous functions vanishing at infinity; if Y is compact, then $CV_0(Y, X) = C(Y, X)$, is the Banach space of continuous functions from Y to X with sup norm and if $Y = \mathbb{R}$, v is a bounded continuous real-valued functions on $\mathbb{R}$ such that $v(t) > 0$ and $V = \{\lambda v : \lambda \in \mathbb{R}^+\}$, then $CV_0(\mathbb{R}, X)$ is a Banach space. The following theorem shows that Mather semigroup is strongly continuous on $CV_0(Y, X)$.

Theorem 4.3. *Let V be system of weights on Y generated by a continuous bounded function v with $v(y) > 0$ for $y \in Y$ and let w be an exponentially bounded strongly continuous cocycle over a dynamical system π on Y such that $V \leq V \circ \pi_t$ for every $t \in \mathbb{R}$. Then (Mather) evolution semigroup E is strongly continuous on $CV_0(Y, X)$.*

Sketch of the proof. Suppose $w : \mathbb{R}^+ \times Y \to B(X)$ is a strongly continuous and exponentially bounded cocycle. Then there exists $\alpha > 0$ such that $\sup\{\|w_t(y)\| : y \in Y \text{ and } t \in [0, 1]\} \leq \alpha$. Let $f \in CV_0(Y, X)$. Then vf vanishes at infinity. Let $p_v(f) = \sup\{v(y)\|f(y)\| : y \in Y\}$. Then p_v is a norm on $CV_0(Y, X)$ and with this norm it is a Banach space. Now,

$$\begin{aligned}&p_v(E(t)f - f) \\ &\quad= \sup\{v(y)\|w_t(\pi_{-t}(y)(f \circ \pi_{-t})(y) - f(y)\| : y \in Y\} \\ &\quad= \sup v(\pi_t(y))\{\|w_t(y)f(y) - f(\pi_t(y))\| : y \in Y\} \\ &\quad= v \circ \pi_t(y) \sup\{\|w_t(y)f(y) - f(\pi_t(y)\| : y \in Y\} \\ &\quad\leq \|v\|_\infty \sup\{\|w_t(y)f(y) - f(\pi_t(y)\| : y \in Y\}\end{aligned}$$

Since $\{C_{\pi_t} : t \in \mathbb{R}^+\}$ is a c_0-semigroup of composition operators, it can be concluded that for every $\epsilon > 0$ there exists $\delta_\epsilon \in (0, 1)$ such that $\|f(\pi_t(y) - f(y)\| < \epsilon$ for every $y \in Y$ and $|t| < \delta_\epsilon$. Let K_ϵ be a compact subset of Y such that $\|f(y)\| < \epsilon$ for every $Y \notin K_\epsilon$. Since $f(K_\epsilon)$ is a compact subset of Y, there exists finitely many points $y_1, y_2, y_3, \ldots, y_n$ in Y such that $\|f(y) - f(y_i)\| < \epsilon$ for some y_i, whenever $y \in K_\epsilon$. The function $(t, y) \to w_t(y)f(y_i)$ is uniformly continuous on the compact set $[0, 1] \times K_\epsilon$ for $i = 1, 2, 3, \ldots, n$. Let $\delta \in (0, \delta_\epsilon)$ such that for $t \in (0, \delta)$ and $y \in K_\epsilon$, $\|w_t(y)f(y_i) - f(y_i)\| < \epsilon$. If $y \notin K_\epsilon$, then

$$\begin{aligned}&\|w_t(y)f(y) - f(\pi_t(y)\| \\ &\quad\leq \|w_t(y)f(y) - f(y)\| + \|f(y) - f(\pi_t(y)\| \\ &\quad\leq 2\alpha\varepsilon + \epsilon\end{aligned}$$

If $y \in K_\epsilon$, then for $t \in [0, \delta]$,

$$\begin{aligned}&\|w_t(y)f(y) - f(\pi_t(y)\| \\ &\quad\leq \|w_t(y)[f(y) - f(y_i)]\| + \|w_t(y)f(y_i) - f(y_i)\| \\ &\qquad+ \|f(y_i) - f(y)\| + \|f(y) - f(\pi_t(y)\| \\ &\quad\leq 3\varepsilon + \alpha\epsilon\end{aligned}$$

Thus $p_v(E(t)f - f) \leq \|v\|_\infty \max\{3\epsilon + \alpha\epsilon, 2\alpha\epsilon + \epsilon\}$. This shows that E is a c_0-semigroup on $CV_0(Y, X)$. This completes the outline of the proof.

Corollary 4.4.

(i) *If $Y = \mathbb{R}$, $v(y) = 1$ for $y \in Y$, $\pi_t(y) = y+t$, and w is an exponentially bounded cocycle, then E is c_0-semigroup on $C_0(\mathbb{R}, X)$.*

(ii) *If $v(y) = e^{-|y|}$ for $y \in \mathbb{R}$ and $\pi_t(y) = y + t$, then $V \leq V \circ \pi_t$ for every $t \in \mathbb{R}$, then it follows that every exponentially bounded cocycle over π gives a strongly continuous evolution semigroup on $CV_0(\mathbb{R}, X)$.*

(iii) *If $v(y) = e^{-y^2}$ for $y \in \mathbb{R}$, then translations does not induce composition operators on $CV_0(\mathbb{R}, X)$ and hence we do not get evolution semigroups.*

Remark 4.5.

(i) The system of weights play significant role in development of evolution semigroups on weighted spaces of continuous functions. It would be nice to characterize the pair (w, π) of cocycles and flows giving rise to c_0-evolution semigroups on weighted spaces of continuous functions.

(ii) If Y is compact, then all weighted spaces on Y coincide to $C(Y, X)$ and it turns out that every pair (w, π) of cocycle and flows gives rise to evolution semigroup on Banach space $C(Y, X)$.

(iii) If X is a locally convex linear topological space, then $CV_b(Y, X)$ and $CV_0(Y, X)$ are also locally convex spaces, where topology is given by a family of seminorms. It would be worthwhile to study evolution semigroups on these spaces.

(iv) In some cases it turns out that E is c_0-semigroup iff cocycle w over π is exponentially bounded and strongly continuous [2].

Definition 4.6 (Evolution semigroups induced by evolution. families). *Let $U : \mathbb{R}^2_{\geq} \to B(X)$ be an exponentially bounded evolution family. Let $\pi_t : \mathbb{R} \to \mathbb{R}$ be the translation map i.e. $\pi_t(s) = s + t$ and $\pi_{-t} = (\pi_t)^{-1}$. Then π is a flow on $\mathbb{R}$. Let $w : \mathbb{R}^+ \times \mathbb{R} \to B(X)$ be defined as $w(t, s) = U(s, s - t)$. It turns out that w is an exponentially bounded strongly continuous cocycle over π. The evolution semigroup E on $CV_0(\mathbb{R}, X)$ is given by*

$$\begin{aligned} E(t)f(s) &= w_t(\pi_{-t}(s))(f \circ \pi_{-t})(s) \\ &= U(s, s - t)f(s - t) \end{aligned}$$

for $s \in \mathbb{R}$. This evoluiton semigroup of weighted composition operators is induced by family U and we denote it by E_U. We shall record some results about E_U in the following theorem.

Theorem 4.7. *Let $V = \{\lambda v : \lambda \geq 0, v \in C_b(\mathbb{R}), v(s) > 0\}$ and let $U : \mathbb{R}^2_{\geq} \to B(X)$ be exponentially bounded evolution family and π be the flow on $\mathbb{R}$ given by translations such that $v \leq v \circ \pi_t$ for every t. Then*

(1) *$E_U : \mathbb{R}^+ \to B(CV_0(\mathbb{R}, X))$ is a c_0-semigroup and gives rise a dynamical system on $CV_0(\mathbb{R}, X)$.*

(2) *A semigroup T on $CV_0(\mathbb{R}, X)$ is an evolution semigroup iff for every $\varphi \in C_0(\mathbb{R})$, and $f \in CV_0(\mathbb{R}, X)$ we have*

$$(T(t)\varphi f)(s) = \varphi(s - t)(T(t)f)(s)$$

for $s \in \mathbb{R}$.

(3) *Infinitesimal generator A of E_U is given by $Af = Mf - f'$, where M is the infinitesimal generator of semigroup of multiplication operators induced by $U(s, s - t)$.*

Outline of the proof: (i) and (iii)

Let $w : \mathbb{R}^+ \times \mathbb{R} \to B(X)$ be defined as $w(t, s) = U(s, s - t)$. Then w is strongly continuous exponentially bounded cocycle over translations on $\mathbb{R}$. Hence by theorem 4.3 the evolution semigroup E induced by w is a c_0-semigroup on $B(CV_0(\mathbb{R}, X))$. It is clear that $E = E_U$ Thus E_U is a c_0-semigroup, and it generates a dynamical system on $CV_0(\mathbb{R}, X)$. This proves (i). Since E_U is product of c_0-semigroup of multiplication operators and c_0-semigroup of composition operators on $CV_0(\mathbb{R}, X)$, the infinitesimal generator A of E is given by $Af = Mf - f'$, where M is multiplication operator induced by $U(s, s - t)$ [3]. This proves (iii).

(ii) If $T = E_U$ for some evolution family U, then

$$\begin{aligned} (T(t)\varphi f)(s) &= U(s, s - t)(\varphi f)(s - t) \\ &= \varphi(s - t)T(t)f(s - t) \\ &= \varphi(s - t)T(t)f(s) \end{aligned}$$

for every $\varphi \in C_0(\mathbb{R})$ and $f \in CV_0(\mathbb{R}, X)$.

Conversely, Suppose T is a c_0-semigroup satisfying the condition. Then define $U(s, t) = T(s - t)$. Using the condition it can be shown that U gives rise to evolution semigroup E_U and $T = E_U$.

Definition 4.8 (Skew-product flow). *Let $w : \mathbb{R}^+ \times Y \to B(X)$ be a strongly continuous cocycle over a flow π on Y. Define $\hat{\pi} : \mathbb{R}^+ \times Y \times X \to Y \times X$ as $\hat{\pi}(t, (y, x)) = (\pi_t(y), w_t(y)x)$. Then $\hat{\pi}$ is a linear semidynamical system on $Y \times X$ and it is called linear skew-product flow induced by pair (w, π). It follows that $\hat{\pi}_t : Y \times X \to Y \times X$ is continuous iff $w_t : Y \to B(X)$ is strongly continuous for $t \in \mathbb{R}^+$.*

In case Y is compact, every strongly continuous cocycle is exponentially bounded. If X is finite dimensional Banach space, then every strongly continuous cocycle is norm continuous and hence can be extended to $\mathbb{R}$ with cocycle properties. Cocycles appear in solutions of some differential equation of non-autonomous nature. For example, if $\{A(y) : y \in Y\}$ is a family of operators on X, then a strongly continuous cocycle $w : \mathbb{R}^+ \times Y \to B(X)$ over a flow π is said to solve the differential equations $\frac{dx}{dt} = A(\pi_t(y))x, y \in Y, t \in \mathbb{R}$, whenever the function $x(t) = w_t(y)x_y, x_y \in D(A(y))$ is differentiable and satisfies the above differential equation. If $V = \{\lambda\chi_F : F$ a compact subset of $\mathbb{R}$ and $\lambda \geq 0\}$, then $CV_0(\mathbb{R}, \mathbb{R}) = C(\mathbb{R}, \mathbb{R})$ with compact open topology which is metrizable. Let $h : \mathbb{R} \to \mathbb{R}^+$ be uniformly continuous function such that $\lim_{t\to\infty} h(t) \geq \lim_{t\to-\infty} h(t)$. Let Y be closure of the set $\{h \circ \pi_t : t \in \mathbb{R}\}$, where $\pi_t(s) = s + t, s \in \mathbb{R}$. Define the flow $\varphi_t : Y \to Y$ as $\varphi(y)s = y(t + s)$ for $s \in \mathbb{R}$. Define

$w : \mathbb{R}^+ \times Y \to B(X)$ as $w_t(y)x = \exp(\int_0^t y(s)ds)x, x \in X$. Then it turns out that w is strongly continuous cocycle over φ. The linear skew-product flow $\hat{\varphi}_t$ is given by

$$\hat{\varphi}_t(y, x) = (\varphi_t(y), w_t(y)x)$$
$$= \left(y \circ \varphi_t, \exp\left(\int_0^t y(s)ds\right) x\right)$$

for $t \in \mathbb{R}^+$.

5. Non-Autonomous Abstract Cauchy Problems and Evolution Semigroups of Weighted Composition Operators

As we know, the abstract Cauchy problems consist of differential equations with initial conditions. Solutions of these problems are hard to find in general. The theory of operators on Banach spaces interact with many Cauchy problems. We shall present the use of some families of operators in description of solutions of some initial-value problems. A linear homogenous abstract Cauchy problem is an initial-value problem of type $\frac{dx}{dt} = A(t)x(t)$ where for every $t \in \mathbb{R}$, $A(t)$ is an operator (bounded or unbounded) on Banach space X. If $A(t) = A$ for every $t \in \mathbb{R}$, then it is called autonomous linear Cauchy problem, otherwise it is called non-autonomous Cauchy problem. In case of autonomous problem induced by a bounded operator A, define $T : \mathbb{R} \to B(X)$ by $T(t) = e^{tA}$. Then T is a strongly continuous semigroup parameterized by t, and hence gives rise to a dynamical system on X. It turns out that the orbit function of x_0 i.e. $x(t) = T(t)x_0 = e^{tA}x_0$ is a unique solution of autonomous Cauchy problem. If A is unbounded and it is generator of a c_0-semigroup T, then $x(t) = T(t)x_0, x_0 \in D(A)$ is a unique solution of autonomous problem. This exhibits that one-parameter family of operators namely c_0-semigroup on X generates dynamical systems and orbit functions are solutions of linear autonomous differential equations with initial conditions. Thus three concepts namely c_0-semigroups of operators on X, the linear dynamical systems on X and solutions of linear autonomous Cauchy problems are related. In this section we shall assume that $V = \{\lambda v : \lambda \geq 0\}$, where $v \in C_b(\mathbb{R})$ and $v(t) > 0$ for every $t \in \mathbb{R}$. Under this condition $CV_0(\mathbb{R}, X)$ is Banach space. We shall record some results in the following theorem.

Theorem 5.1. *Let X be a Banach space and let A be a densely defined operator (not necessarily bounded) on X such that resolvent set $\rho(A)$ is non-empty. Then*

(1) *The initial-value problem $\frac{dx}{dt} = Ax(t), x(0) = x_0$ has unique solution for every $x_0 \in D(A)$ iff A is an infinitesimal generator of a c_0-semigroup T.*
(2) *If T is a c_0-semigroup of linear operators on X, then $\pi_T : \mathbb{R}^+ \times X \to X$, defined by $\pi_T(t, x) = T(t)x$ is a linear dynamical system iff T is strongly continuous.*

Note.

(1) If T is a c_0-semigroup on X, then semigroup properties and continuity of T implies that the orbit functions $x(t) = T(t)x_0$ are differentiable for $x_0 \in D(A)$ and hence satisfy the initial-value problems.
(2) If $A(t) \in B(X)$ for all $t \in \mathbb{R}$ and the map $t \to A(t)$ is bounded and norm continuous, then the homogenous evolution problem has unique solution for every initial value $x \in X$. See [10] for details.

Let $\{A(t) : t \in \mathbb{R}\}$ be a family of linear operators giving rise to abstract Cauchy problem $\frac{dx}{dt} = A(t)x(t), x(s) = x_s, s \in \mathbb{R}$ and $x_s \in D(A(s))$. Let w be the operator-valued function on $\mathbb{R}$ defined as $w(t) = A(t)$. Let M_w be the multiplication operator on $CV_0(\mathbb{R}, X)$ defined as $D(M_w) = \{f \in CV_0(\mathbb{R}, X) : f(s) \in D(A(s))$ for $s \in \mathbb{R}$ and function $s \to A(s)f(s)$ belongs to $CV_0(\mathbb{R}, X)\}$, $M_w f = wf$, where $(wf)(s) = A(s)f(s))$. If $\{A(t) : t \in \mathbb{R}\}$ is a family of bounded operators and $w : \mathbb{R} \to B(X)$ given by $w(t) = A(t)$ is bounded strongly continuous map, then M_w is a bounded operator on $CV_0(\mathbb{R}, X)$ [15,19]. The following theorem gives a characterization of unbounded multiplication operator on $CV_0(\mathbb{R}, X)$.

Theorem 5.2. *Let E with domain $D(E)$ be an operator on $CV_0(\mathbb{R}, X)$. Then E is a multiplication operator M_w induced by an operator-valued map w iff for every $f \in D(E)$ such that $f(s) = 0$ we have $(Ef)(s) = 0$ (Here E is M_w means $D(E) \subseteq D(M_w)$ and $E = M_w$ on $D(E)$).*

Proof. Let E be a multiplication operator and let $f \in D(E)$ with $f(s) = 0$. Then $(Ef)(s) = (wf)(s) = w(s)f(s) = 0$. Conversely, suppose $(Ef)(s) = 0$ whenever $f \in D(E)$ with $f(s) = 0$. Then for $t \in \mathbb{R}$, define $D(A(t))$ as

$D(A(t)) = \{x \in X :$ there exists $f_x \in D(E)$ with $f_x(t) = x\}$

and define $A(t)x = (Ef_x)(t)$ for all $x \in D(A(t))$. If $f_x(t) = f_y(t) = x$, then $(f_x - f_y)(t) = 0$. Hence by above given property $Ef_y(t) = Ef_x(t)$. Thus $A(t)$ is well defined. Let $w(t) = A(t)$ for $t \in \mathbb{R}$ and let M_w be multiplication operator induced by w. Then for $f \in D(E)$, $M_w f = wf$ and $D(E) \subset D(M_w)$.

Definition 5.3. *Let $\dot{x} = A(t)x(t), x(s) = x_s, t \geq s, t, s \in \mathbb{R}$ be abstract Cauchy problem (ACP), where domain $D(A(t))$ is assumed to be dense and $x_s \in D(A(s))$. Let $U : \mathbb{R}^2_{\geq} \to B(X)$ be an evolution family. Then this evolution family is said to solve the abstract Cauchy problem if for every $s \in \mathbb{R}$, there exist a dense subset $X_s \subset D(A(s))$ such that for every $x_s \in X_s$, the function $x(t) = U(t, s)x_s, t \geq s$ is differentiable with $x(t) \in D(A(t))$ and $\frac{dx}{dt} = A(t)x(t)$ for $t \geq s$. The function $x(t)$ is called a classical solution of ACP and X_s is called regularity subspace. The ACP is said to be well-posed if there exists an evolution family which solves it.*

Remark 5.4.

(i) In case of autonomous Cauchy problems the Hille-Yosida theorem characterizes infinitesimal generators and well-posedness of problems. Such a theorem does not exist in case of non-autonomous Cauchy problems.

(ii) Algebraic properties of a semigroup together with strong continuity imply differentiability, which help in finding solution of problem in autonomous case. This is not true in case of evolution families. For example, if p is nowhere differentiable function on $\mathbb{R}$ with $p(t) \neq 0$ for every $t \in \mathbb{R}$, then define $U(s, t) = \frac{p(s)}{p(t)}$ for $s \geq t$. Then U is uniformly continuous, but not differentiable.

(iii) Well-posedness of ACP is also defined in more general form with regularity subspaces and exponentially bounded solution. For details we refer to [8].

The well-posedness of non-autonomous ACP was characterized in [8] in terms of uniqueness of evolution family. We shall record this result in the following theorem.

Theorem 5.5. *Let $\{A(t) : t \in \mathbb{R}\}$ be a family of linear operators with domain $D(A(t))$ and let $\dot{x} = A(t)x(t), x(s) = x_s$ be abstract Cauchy problem. Then ACP is well-posed iff there exists a unique evolution family $U : \mathbb{R}^2_{\geq} \to B(X)$ solving the ACP.*

Let $U : \mathbb{R}^2_{\geq} \to B(X)$ be an exponentially bounded evolution family and let E_U be evolution semigroup of weighted composition operators on $CV_0(\mathbb{R}, X)$ given by $(E_U(t)f)(s) = U(s, s-t)f(s-t)$. We know that E_U is differentiable on domain $D(G)$ of its generator G, whether U is differentiable or not. On $CV_0(\mathbb{R}, X)$ define the operator $(F, D(F))$ as

$$D(F) = \left\{ f \in CV_0(\mathbb{R}, X) : \lim_{t \to 0} \frac{U(s, s-t) - I}{t} \times f(s) \in C_0(\mathbb{R}, X) \right\}$$

$$(Ff)(s) = \lim_{t \to 0} \frac{U(s, s-t) - I}{t} f(s).$$

If $f(s) = 0$. then $(Ff)(s) = 0$. Hence by Theorem 5.2 F is contained in multiplication operator induced by the family $\{B(t) : t \in \mathbb{R}\}$ of operators on X given by $B(t)x = (Ff_x)(t)$, where $D(B(t)) = \{x \in X$: there exist $f_x \in D(F) : f_x(t) = x\}$. It turns out that

(i) $C^1 \cap D(F) = C^1 \cap D(G)$, where $C^1 = \{f \in CV_0(\mathbb{R}, X) : f, f' \in CV_0(\mathbb{R}, X)\}$.

(ii) $Gf = Ff - f'$.

(iii) $\frac{-\partial}{\partial x} U(t, s)f(s) = U(t, s)[B(s)f(s) - f'(s)]$, for $f \in C^1 \cap D(F)$.

Theorem 5.6. *Let $f \in CV_0(\mathbb{R}, X)$. Then the following are equivalent*

(i) *$E_U(t)f \in C^1 \cap D(G)$ for $t \geq 0$.*

(ii) *For $f \in C^1, U(t, s)f(s) \in D(B(t))$ for all $t \geq s$ and $\frac{\partial}{\partial x} U(t, s)f(s) = B(t)U(t, s)f(s)$ holds for $t \geq s$.*

Proof. **(Outline)**
(i) ⇒ (ii) Let $f \in CV_0(\mathbb{R}, X)$ and let $E_U(t)f \in C^1 \cap D(G)$. Since $C^1 \cap D(G) = C^1 \cap D(F)$, $E_U(t)f \in D(F)$. Thus $F(E_U(t)f) = G(E_U(t)f) + (E_U(t)f)'$. Calculating the terms at right hand side we get

$$\begin{aligned} B(s)(E_U(t))f(s) &= F(E_U(t)f)(s) \\ &= B(s)U(s, s-t)f(s-t) \\ &= \frac{\partial}{\partial x} U(t, s)f(s) \end{aligned}$$

(ii) ⇒ (i) Suppose for $f \in C^1$, $U(t, s)f(s) \in D(B(t))$ for $t \geq s$ and $\frac{\partial}{\partial x} U(t, s)f(s) = B(t)U(t, s)f(s)$. Let $f \in CV_0(\mathbb{R}, X)$. Then from above it follows that $E_U(t)f \in D(F)$ for $t \geq 0$. Thus $E_U(t)f \in D(G)$ for $t \geq 0$. Since $E_U(t)f \in$

C^1, $E_U(t)f \in C^1 \cap D(G)$ for all $t \geq 0$. Thus orbit of f under E_U is contained in $C^1 \cap D(G)$, whenever f is in it.

The evolution semigroups of weighted composition operators play significant role in characterization of well-posedness of non-autonomous ACP. This we shall present in the following theoerem.

Theorem 5.7. *Let $\{A(t) : t \in \mathbb{R}\}$ be a family of linear operators on a Banach space X. Then the non-autonomous abstract Cauchy problem induced by family $\{A(t) : t \in \mathbb{R}\}$ is well-posed with exponentially bounded solutions if and only if there exists a unique evolution semigroup $E : \mathbb{R}^+ \to B(CV_0(\mathbb{R}, X))$ with generator G and an invariant core subset S of $C^1 \cap D(G)$ such that $Gf + f' = M_w f$, $f \in S$, where M_w is the multiplication operator induced by the operator-valued map $w : \mathbb{R} \to B(X)$ given by $w(t) = A(t)$.*

Outline of the proof:

Suppose abstract Cauchy problem induced by the family $\{A(t) : t \in \mathbb{R}\}$ is well-posed. Then there exists a unique evolution family $U : \mathbb{R}^2_{\geq} \to B(X)$ solving the ACP. Let E_U be the evolution c_0-semigroup of weighted composition operators induced by family U. Let G be the generator with domain $D(G)$. Let $s \in \mathbb{R}$ and let θ be infinitely differentiable function on $\mathbb{R}$ with compact support contained in interval $[s, \infty)$. Let $y \in X$ such that $x(t) = U(t, s)y$ for $t \geq s$ solves the abstract Cauchy problem. Define the function f as $f(t) = \theta(t)U(t, s)y, t \geq s$ and $f(t) = 0, t < s$. Then $f \in D(G) \cap C^1$. By computing $E(t)f$ and f', it can be shown that $Gf = -f' + M_w f$. Let D be the span of functions of type $\theta(t)U(t, s)y$, where $s \in \mathbb{R}, \theta \in C_c^\infty(\mathbb{R})$ such that $x(t) = U(t, s)y$ gives a solution of ACP. Then it can be shown that D is invariant under action of E_U and it is dense in $CV_0(\mathbb{R}, X)$. Conversely, suppose there exists a unique evolution semigroup with an invariant core set D contained in $C^1 \cap D(G)$ and $Gf + f' = M_w f$ for $f \in D$. Let $X_s = \{x \in X$: there exists $f \in D$ such that $f(s) = x\}$. This is a dense subspace of X for every $s \in \mathbb{R}$. Let $x \in X_s$. Then the function $u(t) = U(t, s)x$ is a unique solution on $[s, \infty)$. This completes the outline of the proof.

Remark 5.8.

(i) If E_U is evolution semigroup on $CV_0(\mathbb{R}, X)$ such that $E_U(t)$ takes $CV_0(\mathbb{R}, X)$ into $D(F)$ for $t > 0$, then it turns out that $E_U(t)$ takes $D(G)$ into $C^1 \cap D(G)$ and $\frac{\partial}{\partial x} U(t, s)x = B(t)U(t, s)x$ for $x \in X$ and $t \geq s$.

(ii) If $D = D(A(t))$ is dense for every $t \in \mathbb{R}$, $\|R(\lambda, A(t))\| \leq \frac{M}{|\lambda|+1}$ for some $M \geq 1$ and Re$\lambda \geq 0$ and $\|(A(t) - A(s))A(0)^{-1}\| \leq L|t - s|^\alpha$ for some $L \geq 0$, $0 < \alpha \leq 1$ and $t, s \in \mathbb{R}$, then it has been shown that ACP is well-posed. For details see [8,10].

Acknowledgement

The authors are thankful to the Department of Science and Technology, Government of India for providing financial assistance for this work.

References

[1] K. D. Bierstedt, Tensor product of weighted spaces, *Bonner Math. Schriften*, **81** (1975) 25–58.

[2] C. Chicone and Y. Latushkin, Evolution semigroups in dynamical systems and differential equations, *Math. Survey, Monograph No. 70, Amer. Math. Soc. Providence* (1999).

[3] K. J. Engel and R. Nagel, One-parameter semigroups of linear evolution equations, Springer-Verlag, N. Y. Berlin (1999).

[4] E. Hille and R. Phillips, Functional analysis and semigroups, *Amer. Math. Soc.*, Col. 31, Providence (1957).

[5] F. Jafri, T. Tonev and E. Tonev, Automatic differentiability and characterisation of cocycle of holomorphic flows, *Proc. Amer. Math. Soc.*, **133** (2005) 3389–3394.

[6] F. Jafri, T. Tonev, E. Toneva and K. Yale, Holomorphic flows, cocycles and coboundaries, *Michigan Math. J.* **44** (1997) 239–253

[7] J. E. Jamison and M. Rajgopalan, Weighted composition operators on $C(X, E)$, *J. Operator theory*, **19** (1988) 307–317.

[8] Gregor Nickel, Evolution semigroups for non-autonomous Cauchy problems, *Mancrop Publishing, Inc.* (1996) 73–95.

[9] L. Nachbin, Elements of approximation theory, *Math. Studies*, **14**, Van Nostrand, New Jersey (1967).

[10] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Springer-Verlag, N. Y. (1983).

[11] Lawrence Perko, Differential equations and dynamical systems, Springer International edition, New York, Berlin (2004).

[12] Y. Peter, H. Pang and Y. Wang, Cauchy problems for non-autonomous non-linear Schrodinger equations, Science in China (series A), Mathematics, Vol. **48** (2005) 522–538.

[13] J. B. Prolla, Weighted spaces of vector-valued continuous functions, *Ann. Mat. Pure. Appl.*, **89(140)** (1971) 145–158.

[14] George R. Sell and Y. You, Dynamics of evolutionary equations, Springer-Verlag, New York (2002).

[15] R. K. Singh and J. S. Manhas, Composition operators on function spaces, North-Holland publishing co., Amsterdam, London, N. Y. (1993).

[16] R. K. Singh and W. H. Summers, Composition operators on weighted spaces of continuous functions, *J. Austral. Math. Soc. series A*, **45** (1988) 303–319.

[17] R. K. Singh, A survey of weighted substitution operators and generalizations of Banach-Stone theorem, *Inter. J. Math. and Math. Sci.* **6** (2005) 937–948.

[18] R. K. Singh and J. S. Manhas, Multiplication operators and dynamical systems, *J. Austral. Math. Soc. (series A)* **53** (1992) 92–102.

[19] R. K. Singh and J. S. Manhas, Multiplication operators on weighted spaces of vector-valued continuous functions, *J. Austral. Math. Soc. (series A)* **50** (1991) 98–107.

[20] R. K. Singh and Bhopinder Singh, A characterization of weighted composition operators, *Rocky Mountain J. Math.* **23** (1993) 1107–1114.

[21] R. K. Singh and Bhopinder Singh, Lamperti-type operators on a weighted space of continuous functions, *Proc. Amer. Math. Soc.* **125** (1997) 1161–1165.

[22] R. K. Singh, P. Singh and V. K. Pandey, Some concrete examples of c_0-semigroups of operators and their properties, *Math. Student* (to appear).

[23] W. H. Summers, Weighted locally convex spaces of continuous functions, Ph.D. dissertation, Lousiana State University (1968).

[24] H. Tanabe, Equations of evolution, Pitman, Boston, London (1979).

[25] J. A. Walker, Dynamical systems and evolution equations, Plenum Press, New York, London (1980).

Group Homomorphisms on $\mathbb{R}$

M. A. Sofi
Department of Mathematics, Kashmir University Srinagar 19000, India
Email: aminsofi@rediffmail.com

Abstract. The main object of study of the present article is Cauchy's functional equation which is used to show surprising connections between ideas arising in diverse areas of mathematical analysis in an effort to understand certain important features of this simple-minded equation when considered in different settings.

1. Introduction

When considered as a group under addition, a homomorphism on $\mathbb{R}$ is a function $f : \mathbb{R} \to \mathbb{R}$ satisfying:

$$f(x+y) = f(x) + f(y), \text{ for all } x, y \in \mathbb{R}. \qquad (1)$$

This is the so called Cauchy's functional equation which has been an object of intensive study since it appeared for the first time in a paper of Hamel way back in 1905. In fact, Hamel was interested in describing all functions satisfying (1). On the contrary, it is fairly easy to describe all homomorphisms on $\mathbb{R}$ when the latter is treated as a topological group (in its usual topology). In fact, continuity of f leads to the relationship

$$f(x) = cx, x \in \mathbb{R}, \text{ for some } c \in R. \tag{2}$$

Indeed, the relationship (1) immediately leads to

$$f(nx) = nx, \forall x \in \mathbb{R}, \forall n \in \mathbb{Z}$$

so that if $r = \frac{m}{n} \in \mathbb{Q}$ ($n \neq 0$), then

$$nf(rx) = f(nrx) = f(mx) = mf(x)$$

which yields

$$f(rx) = rf(x), \forall x \in \mathbb{R}, r \in \mathbb{Q}.$$

Finally, continuity of f combined with the density of $\mathbb{Q}$ in $\mathbb{R}$ gives

$$f(\alpha x) = \alpha f(x), x \in \mathbb{R}, \alpha \in \mathbb{R}$$

which means that f is an $\mathbb{R}$-linear functional on $\mathbb{R}$ and, therefore, there exists $c \in \mathbb{R}$ such that

$$f(x) = cx, x \in \mathbb{R}.$$

Remark. Equation (1) shows that it suffices to assume the continuity of f at some point of $\mathbb{R}$. We now show that the above result is valid under the more general assumption involving the boundedness of f on some proper interval.

Proposition 1.2. *Given that $f : \mathbb{R} \to \mathbb{R}$ satisfies (1) and is*

(i) *bounded on some proper interval or*
(ii) *$f(x)$ is non-negative for sufficiently small positive values of x, then f satisfies (2).*

Proof. (i) Assume that $|f(x)| \leq M, \forall x \in [a, b] \subset \mathbb{R}$. Fix $x \in [0, b-a]$, so that $x + a \in [a, b]$, we have

$$|f(x+a)| \leq M.$$

Thus, for $x \in [0, c]$ where $c = b - a$, we have

$$|f(x)| \leq |f(x+a) - f(a)| \leq 2M.$$

This implies $|f(x)| \leq 2M, \forall x \in [-c \cdot c]$. Finally, given $x \in \mathbb{R}$ and $n \geq 1$, there exists $r \in \mathbb{Q}$ such that $|x - r| < c/n$. It follows that

$$\begin{aligned} |f(x) - xf(1)| &= |(f(x-r) + (r-x)f(1))| \\ &\leq \frac{2M}{n} + |r-x||f(1)| \\ &\leq \frac{2M}{n} + |r-x||f(1)| \\ &\leq \frac{2M}{n} + \frac{cf(1)}{n}. \end{aligned}$$

Finally, the (RHS) above goes to zero as $n \to \infty$. This gives $f(x) = xf(1), \forall x \in \mathbb{R}$.
(ii) As seen in the Introduction, we have

$$f(\alpha x) = \alpha f(x), x \in \mathbb{R}, \alpha \in \mathbb{Q} \tag{3}$$

Let $s \in \mathbb{R}$ and choose sequences $\{r_n\}, \{R_n\}$ in $\mathbb{Q}$ such that $r_n \to s$, $R_n \to s$ and $r_n \leq s \leq R_n, n \geq 1$. Using (1) and (3) combined with (ii) gives:

$$f(x+y) = f(x) + f(y) \geq f(x)$$

for sufficiently small values of y and this leads to

$$r_n f(1) = f(r_n) \leq f(s) \leq f(R_n) = R_n f(1), \quad \text{for sufficiently large } n.$$

Letting $n \to \infty$ yields $f(s) = sf(1)$.

At this stage, it is natural to wonder if (1) admits solutions which are non-trivial, i.e., solutions which are not of the type (2). Below we show that there indeed are such solutions whose existence, however, depends upon the axiom of choice!

Proposition 1.3. *There exist nontrivial solutions to (1).*

Proof. Choose a Hamel basis $\{x_\alpha\}_{\alpha \in \wedge}$ of $\mathbb{R}$ over $\mathbb{Q}$. Then each $x \in \mathbb{R}$ can be written as

$$x = \sum_{\wedge_0} \lambda_\alpha x_\alpha$$

where $\wedge_0$ is a finite subset of $\wedge$. For each $\alpha \in \wedge$, define $f_\alpha(x) = \lambda_\alpha, x \in \mathbb{R}$. Then f_α satisfies (1). However, f_α is not continuous, for $f_\alpha(0) = 0$, $f_\alpha(x_\alpha) = 1$, and since f_α does not assume irrational values, f_α does not have the intermediate value property.

Remark 1.4.

(a) For f given by the above proposition, it follows that for no $c \in \mathbb{R}$, it holds that $f(x) = cx, \forall x \in \mathbb{R}$.
(b) The nontrivial solutions of (1) as guaranteed by Proposition 1.3 have been used in the solution of Hilbert's third problem involving the equidecomposability of the cube and tetrahedron of the same volume. The negative solution to this problem by Max Dehn in 1902 consists in the construction of the so-called Dehn functional associated to a given nontrivial solution of (1) which assigns equal values to polyhedra that are equidecomposable. The choice

of a suitable solution of Cauchy's functional equation for which the associated Dehn functional assigns a positive number to the tetrahedron of unit volume and zero to the unit cube in $\mathbb{R}^3$ yields a complete solution to Hilbert's third problem,

(c) As an amusing application of Proposition 1.2 guaranteeing the existence of trivial solutions of (1) under the assumption of non-negativity of f for positive small values of x, we can show that the area of a rectangle having sides x and y is indeed equal to xy. Thus, letting $\Delta(x, y)$ denote the area of such a rectangle, we see that

  (i) $\Delta(x, y) \geq 0$.
  (ii) $\Delta(x_1 + x_2, y) = \Delta(x_1, y) + \Delta(x_2, y)$
  (iii) $\Delta(x, y_1 + y_2) = \Delta(x, y_1) + \Delta(x, y_2)$, for all non-negative x, y, x_1, x_2, y_1, y_2.

In view of the above, we see that f_y defined by: $f_y(x) = \Delta(x, y)$ is additive and non-negative for $x \geq 0$. Now Proposition 1.2 applies to conclude that $f_y(x) = \Delta(x, y) = c(y)x$, for some $c(y) \in \mathbb{R}$. By (iii), we see that $c(y_1 + y_2) = c(y_1) + c(y_2)$ which together with (i) and another application of Proposition 1.2 yields that $c(y) = cy$ for some non-negative $c \in \mathbb{R}$. Finally, demanding that the unit square have area equal to 1, we get $c = 1$ and this gives $\Delta(x, y) = xy$.

As a counterpart to Proposition 1.2, we have the following classical theorem of Frechet:

Proposition 1.5. *Given $f : \mathbb{R} \to \mathbb{R}$ satisfying (1) which is Lebesgue measurable, we have $f(x) = cx, x \in \mathbb{R}$, for some $c \in \mathbb{R}$.*

Proof. We can write

$$\mathbb{R} = \bigcup_{n=1}^{\infty} f^{-1}(-n, n).$$

Thus, there exists $\ell \geq 1$ such that $m(f^{-1}(-\ell, \ell)) > 0$, where m denotes the Lebesgue measure. By Steinhauss theorem, there exists $\alpha > 0$ such that

$$(-\alpha, \alpha) \subset f^{-1}(-\ell, \ell) - f^{-1}(-\ell, \ell).$$

Since f is additive, we have

$$f(-\alpha, \alpha) \subset (-2\ell, 2\ell).$$

This yields that f is continuous. For, given $\varepsilon > 0$, choose $N \geq 1$ such that $\frac{2\ell}{N} < \varepsilon$. Now, for $|x| < \delta = \frac{\alpha}{N}$, we have

$$|f(Nx)| = N|f(x)| < 2\ell.$$

This gives

$$|f(x)| < \frac{2\ell}{N} < \varepsilon$$

which means that f is continuous at the origin and hence continuous throughout. In other words, there exists $c \in \mathbb{R}$ such that $f(x) = cx$, for all $x \in \mathbb{R}$.

Remark 1.6.

(a) It is relatively easier to derive the same conclusion under the assumption of Lebesgue integrability of f. (Try!)
(b) The above result yields that the nontrivial solution of (1) as guaranteed by Proposition 1.3 is nonmeasurable (in the Lebesgue sense). However, it is possible to show that there exist invariant extensions of the Lebesgue measure on $\mathbb{R}$ with respect to which these nontrivial solutions are measurable.

As is to be expected, nontrivial solutions of (1) exhibit a very weird behaviour in that their graphs 'fill out' the entire plane in the following sense.

Proposition 1.7. *Given $f : \mathbb{R} \to \mathbb{R}$ satisfying (1), then f is nontrivial if and only if $G(f)$, the graph of f is dense as a subset of $\mathbb{R}^2$.*

Proof. The 'if' part is straightforward: if $G(f)$ is dense, then plainly, f is a nontrivial solution, or else $G(f)$ is a straight line through the origin.

Conversely, assume that f is a nontrivial solution of (1). Put $m = f(1)$. Then there exists $x \in \mathbb{R}$ such that $f(x) \neq mx$. Noting that for $r, s \in \mathbb{Q}$, we have $f(r+sx) = rf(1)+sf(x) = rm + sf(x)$, it follows that $(r + sx, rm + sf(x)) \in G(f)$. But we can write

$$(r + sx, rm + sf(x)) = \begin{pmatrix} 1 & x \\ m & f(x) \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix}$$

and using the fact that $\mathbb{Q} \times \mathbb{Q}$ is dense in $\mathbb{R}^2$, we see that $H = M(\mathbb{Q} \times \mathbb{Q})$, being the image of a dense subset of $\mathbb{R}^2$ under an invertible (linear) map M where M is the 2×2 (non-singular) matrix given above, it follows that $G(f) \subset H$ is dense in $\mathbb{R}^2$. This completes the proof.

At this stage, it is natural to ask for solutions to the following variants of Cauchy's functional equation (1). Thus, given $f : \mathbb{R} \to \mathbb{R}$ which is continuous, describe solutions in each of the following cases:

(i) $f(x+y) = f(x)f(y)$, all $x, y \in \mathbb{R}$.
(ii) $f(xy) = f(x) + f(y)$, all $x, y \in \mathbb{R}$.
(iii) $f(xy) = f(x)f(y)$, all $x, y \in \mathbb{R}$.

Let us treat the case of (i) whereas (ii) and (iii) can be disposed of in a similar fashion. Obviously, we are looking for solutions which are not identically zero. For such solutions f, it follows from: $f(x) = f(x - x_0)f(x_0)$ that $f(x) \neq 0$ for all $x \in \mathbb{R}$ if $f(x_0) \neq 0$ for some $x_0 \in \mathbb{R}$. Further, for $x \in \mathbb{R}$, we see that $f(x) = f\left(\frac{x}{2} + \frac{x}{2}\right) = \left(f\left(\frac{x}{2}\right)\right)^2 > 0$, so that it is legitimate to define $g(x) = \log f(x)$, $x \in \mathbb{R}$. From the given condition on f, it follows that g is a continuous function satisfying (1) and, therefore, can be written as $g(x) = cx$, and this gives $f(x) = e^{cx}$, $x \in \mathbb{R}$.

Remark 1.8.

(a) A similar result holds for measurable f. Further, in the absence of measurability, it still holds that $f(x) = e^{g(x)}$, $x \in \mathbb{R}$, for some additive function $g : \mathbb{R} \to \mathbb{R}$.

(b) It is natural to explore a higher dimensional analogue of (1). In other words, given $f : \mathbb{R}^n \to \mathbb{R}$ satisfying (1), does it follow that, at least under the assumption of continuity, a similar result holds as in the case of $n = 1$. The fact that it is indeed so follows by noting that

$$f(x) = \sum_{i=1}^{n} f_i(x),\ x = (x_i)_{i=1}^{n} \in \mathbb{R}^n,$$

where $f_i : \mathbb{R} \to \mathbb{R}(1 \leq i \leq n)$ is given by:

$$f_i(x) = f(0, 0, \ldots 0, x, 0, \ldots 0),\ \text{with } x \text{ in the } i\text{th place}$$

which is obviously additive because so is f. Thus, as noted in the Introduction, we can write

$$f_i(x) = c_i x,\ x \in \mathbb{R},\ (c_i \in \mathbb{R}).$$

This gives:

$$f(x) = \sum_{i=1}^{n} c_i x_i = \bar{c} \cdot x$$

where $c = (c_1, c_2, \ldots c_n) \in \mathbb{R}^n$ and the right hand side denotes the dot product.

Using a similar argument, it can be proved that if f is bounded on some rectangle in $\mathbb{R}^n$ or if f is measurable on $\mathbb{R}^n$, then additivity of f yields only trivial solutions as in the case of $n = 1$.

2. Complex Case

Let us assume that $f : \mathbb{C} \to \mathbb{C}$ is a function satisfying

$$f(z_1 + z_2) = f(z_1) + f(z_2),\ z_1, z_2 \in \mathbb{C} \tag{4}$$

Besides the trivial solution: $f(z) = cz$ to (4), there also exists a (standard) solution to (4) given by:

$$f(z) = c_1 z + c_2 \bar{z},\ c_1, c_2 \in \mathbb{C}.$$

The fact that these are the only standard solutions of (4) is shown in:

Proposition 2.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be a function satisfying (4). Then, assuming that f is*

(i) *continuous, there exist $c_1, c_2 \in \mathbb{C}$ such that $f(z) = c_1 z + c_2 \bar{z}$, $\forall z_1, z_2 \in \mathbb{C}$.*
(ii) *analytic, there exists $c \in \mathbb{C}$ such that $f(z) = cz$, $\forall z \in \mathbb{C}$.*

Proof. Arguing exactly as in Remark 1.8(b), we can decompose f as a sum of real-valued functions of a real variable $f_{ij} : \mathbb{R} \to \mathbb{R}(1 \leq i, j \leq n)$:

$$f(z) = f_{11}(\text{Re}(z)) + f_{12}(\text{im}(z)) + i(f_{21}(\text{Re}(z))) + f_{22}(\text{im}(z))$$

The continuity and additivity of f yields that each f_{ij} is continuous and additive, so that we have

$$f_{ij}(x) = c_{ij} x,\ x \in \mathbb{R},\ (c_{ij} \in \mathbb{R}).$$

This gives

$$\begin{aligned} f(z) &= c_{11}\text{Re}(z) + c_{12}\text{im}(z) + ic_{21}\text{Re}(z) + ic_{22}\text{im}(z) \\ &= (c_{11} + ic_{21})\text{Re}(z) + (c_{12} + ic_{22})\text{im}(z) \\ &= a\text{Re}(z) + b\text{im}(z) \end{aligned}$$

where a and b denote the complex numbers within the parentheses above. To write $f(z)$ in the manner as indicated in (i), we have

$$
\begin{aligned}
&= f(z)\left(\frac{a+bi}{2}\right)\mathrm{Re}(z) + \left(\frac{a-bi}{2}\right)\mathrm{Re}(z) \\
&\quad - \left(\frac{a+bi}{2}\right) i(\mathrm{im}(z)) + \left(\frac{a-bi}{2}\right) i(\mathrm{im}(z)) \\
&= \left(\frac{a-bi}{2}\right)(\mathrm{Re}(z) + i(\mathrm{im}(z))) \\
&\quad + \left(\frac{a+bi}{2}\right)(\mathrm{Re}(z) - i(\mathrm{im}(z))) \\
&= c_1(z) + c_2\bar{z}, \quad \text{where } c_1 = \frac{a-bi}{2}, c_2 = \frac{a+bi}{2}
\end{aligned}
$$

(ii) Using analyticity of f, we can differentiate equation (3) on both sides w.r.t. z_1 to get

$$f'(z_1 + z_2) = f'(z_1), z_1, z_2 \in \mathbb{C}.$$

This gives:

$$f'(z) = c, \forall z \in \mathbb{C}$$

which yields that $f(z) = cz + d(c, d \in \mathbb{C})$. Since $f(0) = 1$, we get $d = 0$ and so

$$f(z) = cz, \forall z \in \mathbb{C}, \ (c \in \mathbb{C}).$$

3. Restricted Domain

Throughout the previous considerations, the domain of definition of f for which the results remain valid has all along been the entire real line $\mathbb{R}$ or $\mathbb{R}^n$. In this section, we shall deal with situations in which the domain of f is restricted to be a finite interval in $\mathbb{R}$ (or a rectangle in $\mathbb{R}^n$) or the equation (1) is assumed to hold almost everywhere (a.e.). It is indeed surprising that the solutions of (1) (or variants thereof) that result under these weaker conditions are again the trivial ones! As a sample of the type of results belonging to this circle of ideas, we have the following.

Theorem A. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is a measurable function such that (1) holds a.e. on $\mathbb{R}$. Then f is trivial almost everywhere: $f(x) = cx, x \in \mathbb{R}$, ae for some $c \in \mathbb{R}$.*

Theorem B. *Let $f : [a, b] \to \mathbb{R}$ be such that (1) holds for all $x, y \in [a, b]$ such that $x + y \in [a, b]$. Assuming that f is bounded. Then f is trivial.*

We shall not provide a complete proof of these statements which, however, do not follow from suitable generalisations of the ideas involved in the proof of analogous results where no restrictions on the domain of definition are involved. However, let us briefly indicate what exactly is involved in the proofs of these statements. In the case of Theorem (A) which answers in the affirmative an old problem of P. Erdos, it turns out that a far more general result is valid in which f is assumed to act between (not necessarily abelian) groups equipped with what are called '*proper linearly invariant ideals*' (PLI ideals, for short). Before we explain the ideas involved in this more general approach to a proof of Theorem (A), let us see how we can use it to characterize multiplicative linear functional on $L_1(\mathbb{R})$, the Banach algebra of all (complex-valued) Lebesgue integrable functions on $\mathbb{R}$ equipped with the convolution product

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy, x \in \mathbb{R}.$$

Let us recall that a multiplicative linear functional (mlf)- also called a complex homomorphism- on a (complex) Banach algebra A is a linear functional $f : A \to \mathbb{C}$ such that $f(xy) = f(x)f(y)$, for all $x, y \in A$. We shall use the fact that an mlf on $L_1(\mathbb{R})$ is given by

$$\varphi_t(f) = \int_{-\infty}^{\infty} f(x)e^{itx}dx, \ f \in L_1(\mathbb{R}), \ \text{for some } t \in \mathbb{R}. \quad (5)$$

Indeed, φ_t is multiplicative, for if $f, g \in L_1(\mathbb{R})$, then Fubini's theorem combined with translation invariance of Lebesgue measure yields

$$
\begin{aligned}
\varphi_t(f * g) &= \int_{-\infty}^{\infty} f * g(x)e^{itx} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)g(y)e^{itx}dydx \\
&= \int_{-\infty}^{\infty} g(y)e^{ity}dy \int_{-\infty}^{\infty} f(x-y)e^{it(x-y)}dx \\
&= \int_{-\infty}^{\infty} g(y)e^{ity}dy \int_{-\infty}^{\infty} f(x)e^{itx}dx \\
&= \varphi_t(f)\varphi_t(g).
\end{aligned}
$$

Conversely, we have the following:

Theorem 3.1. *Given an mlf φ on $L_1(\mathbb{R})$, there exists $t \in \mathbb{R}$ such that (5) holds.*

Proof. By the Reisz Representation Theorem, there exists $h \in L_\infty(\mathbb{R})$ such that

$$\varphi(f) = \int_{-\infty}^{\infty} f(x)h(x)dx,\ f \in L_1(\mathbb{R}).$$

For f and $g \in L_1(\mathbb{R})$, Fubini's theorem yields

$$\begin{aligned}\varphi(f * g) &= \int_{-\infty}^{\infty} f * g(x)h(x)dx \\ &= \int_{-\infty}^{\infty} h(x)\left[\int_{-\infty}^{\infty} f(x-y)g(y)dy\right]dx \\ &= \int_{-\infty}^{\infty} g(y)\left[\int_{-\infty}^{\infty} h(x)f(x-y)dx\right]dy \\ &= \int_{-\infty}^{\infty} g(y)\left[\int_{-\infty}^{\infty} h(u+y)f(u)du\right]dy\end{aligned}$$

which can be written as a double integral over $\mathbb{R}^2$ to get

$$\varphi(f * g) = \iint f(u)g(y)h(u+y)dudy.$$

On the other hand, we have

$$\begin{aligned}\varphi(f * g) = \varphi(f)\varphi(g) &= \int_{-\infty}^{\infty} f(u)h(u)du \int_{-\infty}^{\infty} g(y)h(y)dy \\ &= \iint f(u)g(y)h(u)h(y)dudy\end{aligned}$$

so that, upon comparison, we get

$$\iint f(u)g(y)[h(u+y) - h(u)h(y)]dudy = 0,$$

for each $f, g \in L_1(\mathbb{R})$.

From this, we deduce that

$$h(u+y) = h(u) + h(y), \quad \text{for almost all } u, v \in \mathbb{R} \qquad (6)$$

The last conclusion follows from the fact that the set of all linear combinations of all the products fg for $f, g \in L_1(\mathbb{R})$ is dense in $L_1(\mathbb{R})$ by virtue of generalised Stone-Weirstrauss theorem. Using the argument employed in the paragraph following Corollary 3.8, it follows that (6) leads to $h(x) = e^{\lambda x}$, $x \in \mathbb{R}$, *a.e.* Finally, in view of h being (essentially) bounded, it follows that λ is purely imaginary: $\lambda = it$, $t \in \mathbb{R}$ and, therefore, (5) holds.

Remark 3.2. It is interesting to observe that the above proof can be slightly modified to yield that the only real mlf on (real) $L_1(\mathbb{R})$ is given by:

$$\varphi(f) = \int_{-\infty}^{\infty} f(x)dx,\ f \in L_1(\mathbb{R}).$$

We shall now briefly indicate the ideas involved in the proof of Erdos's question by R. Ger which is valid in a setting that is far more general than that provided by $\mathbb{R}$. To state this theorem in its utmost generality, we need a few definitions.

Definition 3.3. *Given a set X, a non-empty family F of subsets of X is called an ideal if it satisfies the following conditions:*

(i) $A \in \mathcal{F}$ *and* $B \subset A \Rightarrow B \in \mathcal{F}$.
(ii) $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$.

We shall also say that $\mathcal{F}$ is a σ-ideal if $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$, whenever $A_n \in \mathcal{F}, n \geq 1$. Further, a σ-ideal is said to be proper if $X \notin \mathcal{F}$.

Definition 3.4. *An ideal (resp. σ-ideal) $\mathcal{F}$ on a group G is said to be linearly invariant if for all $A \in \mathcal{F}$ and $x \in G$, $x - A \in \mathcal{F}$.*

Definition 3.5. *Given ideals $\mathcal{F}_1$ and $\mathcal{F}_2$ on X and $X \times X$, respectively, we say that $\mathcal{F}_1$ and $\mathcal{F}_2$ are conjugate if for all $A \in \mathcal{F}_2$, we have*

$$A(x) = \{y \in X; (x, y) \in A\} \in \mathcal{F}_1$$

for all $x \in X$, $\mathcal{F}_1$-a.e. In other words, there exists $B \in \mathcal{F}_1$ such that $A(x) \in \mathcal{F}_1$, for all $x \in\in B^c$.

Definition 3.6. *Let $f : G_1 \to G_2$ be a function acting between groups G_1 and G_2 and let and $\mathcal{F}_1, \mathcal{F}_2$ be PIL ideals on G_1 and $G_1 \times G_1$, respectively. We say that f is $\mathcal{F}_2$-almost additive if*

$$f(x+y) = f(x) + f(y)$$

holds $\mathcal{F}_2$-a.e. on $G_1 \times G_1$.

The following result which answers Erdos's question in the affirmative was proved by N.G. de Brujn and generalised to (noncommutative) groups by R. Ger [1].

Theorem 3.7. *Let G_1 and G_2 be groups and assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ are given conjugate PLI ideals on G_1 and $G_1 \times G_1$, respectively. If $f : G_1 \to G_2$ is an $\mathcal{F}_2$-almost additive function, then there exists a unique homomorphism $g : G_1 \to G_2$ such that $f = g$, $\mathcal{F}_1$-ae on G_1.*

Rather than give a complete proof of this result which is rather long but elementary, we shall settle for providing a simple argument for the proof of Theorem (A) based on the above theorem which will completely settle Erdos's question as mentioned earlier.

Corollary 3.8. *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that (1) holds almost everywhere on $\mathbb{R}$. Then there exists $c \in \mathbb{R}$ such that $f(x) = cx$, for all $x \in \mathbb{R}$, ae.*

Proof. Taking $G_1 = G_2 = \mathbb{R}$ and $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) to be the collection of all subsets of $\mathbb{R}$ (resp. $\mathbb{R}^2$ consisting of Lebesgue null sets, which are obviously PIL ideals conjugate to each other, we are in the situation of Theorem 3.7 to conclude that there exists a (unique) additive function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = g(x)$, a.e. on $\mathbb{R}$. However, measurability of f then yields the measurability of g, so that Theorem 1.5 gives $g(x) = cx$ for all $x \in \mathbb{R}$. In conclusion, we have $f(x) = cx$, for all $x \in \mathbb{R}$, a.e. and this completes the argument.

An analogous result for Cauchy's exponential functional equation, i.e, $f(x + y) = f(x)f(y)$, $x, y \in \mathbb{R}$, a.e follows on similar lines.

We will now sketch a proof of Theorem (B) which doesn't, however, follow from a direct generalisation of the method of proof of Proposition 1.2(i) as it is easily seen to break down for functions acting on bounded domains. Our proof, however, makes use of some fundamental facts from noncommutative measure theory including, in particular, a suitable variant of the so-called Mackey-Gleason theorem which identifies a large class of von Neumann algebras A such that each finitely additive measure on the projection lattice of A admits an extension to A. An important and crucial ingredient of the proof of this theorem is the following statement involving the boundedness of frame functions which is actually used in the proof of Theorem (B).

Theorem 3.9 (See [2], Theorem 3.1.4). *A bounded frame function on a Hilbert space H with* $\dim H \geq 3$ *is regular.*

Here the term *frame function* on H refers to a function $f : S_H \to \mathbb{R}$ (defined on the unit sphere S_H of H) such that

$$\sum_{i \in \wedge} f(x_i) = \sum_{i \in \wedge} f(y_i)$$

for all o.n systems $\{x_i\}$, $\{y_i\}$ in H. Also, f is said to be *regular* if there exists a bounded Hermitian nuclear operator $T : H \to H$ such that $f(x) = \langle Tx, x\rangle$, $x \in H$.

Clearly, a regular frame function is always bounded. The theorem stated above says that converse holds for Hilbert spaces of dimension at least 3. For a thorough discussion of frame functions and their role in the proof of Mackey-Gleason theorem, see [2], Chapter 3. Basing our argument on Theorem 3.9, we are now ready for a

Proof of Theorem (B). It suffices to show that, under the assumptions of Theorem 3.2, f is continuous. To this end, consider the 3-dimensional Hilbert space $H = \mathbb{R}^3$ and fix a unit vector $e \in H$. It follows that the mapping $g : S_H \to [0, 1]$ given by

$$g(x) = |\langle e, x\rangle|^2, x \in S_H$$

is surjective. Indeed, choose $e_1 \in S_H$ such that $\langle e, e_1\rangle = 0$ and fix $t \in [0, 1]$. Now, for $\theta \in \left[0, \frac{\pi}{2}\right]$ such that $t = \cos^2\theta$, we see that $x_\theta = \cos\theta \cdot e + \sin\theta \cdot e_1 \in S_H$ and, therefore, $g(x_\theta) = |\langle e, x_\theta\rangle|^2 = t$. We now define $F : S_H \to \mathbb{R}$ by $F(x) = f(|\langle e, x\rangle|^2)$. Clearly, F is a frame function which is bounded in view of the assumed boundedness of f and, therefore, we are in the situation of Theorem 3.9. Thus, it follows that F is regular and, in particular, continuous. To conclude that f is continuous, let $t_n \to t$ in [0, 1]. By the surjectivity of g, there exist θ_n and $\theta \in \left[0, \frac{\pi}{2}\right]$ such that $t_n = |\langle e, \theta_{n'}\rangle|^2$, $t = |\langle e, \theta\rangle|^2$ and $\theta_n \to \theta$. But then $x_{\theta_n} \to x_\theta$ and, therefore, $F(x_{\theta_n}) \to F(x_\theta)$. Putting all these facts together yields that $f(t_n) \to f(t)$ and this completes the proof.

4. Cauchy's Functional Equation in Infinite Dimension

In this section, we shall consider functions f acting on or taking values in an infinite dimensional Banach space. Among other things, here we shall explore the extent to which the validity of Cauchy's functional equation impacts on the geometry of the space in question. We shall also address questions arising in this setting which are motivated by results proved in the previous sections in the scalar case. Thus, let X be a (real) Banach space and $f : X \to \mathbb{R}$ an additive function. As in the case of $X = \mathbb{R}$, continuity of f readily leads to f being linear(and conversely). As regards the Banach space analogue of Proposition 1.5, asserting the continuity of additive functionals f under the assumption of Lebesgue measurability, we shall prove a far-reaching generalisation of this fact in infinite dimension. But before we do that, let us pause to address the question of the existence of additive functional $f : X \to \mathbb{R}$ which are not continuous. The surprising fact is that such functions do not merely exist -even in the infinite-dimensional setting- but they exist in abundance- even in the finite-dimensional case! This statement is made precise in the following

Theorem 4.1. *Given a normed linear space X, the set of additive functional on X which are not continuous (resp. linear) contains, together with the identically zero functional- a subspace which is uncountable dimensional.*

Proof. The construction of such functions follows essentially on the lines of proof of Proposition 1.3. In fact, a Hamel basis $\mathcal{B}$ of X over $\mathbb{Q}$ is obtained by 'gluing' together a Hamel basis H of X over $\mathbb{R}$ with a Hamel basis K of $\mathbb{R}$ over $\mathbb{Q}$. Precisely, we can take $\mathcal{B} = \{kh : k \in K, h \in H\}$ and then define for each $b \in \mathcal{B}$, the function $f_b : X \to \mathbb{Q}$ by

$$f_b(x) = \begin{cases} 1, x = b \\ 0, x \neq b \end{cases}$$

which is then extended ($\mathbb{Q}$-linearly) to the whole of X, denoted again by f_b. It is easily checked that f_b is additive which cannot be $\mathbb{R}$-linear and for the same reason, each f_b is discontinuous. The fact that the set $\{f_b : b \in \mathcal{B}\}$ is linearly independent is straight-forward. We complete the proof by showing that the space $S = \text{span}\{f_b : b \in \mathcal{B}\}$ consists entirely of discontinuous functions. To this end, take $F \in S$. Then there exist $k \geq 1$ and a sequence $\{\lambda_i\}_{i=1}^k$ in $\mathbb{Q}$ such that for each $x \in X$,

$$F(x) = \sum_{i=1}^{k} \lambda_i f_{b_i}(x).$$

Assuming that F is continuous so that, in particular, F restricted to span $\{b_1\}$-denoted by F_1-is also continuous, it follows that F_1 is linear and hence vanishes only at zero. On the other hand, we can write $b_1 = kh, k \in K, h \in H$ and this gives $Kh \subset \text{span}\{b_1\}$. By considering the (uncountably) infinite set $Kh \backslash \{b_i\}_{i=1}^k$, we notice that F_1 vanishes on this entire set which contradicts the above assertion and this completes the proof.

Before we take up the issue involving the behaviour of an additive function having as its domain an infinite dimensional Banach space, let us note that for functions $f : \mathbb{R} \to \mathbb{R}$, the trivial solutions to (1) are precisely those which are linear, or equivalently, continuous. However, in the infinite dimensional case, these two notions are easily seen to be independent of each other. This motivates the question of describing sufficient conditions that would ensure the continuity of an additive function f acting between infinite dimensional Banach spaces and this would lead to a sample of results that can be looked upon as infinite dimensional analogues of results proved in the preceding sections in the scalar case. For functions acting on Euclidean spaces, the following statement, due originally to Sierpinski, is the archetype of such results.

Theorem 4.2. *Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is Lebesgue integrable which is mid-point convex: $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$, for all $x, y \in \mathbb{R}^n$. Then f is continuous (and convex).*

A far reaching generalisation of the above result was proved in 1980 by P. Fischer and Z. Slodkowski in which the mid-term convex function f is allowed to act upon a real linear Polish space and is assumed to be 'measurable in the sense of Christensen'.

For mappings acting between Banach spaces we have the following theorem which is the main result of this section.

Theorem 4.3. *Let $T : X \to Y$ be a linear mapping acting on a Banach space X and taking values in a normed space Y. Assume that T is Borel measurable. Then T is continuous.*

Proof. It suffices to consider the statement for a seminorm defined on X and show that it is continuous under the assumption of Borel measurability. Indeed, for a given Borel measurable linear map $T : X \to Y$, the function $p(x) = \|T(x)\|$ defines a seminorm on X which is Borel measurable and, therefore, continuous by the given assumption. In other words, there exists $c > 0$ such that

$$\|T(x)\| = p(x) \leq c\|x\|, \quad \forall x \in X.$$

which shows that T is continuous. For the proof, we shall make use of the following theorem of Steinhauss on sets of positive measure in a (locally) compact group.

$(*)$ Let G be a (locally) compact group equipped with a left-invariant Haar measure μ and let $S \subset G$ be a Borel set with $\mu(S) > 0$. Then $S - S$ is a neighbourhood of the origin.

Proof of (*): Let χ_S denote the characteristic function of S. Then the convolution $\chi_{-S} * \chi_S$ is a continuous function which vanishes outside of $S - S$. Now, $\chi_{-S} * \chi_S(0) = \int \chi_{-S}(x)\chi_S(x)d\mu(x) = \mu(S) > 0$, so by virtue of continuity of the convolution map at 0 (prove it!), it follows that $S - S$ contains a neighbourhood at the origin.

Let us consider the auxiliary object $C = \{0, 1\}^{\mathbb{N}}$, the countable product of the group $\mathbb{Z}_2 = \{0, 1\}$ which is a compact group in its product topology, with a local base at the origin being given by $U_n = \{x_1, x_2, \ldots, x_n, \ldots); x_i \in \mathbb{Z}_2, i \geq 1,$

$x_i = 0, \forall i = 1, \ldots, n\}$, $n \geq 1$. To show that p is continuous, assume the contrary and pick $h_n \in X$ such that

$$\|h_n\| = 1,\ p(h_n) > n^3,$$

or, without loss of generality,

$$\|h_n\| = \frac{1}{n^2},\ p(h_n) > n,\ \forall n \geq 1.$$

Define $S : C \to X$ by

$$S(x) = \sum_{n=1}^{\infty} x_n h_n, x = (x_n) \in C.$$

Because X is a Banach space, S is a well-defined additive map which is continuous. Therefore, poS is Borel measurable mapping C into $\mathbb{R}$. For $k \geq 1$, let $C_k = \{x \in C;\ poS(x) \leq k\}$, which is a Borel subset of C such that $C = \cup_{k=1}^{\infty} C_k$. Thus, there exists $k_0 \geq 1$ such that $C_0 = C_{k_o}$ has positive Haar measure. Now Steinhauss's theorem $(*)$ applies to yield $j \geq 1$ such that $U_j \subset C_0 - C_0$. Letting δ_k denote the element of C which has 1 at the kth place and zero everywhere else, we see that $\delta_k \in U_j$, for all $k \geq j$. Thus we can write $\delta_k = x - y, x \cdot y \in C_0$, for all $k \geq j$. This yields, in particular, that x and y have the same components, except at the kth place where they differ. Since at the co-ordinate level, action takes place in $\mathbb{Z}_2$, we can assume, without loss of generality, that $x_k = 1$ and $y_k = 0$. This give: $S(x - y) = S(x) - S(y)$ and since $x, y \in C_0$, we get for each $k \geq j$,

$$p(S(\delta_k) = p(S(x - y)) = p(S(x) - S(y))$$
$$\leq p(S(x)) + p(S(y)) \leq 2k_0.$$

Finally, noting that $S(\delta_k) = h_k$ and that $p(h_k) > k$, we arrive at a contradiction. Therefore p, and hence, T is continuous. This completes the proof.

Let us conclude with an amusing but a straightforward application of the above theorem to a category-independent proof of Banach-Steinhauss theorem which is invariably proved using Baire's category theorem.

Corollary 4.4. *Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of bounded linear operators defined on a Banach space X and taking values in a normed space Y. Assume that $T_n(x) \to T(x)$ in Y for each $x \in X$. Then T is bounded.*

Proof. Since each T_n is linear, so is T. Being continuous, each T_n is Borel measurable and so is the pointwise limit T. By Theorem 4.3, T is continuous.

Remark 4.5. A locally convex analogue of Theorem 4.3 is also valid as long as T is assumed to be measurable with respect to the completion of each Radon Gaussian measure on X.

Remark 4.6. Theorem 4.3 is reminiscent of the well known measure-category duality that constitutes the main theme of a famous book on the subject "Measure and Category" by J. C. Oxtoby. On the other hand, this result belongs to the vast pool of results that one comes across in the theory of 'automatic continuity', of which the well-known fact that each mlf on a Banach algebra is continuous, is a typical and one of the first examples of this phenomenon. The following theorem due to J. Kuznetsova is an important recent contribution to this theme of automatic continuity (see: arXiv:1010.0999v1 [math.FA], Oct.5, 2010).

Theorem 4.7. *A unitary representation of a locally compact group which is weakly measurable is automatically continuous.*

We conclude this section with the following result (without proof) which gives an idea of how the geometry of a Banach space is impacted by the behaviour of Cauchy's equation in that space.

Theorem 4.8. *A Banach space X is strictly convex if and only if for each continuous function $f : \mathbb{R} \to X$ such that $\|f(x + y)\| = \|f(x) + f(y)\|$, for all $x, y \in \mathbb{R}$, it holds that $f(x + y) = f(x) + f(y)$, for all $x, y \in \mathbb{R}$.*

Here, strict convexity refers to the geometric property of the unit sphere that each unit vector is an extreme point in the sense that it doesn't lie on a line segment that is completely contained in the unit sphere.

5. Concluding Remarks

In the preceding sections, we have described sufficient conditions on an additive function $f : X \to Y$ that would ensure the continuity of f which, in the case of $X = Y = \mathbb{R}$ leads to f being trivial. As noted previously, the same holds if f is assumed to be linear. However, in the setting in which X and Y are assumed to be infinite dimensional Banach spaces, it makes little sense to prescribe conditions guaranteeing that an additive function is linear. Instead, the idea of 'approximating' a given mapping which is 'nearly additive' in a certain sense by a linear map turns out to be extremely useful. The consideration of such approximation schemes leads to the theory of stability of functional equations which has far reaching implications in Banach space theory. Here, '*approximation*' (of f by g) is meant in the sense that

$$d(f, g) = \sup_{x \in B_X} \|f(x) - g(x)\| < \infty.$$

In this case, we say that the map f is *asymptotically additive* (*linear*) if g is additive (linear). Let us say that f is *quasi-linear* if f is homogeneous and

$$\begin{aligned}&\|f(x+y) - f(x) - f(y)\| \\ &\quad \leq K(\|x\| + \|y\|), \text{ for all } x, y \in X.\end{aligned}$$

The question involving the asymptotic linearity of quasi-linear maps leads to the rich study of the splitting of short exact sequences of Banach spaces and consequently, of the so-called K-*spaces* which arise naturally in this study as the class of those Banach spaces X such that whenever the quotient of a quasi Banach space Z by $\mathbb{R}$ is isomorphic with X, then Z contains $\mathbb{R}$ as a complemented subspace (equivalently Z is a Banach space). Let us recall that a quasi Banach space is defined by the same set of axioms as do a Banach space except that the axiom of triangular inequality comes with a positive constant (which is equal to 1 for Banach spaces). It is surprising that such a mild weakening of the said axiom leads to a theory which is, in certain situations, in stark contrast with that of Banach spaces- the failure of Hahn Banach theorem in the quasi-Banach space setting is one important example of this stark contrast! Typical examples of quasi Banach spaces include L_p or ℓ_p spaces for $0 < p < 1$. On the other hand, the class of K-spaces includes, besides Hilbert spaces, the space c_0 of all null sequences, the space $L_\infty(\Omega)$ of essentially bounded measurable functions and the so-called B-convex Banach spaces, even though it turns out to be a highly nontrivial fact that the space ℓ_1 of absolutely summable sequences does not belong to this class. A celebrated open problem belonging to this circle of ideas asks if K-spaces are precisely those Banach spaces X whose duals have finite cotype. It is intended to talk about these and related issues in a forthcoming article that is expected to appear subsequently.

References

[1] R. Ger, Notes on almost additive functions, *Aequationes Math.* **17**, no. 1, 73–76 (1978).

[2] J. Hamhalter, Quantum measure theory, Kluwer Acad. Publishers (2003).

21st Annual Conference of Jammu Mathematical Society

February 25–27, 2011

Venue: University of Jammu, India.

Description: The 21st Annual Conference of Jammu Mathematical Society (JMS) and a National Seminar on "Analysis and its Applications" will be held on February 25–27, 2011. The programme will consist of plennary lectures of 50 minutes, short talks of 30 minutes and 20 minutes on contributed papers. We also request the life members of the JMS that they may please send their recent correspondence address, e-mail and phone number so that we could inform them about the activities of the Society from time to time. For further details about the activities of the Society.

Contact Address:

Professor S. D. Sharma, (President of JMS)
Dr Romesh Kumar (Secretary, JMS)
Department of Mathematics,
University of Jammu, Jammu 180 006, (India)
E-mail: `jammumathsoc@gmail.com`
Alternative e-mail: `romeshkumar@jammuuniversity.com`
Website: `www.jammumathsoc.org`

Workshop Discrete, Tropical and Algebraic Geometry

(May 5–7, 2011)

Venue: Goethe University, Frankfurt am Main, Germany.

Description: The goal of this workshop is to bring researchers together to discuss current developments in discrete, tropical and algebraic geometry, and also to attract younger people to these areas. For further information visit:

`http://www.math.uni-frankfurt.de/geometry2011/`

SIAM Conference on Optimization

(May 16–19, 2011)

Venue: Darmstadtium Conference Center, Darmstadt, Germany.

Description: The SIAM Conference on Optimization will feature the latest research in theory, algorithms, and applications in optimization problems. In particular, it will emphasize large-scale problems and will feature important applications in networks, manufacturing, medicine, biology, finance, aeronautics, control, operations research, and other areas of science and engineering. The conference brings together mathematicians, operations researchers, computer scientists, engineers, and software developers; thus it provides an excellent opportunity for sharing ideas and problems among specialists and users of optimization in academia, government, and industry. For further information visit:

`http://www.siam.org/meetings/op11/`

IMA Hot Topics Workshop: Strain Induced Shape Formation: Analysis, Geometry and Materials Science

(May 16–20, 2011)

Venue: Institute for Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, Minnesota.

Description: This workshop is devoted to analytical aspects of morphogenesis (shape formation), arising as a consequence of the inelastic effects associated with growth, swelling, shrinkage or plasticity. Such effects result in a local and heterogeneous incompatibility of strains and naturally lead to the non-trivial shapes, seen even in the absence of any external forces, in a variety of science and technology situations. A large body of experimental data, as well as numerous formal derivations, present in physics and mechanics literature, suggest various approaches to coupling between residual strain and the ultimate shape of an object. On the other hand, the rigorous results are sparse, and call for a more detailed analysis. During the workshop, we hope to stimulate discussions and enhance further interactions between applied mathematicians, physicists, analysts and geometers, who pursue research around problems in strain-induced morphogenesis. For further information visit:

`http://www.ima.umn.edu/2010-2011/SW5.16-20.11`

SIAM Conference on Applications of Dynamical Systems (DS11)

(May 22–26, 2011)

Venue: Snowbird Ski and Summer Resort, Snowbird, Utah.

Description: The application of dynamical systems theory to areas outside of mathematics continues to be a vibrant, exciting and fruitful endeavor. These application areas are diverse and multidisciplinary, ranging over all areas of applied science and engineering, including biology, chemistry, physics, finance, and industrial applied mathematics. This conference strives to achieve a blend of application-oriented material and the mathematics that informs and supports it. The goals of the meeting are a cross-fertilization of ideas from different application areas, and increased communication between the mathematicians who develop dynamical systems techniques and applied scientists who use them. For further information visit:

`http://www.siam.org/meetings/ds11/`

Geometric Topology of Knots

(May 25–26, 2011)

Venue: Centro di Ricerca Matematica “Ennio De Giorgi”, Collegio Puteano, Piazza dei Cavalieri 3, Pisa, Italy.

Description: As other branches of 3-dimensional topology, the theory of knots and links was deeply revolutionized by the geometric approach first developed by Thurston. This workshop will focus on the geometry of knots, and particularly on aspects of hyperbolic geometry and the knot invariants associated with it, most notably the volume. Strong emphasis will be put on the relationships between these geometric invariants and the more classical ones, such as the crossing number. The algorithmic and computational methods now available to construct hyperbolic structures, and to compute invariants, will also be considered as central topics. For further information visit:

`http://www.crm.sns.it/event/205/`

Operator Theory and Boundary Value Problems

(May 25–27, 2011)

Venue: University Paris-Sud, Orsay, France.

Description: We are organizing a conference on the applications of the operator theory to the study of the boundary value problems arising in quantum mechanics and other areas. The boundary value problems are understood not only in the classical sence of PDE, but include also systems with singularities or structures composed of pieces of different nature and involving boundary conditions at the interaction set. We are planning to have three days of talks, May 24 being the arrival date and May 28 being the departure date. The aim of the conference is to bring together experts working on various aspects of the topic and to present the state of art to the local mathematical community. For further information visit:

http://www.math.u-psud.fr/ pankrash/ot11/

Finite Groups and Their Automorphisms

(June 7–11, 2011)

Venue: Bogaziçi University, Istanbul, Turkey.

Description: This five-day workshop aims to bring together leading mathematicians and active researchers working on the theory of groups in order to exchange ideas, present new results and identify the key problems in the field, especially but not exclusively, on the relationship between a finite group and another one acted upon by the first. There will be seven mini courses, several invited talks, a limited number of contributed talks and a poster session. For further information visit:

http://istanbulgroup.metu.edu.tr/

Geometric and Nonlinear Analysis, Meeting in Lorraine

(June 12–17, 2011)

Venue: Université Henri Poincar'e, Nancy, France.

Description: This meeting is dedicated to cover large sectors of nonlinear and geometric analysis. For further information visit:

http://www.iecn.u-nancy.fr/ frobert/

49th International Symposium on Functional Equations

(June 19–25, 2011)

Venue: Graz, Austria.

Description: Functional equations and inequalities, mean values, functional equations on algebraic structures, Hyers-Ulam stability, regularity properties of solutions, conditional functional equations, functional-differential equations, iteration theory; applications of the above, in particular to the natural, social, and behavioral sciences. For further information contact:

Jens Schwaiger, Institut für Mathematik, Karl-Franzens-Universität Graz, Heinrichstr. 36, A-8010 Graz, Austria.

Email: jens.schwaiger@uni-graz.at

Metric Measure Spaces: Geometric and Analytic Aspects

(June 27–July 8, 2011)

Venue: Université de Montréal, Pavillon André-Aisenstadt, Montréal, Québec, Canada.

Description: In recent decades, metric-measure spaces have emerged as a fruitful source of mathematical questions in their own right, and as indispensable tools for addressing classical problems in geometry, topology, dynamical systems and partial differential equations. The purpose of the 2011 summer school is to lead young scientists to the research frontier concerning the analysis and geometry of metric-measure spaces, by exposing them to a series of mini-courses featuring leading researchers who will present both the state of the art and the exciting challenges which remain. For further information visit:

http://www.dms.umontreal.ca/~sms/Metric11/index_e.php

International Conference on Special Functions & Their Applications (ICSFA 2011)

(July 28–30, 2011)

Venue: Department of Mathematics & Statistics, J, N. Vyas University, Jodhpur (Rajasthan), PIN-342 005, India.

For Further Details Contact/Visit:

Dr. R. K. Yadav, Organizing Secretary, CSFA 2011
Email: rkmdyadav@gmail.com
http://www.ssfaindia.webs.com/conf.htm

Conference in Harmonic Analysis and Partial Differential Equations in Honor of Eric Sawyer

(July 26–29, 2011)

Venue: Fields Institute, Toronto, Canada.

Description: The meeting will concentrate on Analysis and Partial differential Equations, specifically in relation to the diverse set of topics Dr. Eric T. Sawyer has worked on and influenced during his rich and prolific career: Weighted norm inequalities, Several complex variables, Unique continuation, Singular integrals, fractional integrals, Elliptic equations and systems, Degenerate equations, Monge-Ampère equations, Number theory and Combinatorics. For further information visit:

http://www.fields.utoronto.ca/programs/scientific/11-12/PDE/

Formal and Analytic Solutions of Differential and Difference Equations

(August 8–13, 2011)

Venue: Mathematical Research and Conference Center in Bedlewo, Poland.

Description: Ordinary differential equations in the complex domain; Holomorphic vector fields, normal forms; Summability of WKB solutions; Gevrey solutions, summability of divergent series, Stokes phenomena; Formal solutions of PDEs; Small divisors phenomena; Nonlinear PDEs: semilinear heat, Burgers, KdV, Schrödinger, Navier-Stokes; Summability of formal solutions of difference equations; Analytic and Gevrey hypoellipticity and solvability; Applications to integrable systems and mathematical physics. For further information visit:

http://www.impan.pl/ fasde/index.php.

European Conference on Combinatorics, Graph Theory and Applications 2011

(August 29–September 2, 2011)

Venue: Alfred Renyi, Institute of Mathematics, Budapest, Hungary.

Description: In the tradition of EuroComb01 (Barcelona), EuroComb03 (Prague), EuroComb05 (Berlin), EuroComb07 (Seville) and EuroComb09 (Bordeaux), this conference will cover the full range of combinatorics and graph theory including applications in other areas of mathematics, computer science and engineering. Topics include, but are not limited to: enumerative combinatorics, designs and configurations, graph theory, extremal combinatorics, algebraic combinatorics, topological combinatorics, ordered sets, combinatorial number theory, combinatorial geometry, random methods. For further information visit:

http://www.renyi.hu/conferences/ec11

Study Group Meeting in Industrial Mathematics : SGMIP 2011

(14–26 March, 2011)

A two week meeting, involving a week of mathematical modeling of live industrial problems provided by the GE and GM's research labs in Bangalore followed by a week of problem solving study group activity is scheduled to be held at the IISc during 14-26 March, 2011. The meeting is unique in its format that it'd introduce early career researchers to live industrial problem solving. Resource persons drawn from the IISc, IITs,and the Oxford University UK would actively participate and guide the participants who would break away in a a few groups each handling an industrial problem presented by a

researcher from the GE or GM India Labs. The meeting is funded by the DST, Govt of India. A brain-storming session on industrial mathematics is also planned for the last day of the meeting. More details including how to register can be found at the website:
http://www.serc.iisc.ernet.in/ raha/sgmip2011.html

A poster may be downloaded at

http://www.serc.iisc.ernet.in/ raha/PosterSGMIP.pdf

which may be circulated for wide publicity among applied mathematics researchers.

For Further Details, Contact:

Soumyendu Raha, Co-ordinator, SGMIP 2011, SERC, Indian Institute of Science, Bangalore 560 012.
Email: raha@serc.iisc.ernet.in
Phone: +91-80-2293-2791

Algebraic Representation Theory Conference

(September 1–3, 2011)

Venue: Uppsala University, Uppsala, Sweden.

Support: There is a possibility to apply for financial support for local expenses. PhD students will be supported in the first place.

For Further Information Contact/Visit:

Volodymyr Mazorchuk
Email: art2011@math.uu.se
http://www.math.uu.se/Conference

10TH International Conference on Geometry and Applications

(September 3–9, 2011)

Venue: Geometrical Society "Boyan Petkanchin", Sofia, Bulgaria.

Description: 10th International Conference on Geometry and Applications is organized from Geometrical Society "Boyan Petkanchin" in Bulgaria. The following fields are included: Differential geometry, finite geometries, computer methods in geometry, algebra and analysis, education in the school and university by computers, didactic of mathematics.

For Further Information Contact:

Prof. Dr. Grozio Stanilov
Email: stanilov@fmi.uni-sofia.bg

Chavdar Lozanov
Email: lozanov@fmi.uni-sofia.bg

International Conference "Harmonic Analysis and Approximations, V"

(September 10–17, 2011)

Venue: Tsaghkadzor, Armenia.

Description: The conference will be held at Yerevan State University's guesthouse, Tsaghkadzor (Armenia). The conference is dedicated to the 75th anniversary of academician of NAS of Armenia Norair Arakelian.

For Further Information Visit:

http://math.sci.am/conference/sept2011/conf.html

The readers may download the Mathematics Newsletter from the RMS website at www.ramanujanmathsociety.org