MATHEMATICS NEWSLETTER

Volume 20 March 2011 No. 3

CONTENTS

Visit: www.ramanujanmathsociety.org

Typeset in LaTeX at Krishtel eMaging Solutions Pvt. Ltd., Chennai - 600 017 Phone: 2434 55 16 and printed at United Bind Graphics, Chennai - 600 010. Phone: 2640 1531, 2640 1807

Introduction to Estimation of Survival Model Parameters

M. Pitchaimani
Ramanujan Institute for Advanced Study in Mathematics
University of Madras, Chepauk, Chennai 600 005, India
E-mail: mpitchaimani@yahoo.com

Abstract. In this article we introduce the basic concepts of survival analysis and methods of estimation of survival model parameters.

Keywords. Asymptotic formulae, Exponential law, Estimations, Gompertz law, Hazard rate, Mean, MLE, Power law, Survival rate, Uniqueness

AMS Mathematics Subject Classification: 92D10, 62N02, 62F10, 62L10, 62N05, 62P10

1. Introduction

Survival data consist of a response variable that measures the duration of time until a specified event occurs (event time, failure time, or survival time) and possibly a set of independent variables thought to be associated with the failure time variable. These independent variables (concomitant variables, covariates, or prognostic factors) can be either discrete, such as sex or race, or continuous, such as age or temperature. The system that gives rise to the event of interest can be biological, as for most medical data, or physical, as for engineering data. The purpose of survival analysis is to model the underlying distribution of the failure time variable and to assess the dependence of the failure time variable on the independent variables.

An intrinsic characteristic of survival data is the possibility for censoring of observations, that is, the actual time until the event is not observed. Such censoring can arise from withdrawal from the experiment or termination of the experiment. Because the response is usually a duration, some of the possible events may not yet have occurred when the period for data collection has terminated. For example, clinical trials are conducted over a finite period of time with staggered entry of patients. That is, patients enter a clinical trial over time and thus the length of follow-up varies by individuals; consequently, the time to the event may not be ascertained on all patients in the study. Additionally, some of the responses may be lost to follow-up (for example, a participant may move or refuse to continue to participate) before termination of data collection. In either case, only a lower bound on the failure time of the censored observations is known. These observations are said to be right censored. Thus, an additional variable is incorporated into the analysis indicating which responses are observed event times and which are censored times. More generally, the failure time may only be known to be smaller than a given value (left censored) or known to be within a given interval (interval censored). There are numerous possible censoring schemes that arise in survival analyses.

Another characteristic of survival data is that the response cannot be negative. This suggests that a transformation of the survival time such as a log transformation may be necessary or that specialized methods may be more appropriate than those that assume a normal distribution for the error term. It is especially important to check any underlying assumptions as a part of the analysis because some of the models used are very sensitive to these assumptions.

2. The Hazard and Survival Functions

Let T be a non-negative random variable representing the waiting time until the occurrence of an event. For simplicity we will adopt the terminology of survival analysis, referring to the event of interest as 'death' and to the waiting time as 'survival' time, but the techniques to be studied have much wider applicability. They can be used, for example, to study age at marriage, the duration of marriage, the intervals between successive births to a woman, the duration of stay in a city (or in a job), and the length of life. The observant demographer will have noticed that these examples include the fields of fertility, mortality and migration.

2.1 The Survival Function

We will assume for now that T is a continuous random variable with probability density function (p.d.f.) $f(t)$ and cumulative distribution function (c.d.f.) $F(t) = Pr\{T \leq t\}$, giving the probability that the event has occurred by duration t.

It will often be convenient to work with the complement of the c.d.f, the *survival* function

$$S(t) = Pr\{T > t\} = 1 - F(t) = \int_t^{\infty} f(x)dx, \quad (1)$$

which gives the probability of being alive at duration t, or more generally, the probability that the event of interest has not occurred by duration t.

2.1.1 The Hazard Function

An alternative characterization of the distribution of T is given by the hazard function, or instantaneous rate of occurrence of the event, defined as

$$\lambda(t) = \lim_{dt \to 0} \frac{Pr\{t < T \leq t + dt | T > t\}}{dt}. \quad (2)$$

The numerator of this expression is the conditional probability that the event will occur in the interval $(t, t + dt)$ given that it has not occurred before, and the denominator is the width of the interval. Dividing one by the other we obtain a rate of event occurrence per unit of time. Taking the limit as the width of the interval goes down to zero, we obtain an instantaneous rate of occurrence. The conditional probability in the numerator may be written as the ratio of the joint probability that T is in the interval $(t, t + dt)$ and $T > t$ (which is, of course, the same as the probability that t is in the interval), to the probability of the condition $T > t$. The former may be written as $f(t)dt$ for small dt, while the latter is $S(t)$ by definition. Dividing by dt and passing to the limit gives the useful result

$$\lambda(t) = \frac{f(t)}{S(t)}, \quad (3)$$

which some authors give as a definition of the hazard function. In words, the rate of occurrence of the event at duration t equals the density of events at t, divided by the probability of surviving to that duration without experiencing the event.

Note from equation (1) that $-f(t)$ is the derivative of $S(t)$. This suggests rewriting equation (3) as

$$\lambda(t) = -\frac{d}{dt} \ln[S(t)]. \quad (4)$$

If we now integrate from 0 to t and introduce the boundary condition $S(0) = 1$ (since the event is sure not to have occurred by duration 0), we can solve the above expression to obtain a formula for the probability of surviving to duration t as a function of the hazard at all durations up to t:

$$S(t) = \exp\left\{-\int_0^t \lambda(x)dx\right\}. \quad (5)$$

This expression should be familiar to demographers. The integral in curly brackets in this equation is called the cumulative hazard (or cumulative risk) and is denoted

$$\Lambda(t) = \int_0^t \lambda(x)dx. \quad (6)$$

You may think of $\Lambda(t)$ as the sum of the risks you face going from duration 0 to t.

These results show that the survival and hazard functions provide alternative but equivalent characterizations of the distribution of T. Given the survival function, we can always differentiate to obtain the density and then calculate the hazard using equation (3). Given the hazard, we can always integrate to obtain the cumulative hazard and then exponentiate to obtain the survival function using equation (4). An example will help fix ideas. Example: The simplest possible survival distribution is obtained by assuming a constant risk over time, so the hazard is

$$\lambda(t) = \lambda$$

for all t. The corresponding survival function is

$$S(t) = \exp\{-\lambda(t)\}.$$

This distribution is called the exponential distribution with parameter λ. The density may be obtained multiplying the survivor function by the hazard to obtain

$$f(t) = \lambda \exp\{-\lambda(t)\}.$$

The mean turns out to be $\frac{1}{\lambda}$. This distribution plays a central role in survival analysis, although it is probably too simple to be useful in applications in its own right.

2.2 Expectation of Life or Mean time to Failure

Let μ denote the mean or expected value of T. By definition, one would calculate μ multiplying t by the density $f(t)$ and integrating, so

$$\mu = \int_0^t t\ f(t)dt. \tag{7}$$

Integrating by parts, and making use of the fact that $-f(t)$ is the derivative of $S(t)$, which has limits or boundary conditions $S(0) = 1$ and $S(1) = 0$, one can show that

$$\mu = \int_0^t S(t)dt. \tag{8}$$

In words, the mean is simply the integral of the survival function.

3. Parameter Estimation

Here we review point estimation of parameters, first with an overview of the frequentist approach and its two most important methods, maximum likelihood and least squares, treated in sections 3.2 and 3.3.

An estimator $\hat{\theta}$ (written with a hat) is a function of the data whose value, the estimate, is intended as a meaningful guess for the value of the parameter θ. There is no fundamental rule dictating how an estimator must be constructed. One tries, therefore, to choose that estimator which has the best properties. The most important of these are (a) consistency, (b) bias, (c) efficiency, and (d) robustness.

(a) An estimator is said to be consistent if the estimate $\hat{\theta}$ converges to the true value θ as the amount of data increases. This property is so important that it is possessed by all commonly used estimators.

(b) The bias, $b = E[\hat{\theta}] - \theta$, is the difference between the expectation value of the estimator and the true value of the parameter. The expectation value is taken over a hypothetical set of similar experiments in which $\hat{\theta}$ is constructed in the same way. When $b = 0$, the estimator is said to be unbiased. The bias depends on the chosen metric, i.e., if $\hat{\theta}$ is an unbiased estimator of θ, then $\hat{\theta}^2$ is not in general an unbiased estimator for θ^2. If we have an estimate $\hat{b}$ for the bias, we can subtract it from $\hat{\theta}$ to obtain a new $\hat{\theta}' = \hat{\theta} - \hat{b}$. The estimate $\hat{b}$ may, however, be subject to statistical or systematic uncertainties that are larger than the bias itself, so that the new $\hat{\theta}'$ may not be better than the original.

(c) Efficiency is the inverse of the ratio of the variance $V[\hat{\theta}]$ to the minimum possible variance for any estimator of θ. Under rather general conditions, the minimum variance is given by the Rao-Cramer-Frechet bound,

$$\sigma^2_{\min} = \left(1 + \frac{\partial b}{\partial \theta}\right) / I(\theta), \tag{9}$$

where

$$I(\theta) = E\left[\left(\frac{\partial}{\partial \theta}\sum_i \ln f(x_i;\theta)\right)^2\right] \tag{10}$$

is the Fisher information. The sum is over all data, assumed independent, and distributed according to the p.d.f. $f(x;\theta)$, b is the bias, if any, and the allowed range of x must not depend on θ.

The *mean-squared error*,

$$MSE = E[(\hat{\theta} - \theta)^2] = V[\hat{\theta}] + b^2 \tag{11}$$

is a measure of an estimators quality which combines the uncertainties due to bias and variance.

(d) Robustness is the property of being insensitive to departures from assumptions in the p.d.f., e.g., owing to uncertainties in the distributions tails.

Simultaneously optimizing for all the measures of estimator quality described above can lead to conflicting requirements. For example, there is in general a trade-off between bias and variance. For some common estimators, the properties above are known exactly. More generally, it is possible to evaluate them by Monte Carlo simulation. Note that they will often depend on the unknown θ.

3.1 Estimators for Mean, Variance and Median

Suppose we have a set of N independent measurements, x_i, assumed to be unbiased measurements of the same unknown quantity μ with a common, but unknown, variance σ^2. Then

$$\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i \tag{12}$$

$$\hat{\sigma^2} = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \hat{\mu}) \tag{13}$$

are unbiased estimators of μ and σ^2. The variance of $\hat{\mu}$ is $\sigma^2/N$and the variance of $\hat{\sigma^2}$ is

$$V[\hat{\sigma^2}] = \frac{1}{N}\left(m_4 - \frac{N-3}{N-1}\sigma^4\right) \qquad (14)$$

where m_4 is the fourth central moment of x. For Gaussian distributed x_i, this becomes $2\sigma^4/(N-1)$ for any $N \geq 2$, and for large N, the standard deviation of $\hat{\sigma}$ (the error of the error) is $\sigma/\sqrt{2N}$. Again, if the x_i are Gaussian, $\hat{\mu}$ is an efficient estimator for μ, and the estimators $\hat{\mu}$ and $\hat{\sigma^2}$ are uncorrelated. Otherwise the arithmetic mean (12) is not necessarily the most efficient estimator.

If σ^2 is known, it does not improve the estimate $\hat{\mu}$, as can be seen from equation (12); however, if μ is known, substitute it for $\hat{\mu}$ in equation (13) and replace $N-1$ by N to obtain an estimator of σ^2 still with zero bias but smaller variance. If the x_i have different, known variances σ_i^2, then the weighted average

$$\hat{\mu} = \frac{1}{w}\sum_{i=1}^{N} w_i x_i \qquad (15)$$

is an unbiased estimator for μ with a smaller variance than an unweighted average; here $w_i = 1/\sigma_i^2$ and $w = \sum_{i=1}^{N} w_i$. The standard deviation of $\hat{\mu}$ is $1/\sqrt{w}$.

As an estimator for the median x_{med}, one can use the value $\hat{x}_{\mathrm{med}}$ such that half the x_i are below and half above (the sample median). If the sample median lies between two observed values, it is set by convention halfway between them. If the p.d.f. of x has the form $f(x-\mu)$ and μ is both mean and median, then for large N the variance of the sample median approaches $1/[4Nf^2(0)]$, provided $f(0) > 0$. Although estimating the median can often be more difficult computationally than the mean, the resulting estimator is generally more robust, as it is insensitive to the exact shape of the tails of a distribution.

3.2 The Method of Maximum Likelihood

Suppose we have a set of N measured quantities $x = (x_1, \ldots, x_N)$ described by a joint p.d.f. $f(x;\theta)$, where $\theta = (\theta_1, \ldots, \theta_n)$ is set of n parameters whose values are unknown. The likelihood function is given by the p.d.f. evaluated with the data x, but viewed as a function of the parameters, i.e., $L(\theta) = f(x;\theta)$. If the measurements x_i are statistically independent and each follow the p.d.f. $f(x;\theta)$, then the joint p.d.f. for x factorizes and the likelihood function is

$$L(\theta) = \prod_{i=1}^{N} f(x_i;\theta) \qquad (16)$$

The method of maximum likelihood takes the estimators $\hat{\theta}$ to be those values of θ that maximize $L(\theta)$.

Note that the likelihood function is not a p.d.f. for the parameters θ; in frequentist statistics this is not defined. In Bayesian statistics, one can obtain from the likelihood the posterior p.d.f. for θ, but this requires multiplying by a prior p.d.f.

It is usually easier to work with $\ln L$, and since both are maximized for the same parameter values θ, the maximum likelihood (ML) estimators can be found by solving the likelihood equations,

$$\frac{\partial \ln L}{\partial \theta_i} = 0 \quad i = 1, \ldots, n \qquad (17)$$

Often the solution must be found numerically. Maximum likelihood estimators are important because they are approximately unbiased and efficient for large data samples, under quite general conditions, and the method has a wide range of applicability.

In evaluating the likelihood function, it is important that any normalization factors in the p.d.f. that involve θ be included. However, we will only be interested in the maximum of L and in ratios of L at different values of the parameters; hence any multiplicative factors that do not involve the parameters that we want to estimate may be dropped, including factors that depend on the data but not on θ.

Under a one-to-one change of parameters from θ to η, the ML estimators $\hat{\theta}$ transform to $\eta(\hat{\theta})$. That is, the ML solution is invariant under change of parameter. However, other properties of ML estimators, in particular the bias, are not invariant under change of parameter.

The inverse V^{-1} of the covariance matrix $V_{ij} = \mathrm{cov}[\hat{\theta}_i, \hat{\theta}_j]$ for a set of ML estimators can be estimated by using

$$(\hat{V}^{-1})_{ij} = -\left.\frac{\partial^2 \ln L}{\partial\theta_i \partial\theta_j}\right|_{\hat{\theta}}. \qquad (18)$$

For finite samples, however, equation (18) can result in an underestimate of the variances. In the large sample limit (or in a linear model with Gaussian errors), L has a Gaussian form and $\ln L$ is (hyper) parabolic. In this case, it can be seen that a numerically equivalent way of determining s-standard-deviation errors is from the contour given by the θ' such that

$$\ln L(\theta') = \ln L_{\max} - s^2/2 \tag{19}$$

where $\ln L_{\max}$ is the value of $\ln L$ at the solution point. The extreme limits of this contour on the θ_i axis give an approximate s-standard-deviation confidence interval for θ_i.

In the case where the size n of the data sample $x_1, \ldots, x_n$ is small, the unbinned maximum likelihood method, i.e., use of equation (16), is preferred since binning can only result in a loss of information, and hence larger statistical errors for the parameter estimates. The sample size n can be regarded as fixed, or the user can choose to treat it as a Poisson-distributed variable; this latter option is sometimes called "extended maximum likelihood [1–3].

If the sample is large, it can be convenient to bin the values in a histogram, so that one obtains a vector of data $n = (n_1, \ldots, n_N)$ with expectation values $\nu = E[n]$ and probabilities $f(n; \nu)$. Then one may maximize the likelihood function based on the contents of the bins (so i labels bins). This is equivalent to maximizing the likelihood ratio $\lambda(\theta) = f(n; \nu(\theta))/f(n; n)$, or to minimizing the equivalent quantity $-2\ln\lambda(\theta)$. For independent Poisson distributed n_i this is [4].

$$-2\ln\lambda(\theta) = 2\sum_{i=1}^{N}\left[\nu_i(\theta) - n_i + n_i \ln\frac{n_i}{\nu_i(\theta)}\right], \tag{20}$$

where for bins with $n_i = 0$, the last term in (20) is zero. The expression (20) without the terms $\nu_i - n_i$ also gives $-2\ln\lambda(\theta)$ for multinomially distributed n_i, i.e., when the total number of entries is regarded as fixed. In the limit of zero bin width, maximizing (20) is equivalent to maximizing the unbinned likelihood function (16).

A benefit of binning is that it allows for a goodness-of-fit test. According to Wilks theorem, for sufficiently large ν_i and providing certain regularity conditions are met, the minimum of $-2\ln\lambda$ as defined by equation (20) follows a χ^2 distribution [5]. If there are N bins and m fitted parameters, then the number of degrees of freedom for the χ^2 distribution is $N - m$ if the data are treated as Poisson-distributed, and $N - m - 1$ if the n_i are multinomially distributed.

Suppose the n_i are Poisson-distributed and the overall normalization $\nu_{\text{tot}} = \sum_i \nu_i$ is taken as an adjustable parameter, so that $\nu_i = \nu_{\text{tot}} p_i(\theta)$, where the probability to be in the ith bin, $p_i(\theta)$, does not depend on ν_{tot}. Then by minimizing equation (20), one obtains that the area under the fitted function is equal to the sum of the histogram contents, i.e., $\sum_i \nu_i = \sum_i n_i$. This is not the case for parameter estimation methods based on a least-squares procedure with traditional weights [3].

3.3 The Method of Least Squares

The method of least squares (LS) coincides with the method of maximum likelihood in the following special case. Consider a set of N independent measurements y_i at known points x_i. The measurement y_i is assumed to be Gaussian distributed with mean $F(x_i; \theta)$ and known variance σ_i^2. The goal is to construct estimators for the unknown parameters θ. The likelihood function contains the sum of squares

$$\chi^2(\theta) = -2\ln L(\theta) + \text{constant} = \sum_{i=1}^{N}\frac{(y_i - F(x_i;\theta))^2}{\sigma_i^2}. \tag{21}$$

The set of parameters θ which maximize L is the same as those which minimize χ^2.

The minimum of equation (21) defines the least-squares estimators $\hat{\theta}$ for the more general case where the y_i are not Gaussian distributed as long as they are independent. If they are not independent but rather have a covariance matrix $V_{ij} = \text{cov}[y_i, y_j]$, then the LS estimators are determined by the minimum of

$$\chi^2(\theta) = (y - F(\theta))^T V^{-1}(y - F(\theta)). \tag{22}$$

where $y = (y_1, \ldots, y_N)$ is the vector of measurements, $F(\theta)$ is the corresponding vector of predicted values (understood as a column vector in (22)), and the superscript T denotes transposed (i.e., row) vector.

In many practical cases, one further restricts the problem to the situation where $F(x_i; \theta)$ is a linear function of the parameters, i.e.,

$$F(x_i; \theta) = \sum_{j=1}^{N}\theta_j h_j(x_i). \tag{23}$$

Here the $h_j(x_i)$ are m linearly independent functions, e.g., $1, x, x^2, \ldots, x^{m-1}$, or Legendre polynomials. We require $m < N$ and at least m of the x_i must be distinct.

Minimizing χ^2 in this case with m parameters reduces to solving a system of m linear equations. Defining $H_{ij} = h_j(x_i)$

and minimizing χ^2 by setting its derivatives with respect to the θ_i equal to zero gives the LS estimators,

$$\hat{\theta} = (H^T V^{-1} H)^{-1} H^T V^{-1} y \equiv D_y. \tag{24}$$

The covariance matrix for the estimators $U_{ij} = \text{cov}[\hat{\theta}_i, \hat{\theta}_j]$ is given by

$$U = DVD^T = (H^T V^{-1} H)^{-1}, \tag{25}$$

or equivalently, its inverse U^{-1} can be found from

$$(U^{-1})_{ij} = \frac{1}{2} \frac{\partial^2 \chi^2}{\partial \theta_i \partial \theta_j}\bigg|_{\theta=\hat{\theta}} = \sum_{k,l=1}^{N} h_i(x_k)(V^{-1})_{kl} h_j(x_l). \tag{26}$$

The LS estimators can also be found from the expression

$$\hat{\theta} = Ug \tag{27}$$

where the vector g is defined by

$$g_i = \sum_{j,k=1}^{N} y_j h_i(x_k)(V^{-1})_{jk}. \tag{28}$$

For the case of uncorrelated y_i, for example, one can use (27) with

$$(U^{-1})_{ij} = \sum_{k=1}^{N} \frac{h_i(x_k) h_j(x_k)}{\sigma_k^2} \tag{29}$$

$$g_i = \sum_{k=1}^{N} \frac{y_k h_i(x_k)}{\sigma_k^2} \tag{30}$$

Expanding $\chi^2(\theta)$ about $\hat{\theta}$, one finds that the contour in parameter space defined by

$$\chi^2(\theta) = \chi^2(\hat{\theta}) + 1 = \chi^2_{\min} + 1 \tag{31}$$

has tangent planes located at approximately plus or minus one standard deviation $\sigma_{\hat{\theta}}$ from the LS estimates $\hat{\theta}$.

In constructing the quantity $\chi^2(\theta)$, one requires the variances or, in the case of correlated measurements, the covariance matrix. Often these quantities are not known a priori and must be estimated from the data; an important example is where the measured value y_i represents a counted number of events in the bin of a histogram. If, for example, y_i represents a Poisson variable, for which the variance is equal to the mean, then one can either estimate the variance from the predicted value, $F(x_i; \theta)$, or from the observed number itself, y_i. In the first option, the variances become functions of the fitted parameters, which may lead to calculational difficulties. The second option can be undefined if y_i is zero, and in both cases for small y_i, the variance will be poorly estimated. In either case, one should constrain the normalization of the fitted curve to the correct value, i.e., one should determine the area under the fitted curve directly from the number of entries in the histogram. A further alternative is to use the method of maximum likelihood; for binned data this can be done by minimizing equation (20)

As the minimum value of the χ^2 represents the level of agreement between the measurements and the fitted function, it can be used for assessing the goodness of fit.

4. Analytic Estimation of Survival Model Parameters

Studies in the evolutionary biology of aging require good estimates of the age-dependent mortality rate coefficient. In this section, we provide uniqueness of mortality (hazard) rate parameters, interval of existence of mortality rate parameters M_r and b and their asymptotic expressions in Allometry survival model, in the absence of age-specific mortality data.

In the presence of mortality data by age, for example, the Allometric scaling parameters M_r and b have been estimated by using various statistical methods like maximum likelihood, least square, linear regression, and nonlinear regression. Usually, an experimentalist knows the lifespan of each individual in a given population and can make use of standard techniques such as MLE or linear regression to estimate the model parameters.

In the absence of age specific mortality data, in this section we have developed a method to estimate b from the instantaneous mortality rate at reference length, i.e., M_r; the original population size, N_0; and the maximum lifespan, t_m.

Julian Huxley and Georges Teissier coined the term allometry in 1936. In a joint paper, simultaneously published in English and French (Huxley and Teissier [6]), they agreed to use this term in order to avoid confusion in the field of relative growth. They also agreed on the symbols to be used in the algebraic formula of allometric growth: $Y = am^b$

This makes body size a good choice for baseline analysis, using the scaling (heterogonic or allometric) equation of Huxley [7] and Kleiber [8]:

$$Y = am^b, \tag{32}$$

in which Y is a physiological, morphological, or ecological variable; the coefficient a is characteristic of a class or order of animals and the physical dimensional units (if any) being used in the measurement of Y; m is body mass (kg); and the exponent b is the ratio of changes in orders of magnitude for Y compared to m, thus expressing the effect of body mass changes on Y.

We should point out that although keeping variables in familiar physical units does have certain conceptual advantanges, an analysis is much more flexible and robust if the variable Y is scaled by a related intrinsic parameter of similar physical dimension such that the dependent variable is an intrinsic dimensionless entity; and likewise if the variable m with a physical dimension of mass is similarly scaled by an intrinsic mass characteristic of the system such that the size-related independent variable is also a dimensionless entity. In such circumstances the coefficient a is entirely characteristic of the taxonomic classification of the organism and the dimensionless variables can be used as arguments of mathematical functions such as logarithms which transform pure numbers without physical dimension. Furthermore, semantic confusion over the use of kg as a weight or as a mass is avoided.

4.1 Linear Growth

Stocking is widely used in the management of freshwater and, to a lesser extent, coastal-marine fisheries (e.g., Heidinger [9]). A key problem in the management of stocked fisheries is the optimization of release size (e.g., Cowx [10]). The optimal release size depends on the contribution that fish of a particular size will make to the catch or fishable stock and on the resources required to produce seed fish of that size. Of the data required to assess optimum size, the survival of seed fish of different sizes to the fishable stock (and/or contribution to the catch) are the most difficult to obtain. Systematic assessments have been either entirely empirical (release-recapture of marked seed fish of different sizes) or based on detailed ecological studies (Wahl *et al.* [11]). However, the costs and effort involved in both approaches restrict their use to a small number of fisheries, and the results are not readily generalized. An alternative approach that implies a simple generalization is the use of allometric mortality-size relationship (Lorenzen [12]). Provided that natural mortality in stocked fish is subject to a consistent allometry, then an estimate of mortality for a single reference size is sufficient to predict survival for a range of different release sizes.

Theoretical and empirical studies (Peterson and Wroblewski [13], Lorenzen [14]) point to the existence of an allometric relationship between natural mortality and body weight, of the form

$$M_w = M_u W^b \tag{33}$$

where M_w is natural mortality at weight W; M_u is mortality at unit weight; b is the allometry exponent; and where there is an implied RHS coefficient of (unit weight)$^{-b}$. Note that a mathematical structure of this form would also apply to a system transformed to corresponding dimensionless variables as mentioned in the Introduction.

4.1.1 Survival Model

The survival model follows that developed by Lorenzen [15] in which the allometric relationship between natural mortality and body length may be described by the equation

$$M(l) = M_r \left(\frac{l}{l_r}\right)^b \tag{34}$$

where $M(l)$ is the mortality rate at length l, M_r is the instantaneous mortality rate at reference length l_r (e.g., 15 cm – as used by Lorenzen [15]), and b is the allometric exponent of the mortality-length relationship. This reference length, l_r, needs to be chosen as a parameter such that it is smaller than another parameter, l_0, the length at stocking.

If this equation accurately describes mortality in the stocked population, then the decline in population size of a stocked cohort (organisms of the same age and size) of original population size N_0, while sufficiently large enough to be approximated as a continuous variable, is described by the differential equation

$$\frac{dN(t)}{dt} = -N(t) M_r \left(\frac{l(t)}{l_r}\right)^b \tag{35}$$

where $l(t)$ is length at time t. This differential equation may be solved explicitly if a linear growth model is substituted for $l(t)$. A linear length growth model is reasonably used in the empirical analysis, because time at large (i.e., the time interval between release at stocking and estimated survival age at death or recapture) is short and the size of the fish is small relative to the reported maximum sizes in all stocking experiments analyzed in [15]. A model of the form

$$l(t) = l_0 + ut \tag{36}$$

is used where t is the time since stocking, and u is the linear length growth rate. Substitution of equation (36) into equation (35), integration, and division by N_0 on both sides gives the following equation to predict survival, $S(t)$ (proportion of stocked fish surviving), from the time of stocking to time t:

$$S(t) = \frac{N(t)}{N_0} = e^{-\frac{M_r\left(-l_0\left(\frac{l_0}{l_r}\right)^b + (l_0+ut)\left(\frac{l_0+ut}{l_r}\right)^b\right)}{(b+1)u}}. \tag{37}$$

as was derived by Lorenzen [14] for the case where $b \neq -1$.

The two parameters M_r and b are of interest to many investigators in biogerontology and the evolutionary biology of aging [16–20]. Species comparisons in mortality rates are aided by calculations of MRD (mortality rate doubling time) which changes in the same direction as lifespan and is given by

$$MRD = \frac{2^{\frac{1}{b}} l_r - l_0}{u}. \tag{38}$$

In the presence of mortality data by age, the Allometric scaling parameters M_r and b have been estimated by using various statistical methods like maximum likelihood, linear regression, and nonlinear regression [21–27]. Usually, an experimentalist knows the lifespan of each individual in a given population and can make use of standard techniques such as MLE or linear regression [21,22] to estimate the model parameters.

In the absence of age specific mortality data, in this paper we have developed a method to estimate b from the instantaneous mortality rate at reference length, i.e., M_r; the original population size, N_0; and the maximum lifespan, t_m.

4.1.2 Estimation of Parameters

When attempting to estimate Allometric scaling parameters it is difficult, in general, to decide upon a particular value of t to use in the expression for $(S(t))$ given in equation (37). However if we are examining the issue of evolution of longevity, then choosing $t = t_{\max}$, a known maximum lifespan for the species, is a reasonable starting value. Finally for ease of analysis we may use the finite lifespan procedure of Witten and Satzer [22] and set $S(t_m) = \frac{1}{N_0}$ (the population contains only one individual left from an original population size N_0). We obtain the following equation for t_m (the time at which the population has only one individual and which approximates the maximum lifespan t_m^*)

$$t_m^* \simeq t_m = \frac{1}{u}\left[(l_r)^{\frac{b}{b+1}}\left(\frac{(b+1)u\ln N_0}{M_r} + l_0\left(\frac{l_0}{l_r}\right)^b\right)^{\frac{1}{b+1}} - l_0\right]. \tag{39}$$

The average mortality rate of a steady state population subject to age specific mortality rates of equation (3) is [19,20]

$$A_{av} = \frac{1}{\int_0^\infty S(t)dt}. \tag{40}$$

Equation (39) gives,

$$\frac{(b+1)u}{M_r} = \frac{-l_0\left(\frac{l_0}{l_r}\right)^b + (l_0+ut)\left(\frac{l_0+ut}{l_r}\right)^b}{\ln N_0}$$

or,

$$\frac{M_r}{(b+1)u} = \frac{\ln N_0}{-l_0\left(\frac{l_0}{l_r}\right)^b + (l_0+ut)\left(\frac{l_0+ut}{l_r}\right)^b}$$

and from equation (40), we get

$$\frac{1}{A_{av}} = \int_0^\infty e^{-\frac{M_r\left(-l_0\left(\frac{l_0}{l_r}\right)^b + (l_0+ut)\left(\frac{l_0+ut}{l_r}\right)^b\right)}{(b+1)u}} dt$$

A simple substitution of $z \equiv (l_0 + ut)^{b+1}$ in the above integral gives the same transcendental parametric relationship

$$b+1 = \frac{A_{av}}{u}\, e^{\frac{M_r l_0\left(\frac{l_0}{l_r}\right)^b}{(b+1)u}} \int_{l_0^{b+1}}^\infty \frac{e^{-\frac{M_r z}{l_r^b (b+1)u}}}{z^{\frac{b}{b+1}}} dz$$

or,

$$\begin{aligned} b+1 &= A_{av}\, e^x \int_x^\infty \left(\frac{(b+1)l_r^b}{u^b}\right)^{\frac{1}{b+1}} \frac{e^{-z}}{z^{\frac{b}{b+1}}} dz, \\ &\approx A_{av}\frac{l_r}{u}\, e^x \int_x^\infty \frac{e^{-z}}{z} dz, \quad \text{see [25]} \end{aligned} \tag{41}$$

where $x = \frac{M_r l_0\left(\frac{l_0}{l_r}\right)^b}{(b+1)u} = \frac{\ln N_0}{\left(\frac{l_0+ut_m}{l_0}\right)^{b+1} - 1}$.

The basic equation (41) is transcendental, involving exponential integral, and hence, its solution may not be unique. Hence, it is necessary to investigate the uniqueness of solution of (41). In [27] we have proved the following uniqueness theorems.

Theorem 1 [27]. *Equation (41) has a unique solution, if* $2\,A_{av}\,\frac{l_r}{u}\left(\frac{l_0+ut_m}{l_0}\right) < 1$.

We note that uniqueness condition $2\,A_{av}\,\frac{l_r}{u}\left(\frac{l_0+ut_m}{l_0}\right) < 1$ is independent of population size N_0.

A Necessary Condition for Uniqueness

Theorem 2 [27]. *To have a unique solution of equation (41), it is necessary that* $\frac{A_{av}t_m}{\ln N_0} < 1$.

4.1.3 Interval of Existence of Numerical Solution

In [28] we obtained the following asymptotic formulae for b and M_r, when age specific mortality data were absent.

Theorem 3 [28]. *For every fixed* A_{av}, t_m, l_0, u *and* N_0, *let* I *be interval defined by*

$$I = \left[b^* + 1, \frac{1}{\ln\left(\frac{l_0+ut_m}{l_0}\right)} \times \ln\left[1 + e^{A_{av}\frac{l_r}{u}\ln\left(\frac{l_0+ut_m}{l_0}\right)C/\left(A_{av}\frac{l_r}{u}\ln\left(\frac{l_0+ut_m}{l_0}\right)-1\right)} \times (\ln N_0)^{A_{av}\frac{l_r}{u}\ln\left(\frac{l_0+ut_m}{l_0}\right)/\left(A_{av}\frac{l_r}{u}\ln\left(\frac{l_0+ut_m}{l_0}\right)-1\right)}\right]\right]$$

with $b^* + 1 \geq \frac{\ln\left(\frac{\ln N_0}{A_{av}t_m}\right)}{ut_m \ln(l_0+ut_m)}$.

Suppose there exists a unique solution of equation (41) in I when

$$A_{av}\frac{l_r}{u}\ln\left(\frac{l_0+ut_m}{l_0}\right) \neq 1.$$

Then it sis necessary that $\frac{A_{av}t_m}{\ln N_0} < 1$. Moreover the asymptotic solution of equation (10) for a large N_0 is given by (18) when $A_{av}\frac{l_r}{u}\ln\left(\frac{l_0+ut_m}{l_0}\right) \neq 1$.

4.2 Exponential Growth

Two modes of growth have been proposed in the ecdysozoan: "saltational," in taxa in which a tanned cuticle permits size increase only at molts, and "continuous" in taxa with stretchable, collagenous cuticles [29]. Research into these methods of growth has been limited almost exclusively to the arthropods (saltational growth) and nematodes (continuous growth), and even here, despite long standing interest in the details of the saltational growth of arthropod taxa (Alpatov [30], Rice [31]), continuous growth has rarely been investigated closely (Howells and Blainey [32], Wilson [33]). Specificaly, little is known of how continuous growth is achieved at a fine scale, the role of cuticle, and the cells that secret it. Understanding the details of growth has important implications for understanding the significance of molting as an evolutionary conserved feature of the ecdysozoa (Wilson [33]) and for interpreting the increasing number of studies that seek to identify the molecular and cellular controls of the ecdysozoan growth (Estevez *et al.* [34], Johnston *et al.* [35], Oldham *et al.* [36]).

In [29], Knight *et al.* used the free living nematode *Canenorhabditis elegans* as the best characterized example of continuously growing ecdysozoan (Riddle *et al.* [37]). The hatchling worm is 0.25 mm long and grows to 1.4 mm within 5 days, a 6-fold increase in length and over a 100-fold increase in volume. *C. elegans* have an S-shaped growth curve an exponential phase of larval growth and a gradual approach to a plateau in late adulthood (Byerly *et al.* [38]). In view of this we assume that the growth variable $l(t)$ (*see equation* (4)) is exponential. That is

$$l(t) = l_0 e^{vt} \tag{42}$$

where v is the allometric exponent.

4.2.1 Survival Model

A model of the form given in (42) is used. Substitution of equation (42) into equation (35), integration, and division by N_0 on both sides gives the following equation to predict survival, $S(t)$,

$$S(t) = e^{-M_r\left(\frac{l_0}{l_r}\right)^b\left(\frac{e^{bvt}-1}{bv}\right)} \tag{43}$$

The two parameters M_r and b are of interest to many investigators in biogerontology and the evolutionary biology of aging [16–20]. Species comparisons in mortality rates are aided by calculations of MRD (mortality rate doubling time) which changes in the same direction as lifespan and is given by

$$MRD = \frac{1}{v}\ln\left(\frac{2^{\frac{1}{b}}l_r}{l_0}\right). \tag{44}$$

In the absence of age-specific mortality data, we have developed a method to estimate b from the instantaneous mortality rate (M_r), original population size (N_0), and maximum lifespan (t_m).

4.2.2 Estimation of Parameters

When attempting to estimate allometric scaling parameters it is difficult, in general, to decide upon a particular value of t to use in the expression for $(S(t))$ given in equation (37). However if we are examining the issue of evolution of longevity, then choosing $t = t_m^*$, a known maximum lifespan for the species, is a reasonable starting value. Finally for ease of analysis we may use the finite lifespan procedure of Witten and Satzer [22] and set $S(t_m) = \frac{1}{N_0}$ (the population contains only one individual left from an original population size N_0). We obtain the following equation for t_m (the time at which the population has only one individual and which approximates the maximum lifespan t_m^*)

$$t_m^* \simeq t_m = \frac{1}{bv} \ln \left[1 + bv \frac{\ln N_0}{M_r} \left(\frac{l_r}{l_0} \right)^b \right]. \tag{45}$$

The average mortality rate of a steady state population subject to age specific mortality rates of equation (14) is [16,17]

$$A_{av} = \frac{1}{\int_0^\infty S(t)dt}, \tag{46}$$

Equation (45) gives,

$$\frac{b\ v\ \ln N_0}{M_r} = \left(\frac{l_0}{l_r} \right)^b (e^{bvt_m} - 1)$$

or,

$$\frac{M_r}{bv} \left(\frac{l_0}{l_r} \right)^b = \frac{\ln N_0}{(e^{bvt_m} - 1)}, \tag{47}$$

and from equation (18) we get,

$$\frac{1}{A_{av}} = \int_0^\infty e^{-M_r \left(\frac{l_0}{l_r}\right)^b \left(\frac{e^{bvt}-1}{bv}\right)} dt.$$

A simple substitution in the above integral gives

$$b = \frac{A_{av}}{v}\, e^{\frac{M_r}{bv}\left(\frac{l_0}{l_r}\right)^b} \int_{\frac{M_r}{bv}\left(\frac{l_0}{l_r}\right)^b}^\infty \frac{e^{-\tau}}{\tau} d\tau,$$

and using (19), we get

$$b = \frac{A_{av}}{v}\, e^{\frac{\ln N_0}{e^{bvt_m}-1}} \int_{\frac{\ln N_0}{e^{bvt_m}-1}}^\infty \frac{e^{-\tau}}{\tau} d\tau. \tag{48}$$

The basic equation (25) is transcendental, involving exponential integral, and hence, its solution may not be unique. Therefore, it is necessary to investigate the uniqueness of solution of (25). In [27] we have proved the following uniqueness theorems for equation (25).

Theorem 4 [27]. *Equation (41) has a unique solution, if* $2\ A_{av} \frac{l_r}{u} \left(\frac{l_0 + ut_m}{l_0} \right) < 1.$

A Necessary Condition for Uniqueness

Theorem 5 [27]. *To have a unique solution of b in equation (25), it is necessary that* $\frac{A_{av} t_m}{\ln N_0} < 1$.

4.2.3 Interval of Existence of Numerical solution

In [28] we obtained the following asymptotic formulae for b and M_r, when age specific mortality data were absent.

Theorem 6 [28]. *For every fixed A_{av}, t_m, v and N_0, let I_1 and I_2 be intervals defined by*

$$I_1 = \left[b^*, \frac{1}{vt_m} \ln \left[1 + e^{\frac{A_{av}t_m C}{A_{av}t_m - 1}} \ln(\ln N_0)^{\frac{A_{av}t_m}{A_{av}t_m - 1}} \right] \right],$$

$$b^* \geq \left[\frac{1}{vt_m} \ln \left(\frac{\ln N_0}{A_{av} t_m} \right) \right],$$

$$I_2 = \left[b^{**}, \frac{1}{vt_m} \ln \left[1 + \frac{\ln N_0 - C \ln N_0 - 1}{\ln(\ln N_0) + C} \right] \right],$$

$$b^{**} \geq \frac{1}{vt_m} \ln(\ln N_0),$$

where $C = 0.577215$, Euler's constant.

Suppose there exists a unique solution of (25) in I_1, I_2 respectively, when $A_{av} t_m \neq 1$ and $A_{av} t_m = 1$. Then it is necessary that $\frac{A_{av} t_m}{ln N_0} < 1$.

Remark 1. From (24) of [28] it is easy to get the asymptotic formula of initial mortality rate using (30) of [28] and (31) of [28].

Conclusion

The purpose of this study is to estimate the two parameters M_r and b. The two parameters M_r and b are of interest to many investigators in bio gerontology and the evolutionary biology of aging [16–20].

In the presence of mortality data by age, the Allometry/Gompertz/Weibull/Frailty scaling parameters M_r and b have been estimated by using various statistical methods like maximum likelihood, linear regression, and nonlinear regression discussed in Section 3 [21–27]. Usually, an experimentalist knows the lifespan of each individual in a given population and can make use of standard techniques such as MLE or linear regression [21,22] to estimate the model parameters.

Studies in the evolutionary biology of aging require good estimates of the age dependent mortality rate coefficient. Results of the previous study [12] indicate that survival model based on allometry mortality length relationship and population specific M_r provide practical tool for assessing optimal release size in stocking programmes. The applications of these (linear and exponential) survival models to the analysis and prediction of stocking outcomes requires an estimate of M_r in the population in question (as well as the growth parameters) [12]. Hence In Section 4, we provide an interval of existence of mortality rate parameters M_r and b and their asymptotic expressions in Allometry survival model, in the absence of age-specific mortality data. Furthermore the asymptotic solution is useful in the study of qualitative behaviour of solution. Finally, this article provides the basic concepts of survival models and methods of estimation of survival model parameters through Statistically and Analytically.

References

[1] L. Lyons, Statistics for Nuclear and Particle Physicists, (Cambridge University Press, New York, 1986).

[2] R. Barlow, *Nucl. Instrum. Methods* **A297**, 496 (1990).

[3] G. Cowan, Statistical Data Analysis, (Oxford University Press, Oxford, 1998).

[4] For a review, see S. Baker and R. Cousins, Nucl. Instrum. Methods **221**, 437 (1984).

[5] A. Stuart, J. K. Ord and S. Arnold, Kendalls Advanced Theory of Statistics, Vol. 2A: Classical Inference and the Linear Model, 6th Ed., Oxford Univ. Press (1999).

[6] J. S. Huxley and G. Teissier, Terminologie et notation dans la description de la croissance relative, *Comptes Rendus Seances Soc. Biol. Fil.* **121**, 934–937 (1936).

[7] J. Huxley, Problems of relative growth, London: Methuen (New York: Dover), 1972.

[8] M. Kleiber, The fire of life, New York: Wiley, 1961.

[9] R. C. Heidinger, Stocking for sport fisheries enhancement. In Inland fisheries management in North America. 2nd ed. Edited by C. C. Kohler and W. A. Hubert, American fisheries society, Bethesta, Md. 375–401 (1999).

[10] I. G. Cowx, Stocking strategies, *Fish. Manag. Ecol.* **1**, 15–31 (1994).

[11] D. H. Wahl, R. A. Stein and D. R. DeVries, An ecological frame work for evaluating the success and effects of stocked fishes, *Am. Fish. Soc. Symp.* **15**, 176–189 (1995).

[12] K. Lorenzen, Population dynamics and management of culture based fisheries, *Fish. Manag. Ecol.* **2**, 61–73 (1995).

[13] I. Peterson and J. S. Wroblewski, Mortality rate of fishes in the pelagic ecosystem, *Can. J. Fish. Aquat. Sci.* **41**, 1117–1120 (1984).

[14] K. Lorenzen, The relationship between body weight and natural mortality in fish: A comparison of natural ecosystem and aquaculture, *J. Fish Biol.* **49**, 627–647 (1996).

[15] K. Lorenzen, Allometry of natural mortality as a basis for assessing optimal release size in fish-stocking programmes, *Can. J. Fish. Aquat. Sci.* **57**, 2374–2381 (2000).

[16] M. Witten, Reliability theoretic methods and aging: Critical elements, hierarchies and longevity – interpreting biological survival curves, Molecular biology of aging, Eds., A. Woodhead, A. Blachett and A. Hollaender, Plenum press New York, 1985.

[17] E. S. Lakshminarayanan and M. Pitchaimani, Unique estimation of mortality rates in Gompertz survival model parameter, *Appl. Math. Lett.* **16**, 211–219 (2003).

[18] C. E. Finch, M. C. Pike and M. Witten. Slow mortality rate accelerations during aging in some animals approximate that of humans, *Science* **249**, 902–905 (1990).

[19] M. Pitchaimani and T. Eakin, Unique estimation of Gompertz parameters with mortality deceleration rate, *Mathematical and Computer Modelling* **47**, 104–114 (2008).

[20] M. Witten, A return to time, cells, systems and aging: Relational and reliability theoretic approaches to the study of senescence in living system, *Mech. Aging and Dev.* **27**, 323–340 (1984).

[21] J. F. Lawless, Statistical models and methods for lifetime data, John Wiley and Sons, New York, 1982.

[22] M. Witten and W. Satzer, Gompertz model survival parameters: Estimation and sensitivity, *Appl. Math. Lett.* **5**, 7–12 (1992).

[23] S. S. Heppell, C. Pfister and H. de Kroon, Elasticity analysis in population biology: methods and applications, *Ecology* **81**, 606 (2000).

[24] M. Fujiwara and H. Caswell, Demography of the endangered North Atlantic right whale, *Nature*, **414**, 537–541 (2001).

[25] A. Lopez, Problems in stable population theory, Princeton University Press, Princeton, 1961.

[26] S. D. Tuljapurkar, population dynamics in variable environments. Springer–Verlag, New York, 1990.

[27] M. Pitchaimani, Uniqueness of allometry model parameters, *J. Appl. Math. Comput.* **28**, 485–500 (2008).

[28] M. Pitchaimani, Existence of allometry model parameters and their asymptotic formulae for a large population, *J. Appl. Math. Comput.* **35**, 143–159 (2011).

[29] C. G. Knight, M. N. Patel, R. B. R. Azevedo and A. M. Leroi, A novel mode of ecdysozoan growth in Canenorhabditis elegans, *Evol. & Devel.* **4:1**, 16–27 (2002).

[30] W. W. Alpatov, Growth and variation of the larvae of Drosophila melanogaster, *J. Exp. Zool.* **52**, 407–437 (1929).

[31] A. L. Rice, Growth rules and the larvae of decapod crustaceans, *J. Nat. Hist.* **2**, 525—530 (1968).

[32] R. E. Howells and L. J. Blainey, The moulting process and the phenomenon of intermoult growth in the filarial nematode, Brugia pahangi, *Parasitology*, **87**, 493–505 (1983).

[33] A. G. Wilson, Nematode growth patterns and the moulting cycle: The population growth profile, *J. Zool.* **179**, 135–151 (1976).

[34] M. Estevez, L. Attisano, J. L. Wrana, P. S. Albert, J. Massagu and D. L. Riddle, The daf-4 gene encodes a bone morphogenetic protein – receptor controlling C. elegans dauer larva development, *Nature* **365**, 644–649 (1993).

[35] L. A. Johnston, D. A. Prober, B. A. Edgar, R. N. Eisenman and P. Gallant, Drosophila myc regulates cellular growth during development, *Cell* **98**, 779–790 (1999).

[36] S. Oldham, R. Bohni, H. Stocker, W. Borgiolo and E. Hafen, Genetic control of size in Drosophila, *Phil. Trans. R. Soc. Lond. B Biol. Sci.* **355**, 945–952 (2000).

[37] D. L. Riddle, T. Blumenthal and J. R. Priess, C. elegans II. Cold Spring Harbor Laboratory Press, Cold Spring Harbor, MA., 1997.

[38] L. Byerly, R. C. Cassada and R. L. Russell, The life cycle of the nematode Canenorhabditis elegans, *Dev. Biol.* **51**, 23–33 (1976).

An application of Graph Theory to Algebra

Keerti Vardhan Madahar
Panjab University, Chandigarh
E-mail: Keerti1vardhan@gmail.com

Introduction

We give here an exposition of Richard G. Swan's proof of Amuitsar and Livitsky Theorem, which states that the commutator of $2n$ matrices of order n over a commutative ring R is zero. Swan, published his proof in 1963 (first time) and later he published a revised version in 1969 (see [1,2]), since the earlier version had a gap in it. A complete proof combining both of these papers is being presented here. Swan proves this theorem by using simple Graph Theory while the original proof of Amuitsar's and Livitsky is based on complicated algebraic definitions. To make the article self contained, some definitions and results from the graph theory are being recalled here.

Definition. *The commutator of k square matrices of order n is defined as*

$$[A_1, A_2, A_3, \dots, A_k] = \sum \operatorname{sign}(\sigma) A_{\sigma(1)}, A_{\sigma(2)}, A_{\sigma(3)}, \dots, A_{\sigma(k)}$$

The summation in the above equation runs over all elements σ of the symmetric group on k letters and $\operatorname{sign}(\sigma)$ is either $+1$ or -1 depending on whether is an even or odd permutation.

An **Eulerian trail** from a vertex P to a vertex Q in an oriented graph G is a method of walking from P to Q along all the edges of G so that every edge is traversed exactly once and in a proper pre-assigned direction e.g. in the following graph there are only two Eulerian trails from P to Q i.e. $\omega_1 = (e_1, e_2, e_3, e_4, e_5)$ and $\omega_2 = (e_5, e_3, e_4, e_1, e_2)$.

Notice that not every oriented graph contains an Eulerian trail e.g. in the figure 1 below if we reverse the orientation of the edge e_1 then there will not be any Eulerian trail. It follows easily that a directed graph G contains an Eulerian trail if and only if either, at each vertex, *in degree* is equal to the *out degree* or there is one vertex where in degree exceeds out degree by one and there is a vertex where out degree exceeds in degree by one.

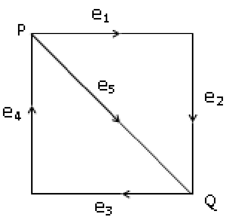

Figure 1.

Suppose an oriented graph G contains Eulerian trails. We then say two trails are equivalent if and only if one can be obtained from the other by an even permutation of the edges. For instance the Eulerian trails $\omega_1 = (e_1, e_2, e_3, e_4, e_5)$ and $\omega_2 = (e_5, e_3, e_4, e_1, e_2)$ of the figure 1 are equivalent because ω_2 can be obtained from ω_1 by using an even permutation $\sigma = (15234)$ and we may write $\omega_2 = (e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}, e_{\sigma(4)}, e_{\sigma(5)}) = \sigma(\omega_1)$. This means Eulerian trails contained in an oriented graph can be divided into two classes. If $(e_1, e_2, e_3, \dots, e_{2n})$ represents one of the trails then the class containing this trail will be denoted by ϵ_+ and other class will be denoted by ϵ_-.

An **Eulerian circuit** is an Eulerian trail which begins and ends at same vertex. This particular vertex will be denoted by R in this article.

Lemma. *Suppose an oriented graph G has m edges and n vertices such that* $m = 2n$ *then*

$$|\epsilon_+| = |\epsilon_-|; \text{ here } |X| \text{ denotes cardinality of the set } X$$

Remarks.

1. The lemma is not true for the graph of figure 1 because the condition $m = 2n$ is not met there.
2. It is enough to prove the result for Eulerian circuits instead of Eulerian trails because each Eulerian trail which starts at P and ends at Q may be modified to an Eulerian circuit by adding a vertex R and two directed edges RP and QR as shown in figure 2 below. This gives a one-to-one correspondence between the Eulerian trails from P to Q and the Eulerian circuits from R to R. This correspondence preserves the equivalence relation i.e. two trails from P to Q are equivalent if and only if the corresponding circuits from R to R are equivalent.

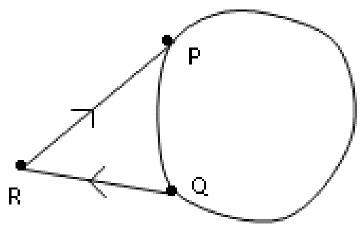

Figure 2.

Proof. Induction on the number of vertices of G will be used to prove the lemma. Clearly the result is true for a graph having one vertex and two edges. And suppose the result is true for a graph having less than n vertices. Now we are to prove the result for a graph having n vertices. We do this by considering the following three possible cases.

1. The graph G contains a loop at a vertex of degree 4

Suppose that the vertex b (shown in figure 3 below) has degree 4. If $b = R$ then every Eulerian circuit will either begin with an edge e_1 or end at the edge e_1. By moving the edge e_1 from beginning to end (in any Eulerian circuit) and vice-versa, we get 1-1 correspondence between Eulerian circuits beginning at e_1 and circuits ending at e_1. This correspondence maps an element of ϵ_+ to an element of ϵ_- and vice-versa because any circuit $(e_{i_1}, e_{i_2}, e_{i_3}, \ldots, e_{i_{2n}})$ can be transformed into another circuit $(e_{i_2}, e_{i_3}, \ldots, e_{i_{2n}}, e_{i_1},)$ by using an odd permutation $(i_1 i_2 i_3 \ldots i_{2n})$, so the lemma holds in this case.

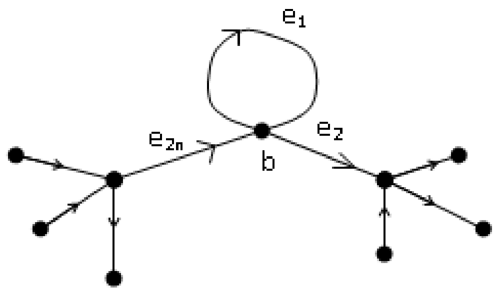

Figure 3.

If $b \neq R$, then by replacing the vertex b and three edges associated with it by a single edge e_j we get a new graph G' as shown below in figure 4. There is a 1-1 correspondence between the Eulerian circuits in G and Eulerian circuits in the modified graph G' and this 1-1 correspondence preserves the equivalence relation among the Eulerian circuits. Because if there is an even permutation which sends one Eulerian circuit, in G, onto the other then the permutation obtained by deleting two letters (corresponding to the edges e_1 and e_2 say) will again be an even permutation, so the new Eulerian circuits will also be equivalent. Since the lemma is true for G' by induction step, so is true for G due to said correspondence.

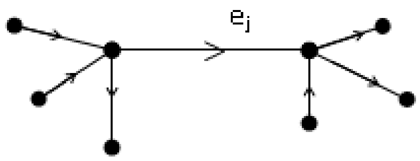

Figure 4.

2. There is a vertex $b \neq R$ of degree 2

Suppose G has a vertex $b \neq R$ of degree 2 then following two situations arise (see figure 5). In the first case observe that there is a 1-1 correspondence, which preserves equivalence relation, between Eulerian circuits in G and Eulerian circuits in $G - b$.

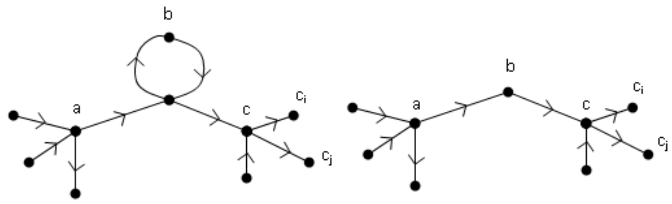

Figure 5.

By induction lemma holds for the graph $G - b$ and due to the said 1-1 correspondence, it also holds good for G.

In the second situation we can not replace edges ab and bc by an edge ac to use the induction step because we need to reduce one vertex and two edges in order to use the induction step. So in this case, for each edge cc_i, we form a graph G_i by replacing the edges ab, bc and cc_i by a new edge ac_i as shown in figure 6 below. Notice that each Eulerian circuit in G is an Eulerian circuit in exactly one of the G_i's. By induction the result holds for each G_i, so it also holds for G.

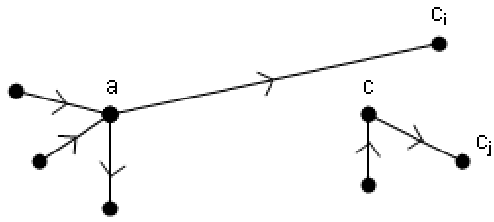

Figure 6.

3. Neither of the above two cases

Suppose G contains no loop over a vertex of degree 4 and there is no vertex other than R of degree 2. Since average degree of each vertex in G is 4 (follows from $m = 2n$) therefore either deg. $R = 2$ and one vertex will have degree 6 or all the vertices of G have degree 4. In either case we can find adjacent vertices, in G, of degree 4 as shown below in figure 7.

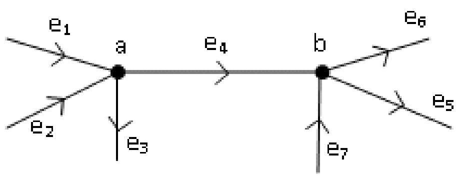

Figure 7.

Now we construct two pairs of graphs (G_1, G_2) and (H_1, H_2) from the graph G. And notice that any Eulerian circuit in G is Eulerian in exactly one of the G_i's.

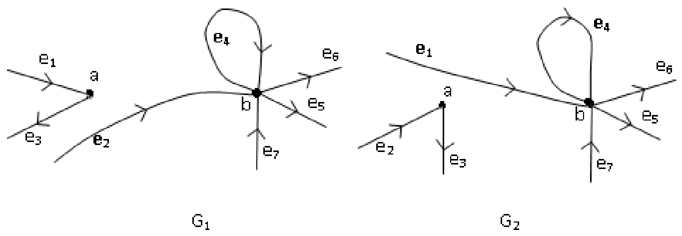

Figure 8.

Further notice that the graphs G_1 and G_2 contains some Eulerian circuits which are not coming from the Eulerian circuits of G e. g. the Eulerian circuits $\ldots e_i e_j \ldots$ for $i = 1, 2$ and $j = 5, 6$ of G_1, G_2 are not there in G. These extra circuits lie in exactly one of the H_i's given below, so they exhaust all Eulerian circuits.

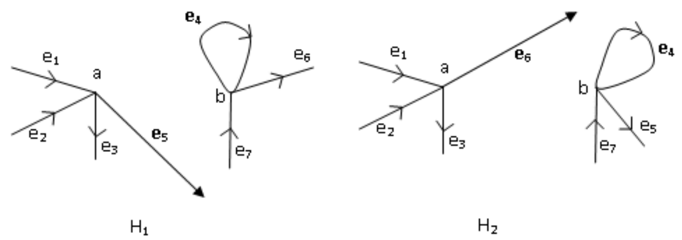

Figure 9.

The lemma is true for G_1, G_2 because of the case 2 and is true for H_1, H_2 because of the case 1.

So we have

$$\begin{aligned}|\epsilon_+(G)| &= |\epsilon_+(G_1)| + |\epsilon_+(G_2)| - |\epsilon_+(H_1)| - |\epsilon_+(H_2)| \\ &= |\epsilon_-(G_1)| + |\epsilon_-(G_2)| - |\epsilon_-(H_1)| - |\epsilon_-(H_2)| \\ &= |\epsilon_-(G)|\end{aligned}$$

Theorem. *Let $A_1, A_2, A_3, \ldots, A_{2n}$ be $n \times n$ matrices over a commutative ring R, then their commutator will be zero.*

Proof. Since the matrix units E_{ij}, i.e. the matrices with (i, j)th entry one and all other entries zeros, form a basis for $M_n(R)$ and the commutator $[A_1, A_2, A_3, \ldots, A_{2n}]$ is a multi-linear function, so it is enough to prove the theorem for the matrix units only. We construct a directed graph G corresponding to a given set of $2n$ matrix units as follows.

Let $P_1, P_2, \ldots, P_n$ denote n vertices of G and $P_a P_b$ is a directed edge of G if and only if the matrix E_{ab} is in our chosen list of $2n$ matrices.

Note that the product rule $E_{ij} E_{jk} = E_{ik}$ implies that if the matrices E_{ij} and E_{jk} are in our list then there is a path, in G, from a vertex P_i to the vertex P_k. Similarly the product of $2n$ such matrices is E_{ij} if and only if the corresponding sequence of edges form an Eulerian trail from a vertex P_i to the vertex P_j. By above lemma the number of such trails from P_i to P_j are even in number; half of which lies in ϵ_+ and half lies in ϵ_-. It amounts to state that the number of times the matrix E_{ij} occurs, in the expansion of the commutator $[E_{ab}, E_{cd}, E_{fg}, \ldots, E_{mn}]$, with coefficient $+1$ is same as the number of times it occurs with coefficient -1. The terms of the commutator which don't correspond to any Eulerian trail will be zero. Hence the result follows.

References

[1] R. G. Swan, An application of graph theory to algebra, *Proc. Amer. Math. Soc.* **14**, 367–373 (1963).

[2] R. G. Swan, Correction to an application of graph theory to algebra, *Proc. Amer. Math. Soc.* **21** 379–380, (1969).

[3] Bela Bollobas, Modern Graph Theory, Springer-Verlag (2002 Ed.)

Alexandre Grothendieck (1928 –)

R. Anantharaman

Retired from: SUNY/College,

Old Westbury, NY11568-0210

E-mail: rajan_a2000@yahoo.com

Dedicated to the memory of Dr. P. C. Vaidya, mentor to many

Abstract. This is a modest attempt to write about a most extraordinary mathematician: his youthful brilliance discovered by Jean Diendonne; his setting new directions for Banach spaces due to his famous "Resume": Work on Algebraic Geometry; Peace movement and his somewhat nonconformal acts in "real life". In an addendum we compare him with the geniuses Galois and Ramanujan.

1. Introduction

According to competent authorities Alexandre Grothendieck ranks among the very best mathematicians of the twentieth century. We refer to [4], for more complete accounts of his mathematics and other activities. Let us single out the following with little or no mathematics details:

(1) Self-taught genius while a student of J. Dieudonne (see [2]);
(2) Changed future direction of Banach Space Theory by his now famous "Resume" (1956) [9];
(3) Switched to Algebraic Geometry [7]; Fields medal in 1966 (Moscow);
(4) Started "Peace Movement" or "Survival" in protest of military like organizations in the West, for example NATO
(5) His student Alain Connes got Fields medal; both jointly awarded the Crafoord Prize (1988). Grothendieck returns his share in $200,000.
(6) Nonconformal approach to life.
(7) Conclusion
(8) Addendum: Comparison of Grothendieck with Galois & Ramanujan.

2. Self-taught Genius

Alexandre Grothendieck was born in Berlin in 1928. His father was jewish and actively worked to oppose Nazism in Hitler's Germany; this was a dangerous activity then, and he most likely died in a concentration camp [4]. The boy Alexandre and mother Hanka were (escaped?) luckily in Southern France. The country was occupied by Nazi Germany during World War II; probably they lived in hiding.

Sometime later Grothendieck studied in a University (in Southern France). The Professor, Jean Dieudonne was by all accounts an inspiring one. The following is from hearsay; I heard it from Bob Bartle [2]: The young Grothendieck was in one of Dieudonne's "first courses". A certain student (who?) pestered the good professor "about dong research now (in math.)". As this continued for sometime, Dieudonne gave the student (to study) a long joint paper of his with Laurent Schwartz (on LCS), thinking, "this will surely shut him up". Imagine the good professor's astonishment when this student returned in a month or so with answers to open problems stated at the end of the said paper!! Needless to say, A. Grothendieck was recognized from that point on for what he has been: a full fledged mathematician with enormous native talent.

Grothendieck's thesis [9] was a "Doctorate d' Etat" from the University of Paris. The dedication to his mother, Hanka Grothendieck, is in German. Also in the Introduction he thanks both Messieurs Jean Dieudonne and Laurent Schwartz not only for their advice in mathematical research but also "pour ma formation generale".

Let us mention some of his results in Banach spaces. A nodding acquaintance of Chapters II and IV of [8], Chapter I–IV of [20] is enough to understand statements of the Theorems below. Especially we mention the spaces $L^p(\lambda)$ of measurable functions f that satisfy the condition $|f(x)|^p$ is Lebesgue integrable $(d\lambda)$; later we mention L^2 and the space l^2.

We may now state an easy example of Grothendieck's early results is in the unbeatable Rudin ([20, Chapter 5]):

Let (S, Σ, λ) be a probability space $0 < p < \infty$ and X be a closed subspace of L^p. If $X \subset L^\infty$ then X is finite dimensional.

That is, any closed infinite dimensional subspace of L^p must contain subspace (many) unbounded functions.

Let us cite an example that illustrates Grothendieck's gift to generalize earlier results in a highly non-trivial way. We state it roughly avoiding technicalities; see ([8], Chapter VI) for precise statements. For the next concept let us recall that a (bounded linear operator) $T : X \to Y$ is called weakly compact if T maps the closed unit ball of X on a set whose closure is weakly compact in Y. This is stronger than bounded operator, and "absolutely 1-summing" (below) is stronger than weakly compact [6,16,19]. Let us now state an important notion due to two classical masters: In 1940 N. Dunford and B. J. Pettis proved a theorem that has the following curious consequences ([8, Chapter VI]): Let X be either a "$C(K)$

space" or an $L^1(\lambda)$ space. Then every weakly compact operator $T : X \to Y$ sends weakly null sequences to norm-null ones (here Y is an arbitrary Banach space). This property in the previous sentence was called by Grothendieck "Dunford Pettis Property" and he investigated thoroughly and abstractly such Banach spaces X that enjoy this property (i.e. for weakly compact operators T as above). Actually this property was also studied independently by J. W. Brace ([8, Chapter VI]). This property has been extensively studied since then. Later work has studied in more detail, a subclass of (such spaces) and operators T called "Absolutely Summing Operators on Banach spaces" see for eg. [16,18,19]. There are also more modern ones [6]. To explain this last concept let us recall [6,19] that a sequence (x_n) in X is said to be weakly (norm) summable if for every x^* in the dual X^* of X the scalar series $\sum |x^*(x_n)|$ (resp. $\sum \|x_n\|$) converges. Roughly speaking absolutely 1-summing operators send weakly summable sequences in X to norm summable ones in Y. Even this last concept was essentially known to Grothendieck in the Resume [11]. Returning to the Dunford–Pettis Property, it is in a section of [3], called "Le Proprite du Dunford et Pettis".

Resume. The "Resume" [11] as it is called affectionately was (in 1956) a very hard piece of mathematics to read, even by experts. It was only in 1968 that J. Lindenstrauss and P. Pelczynski [16] revived it, rewrote many parts in it without using Tensor products and added their own results on Banach space Theory. They say in their Introduction: "The paper by Grothendieck is hard to read and its contents are not known even to experts in Banach space Theory. In facts using the results in it many problems in this subject can be easily solved" and they show how. Let us quote one famous now classic result that follows from a Theorem in the Resume:

Theorem [11,16,17]. *Let X be a Banach space such that both X and its dual X^* embed isomorphically into an L^1 space. Then X is isomorphic to Hilbert space. Thus, if X embeds into L^1 and X is not isomorphic to Hilbert space, then X^* cannot embed into L^1.*

The proof of this Corollary to Grothendieck's Theorem (GT) ([6,19]) is not easy. One version of GT says that every bounded operator $T : L^1 \to l^2$ is absolutely 1-summing ([6,18,19]). According to experts [4] the directions set by Grothendieck for Banach space Theory in the Resume (1956) has continued and is doing so even now.

3. Algebraic Geometry/Fields Medal

Next Grothendieck turned his attention to Algebraic Geometry and was making fundamental contributions. We refer to [4,7,14] for a competent description of the ideas involved. Let us only remark that this subject is as old as Rene DesCartes (more than 350 years ago!). Naturally it has seen changes, as did the Calculus of Newton and Leibnitz. The latter subject wears the "dress of Measure theory". Once again he sought to change the subject "in its depths". He was doing this in the University of Paris Seminar conducted by him and jointly with others. The culmination of these efforts are in [7]. In fact, some of the people (including Grothendieck himself) involved in those seminars formed the core of famous 'Bourbaki'; one may also see a recent book [1] in which the first chapter titled 'Disappearance' is devoted entirely to Grothendieck.

Grothendieck was awarded the Fields medal at the International Congress of Mathematicians held at Moscow in 1966. Hearsay report [2] has it that Jean Dieudonne mentioned the extra ordinary precocity exhibited by Grothendieck even as an undergraduate (see Section 1 above).

4. Peace Movement

Alexandre Grothendieck has been a pacifist all through. About the time he was awarded the Fields medal he strongly felt that countries in the West were as much (or more?) in error as the opposite side. To remind ourselves: those were the years of the Cold War; the two superpowers were USA and Soviet Russia. Western countries were allied to USA; France was one of them. The Eastern block allied to Soviet Russia had a good number too, notably East Germany. Both sides had/were developing nuclear arsenals "as protection against sudden attack by the opposite side".

Grothendieck wanted to start a protest or Peace movement, that he called "Survival". Needless to say, he was thinking of the mathematicians in Western countries; any such open protest in the Soviet block was (i) not allowed (ii) deadly game as far as your life was concerned.

He argued that all the mathematicians (in the West) have a moral obligation to protest to their governments against the dangerous stock pile of (nuclear) weapons. After all, there are other problems like poverty. Further, the West is essentially democratic and hence “mathematicians can and must voice their objections; if needed we may use Civil Disobedience as taught by Mahatma Gandhi”.

For starters, Grothendieck wanted mathematicians in the West to refuse grant moneys from military agencies such as the Dept. of Defense or NATO. The author humbly feels that Grothendieck reckoned without considering human nature, even among mathematicians (in the matter of getting grant money legally/regardless of source) (see [4]).

Grothendieck practised (late 60’s – early 70’s) what he preached: he said “he was not doing mathematics now”, and threw his energy into the Peace movement (see [4]). There is a report/talk by him in the early 70’s to Mathematics Dept. at McGill Univ. (Canada) and the title is “Survival”.

(Un)fortunately Grothendieck had only a few followers in his “Survival”, though all admired his courage of conviction as much as they did his achievements in Mathematics.

5. Connes, Crafoord Prize; Grothendieck’s Style

Grothendieck was Professor of Mathematics at the University of Paris. His student Alan Connes was quite well known first for his work on (cohomology of) Operator Algebras. Subsequently he solved a long standing open problem (see [4]). In 1988 the Crafoord Prize (in Mathematics) worth $200,000 was awarded jointly to A. Connes and A. Grothendieck. Grothendieck returned his share of prize saying that “the money I get from the University is sufficient for me and those who depend on me” [12].

Grothendieck’s style. Even as a student under Dieudonne (see Section 1) Grothendieck was showing his talent for extreme (but highly non trivial!) abstraction that has been his hall mark all along. May be due to this and the density (as well as the speed?) of ideas he made a few errors (see [10]). Grothendieck has “Addenda” in [10] where he states and then fixes these errors pointed out by Monsieur Dieudonne. It is interesting that in his long papers (nearly 80+ pages) (for example [11]) Grothendieck’s list of references does not exceed two pages.

Another characteristic of Grothendieck was that he worked on one subject at a time with all the power and concentration he could muster rich ideas that was a great lot. When he had made ground breaking discoveries he left the subject – it seems so in respect of Banach Spaces/Tensor Products after the “Resume”. Others have developed his very rich ideas. Accounts of working with Grothendieck are in the very recent [15].

6. Nonconformal Acts in “Real” Life

We saw earlier (see Section 1) that Alexandre Grothendieck acknowledged the influence of his early mentors, Jean Dieudonne and Laurent Schwartz “pour ma formation generale”. From hearsay, Laurent Schwartz was in the Communist party of France. That is, Grothendieck’s (personal) ideas were, if anything, “to the Left”.

His sympathies were with the poor and downtrodden. His apartment in Paris was open to all sorts of people ([4]); sometimes he let in the ones “not desirable/successful in society”. Apparently this last was illegal in France and he got into trouble with authorities [4].

Another act of his (from hearsay) was to use footwear (shoes?) made from rubber tires useless to cars. This was probably to promote conservation.

Report has it (see [4]) that Grothendieck lived in seclusion for a period (2002?) in Tibet (?) or Himalaya and that he wanted to change the current velocity of light (see [4]). It seems that he had “no political skill” in Committees. “He ruined in ten minutes the work of months of political preparation” (see [4]).

Grothendieck apparently made a list of Teachers that the World has seen. For example: Buddha, Jesus Christ etc. to Mahatma Gandhi.

7. Conclusion

The exceptions in “daily life” do not detract a bit from Alexandre Grothendieck’s place among the “all time very bests” in mathematics. His ideas and his mathematics are a source of inspiration to all mathematicians.

8. Addendum: Comparison of Grothendieck with Galois and Ramanujan

Let us discuss this topic very briefly.

Evariste Galois is recognized for quite some time as an incredible genius who set directions for (abstract) algebra.

Indeed Galois theory ranks very high if not at the top in its intrinsic "aesthetic beauty". It is also quite abstract, though it answers for the first time (in centuries!) a basic "concrete" question: When can a polynomial equation (over the rationals) be solved using only radicals? In terms of extreme abstraction, Galois is closer to Grothendieck than the other.

In contrast Ramanujan's results on number theory are "more concrete"; at least the statements of his Theorems are so. He excels (see [13]) in "nice looking" formulas. It is still beautiful mathematics according to experts (for example [13] among others) though not the same kind as Galois theory. Apparently he provided very few proofs. Ramanujan's genius is of a different nature from the other two in this respect. One reason is that he had very little formal training (see [13]), however, learned mathematicians have worked (and are still working) on his results (see for example [5]).

What Grothendieck has in common with the other two is a very highly developed intuition and ability to open new paths; for example his "Resume". As another with this gift we may mention Bernhard Riemann. In his case it was (also) deep geometric insight as well as in number theory. All of these were in a sense "true" mathematicians, not mainly logicians. Luckily Grothendieck's genius is recognized in his lifetime; his case is different from the other two in this respect.

Acknowledgements

The author is grateful to colleagues at IIT Madras for moral and material support. He is especially indebted to Prof. S. Sundar and P. V. Subrahmanyam at IIT Madras. The author is grateful to Prof. Aravinda and Ponnusamy for their suggestions, and help with the editing.

References

[1] Amir D. Aczel, The Artist and the Mathematician: The story of Nicolas Bourbaki, Basic Books, 2007.

[2] R. G. Bartle, Personal Communication, Kent, Ohio, 1975.

[3] N. Bourbaki, Integration, Che VI, Hermann-Paris, 1959.

[4] P. Cartier, Alexandre Grothendieck, Bulletien of A.M.S., 2004.

[5] F. Dyson, A walk in Ramanujan's garden, Univ. of Illinois, Urbana Ill. Ramanujan Centenary Conference, Academic Press, 1987.

[6] J. Diestel, H. Jarchow and A. Tonge, Absolutely summing operations on Banach spaces, Cambridge University Press, 1995.

[7] J. Dieudonne et A. Grothendieck, Elements de la Geometre Algebraique, Hermann-Paris, 1970.

[8] N. Dunford and J. Schwartz, Linear Operators I, Inter science, New York, 1958.

[9] A. Grothendieck, Produits Tenseriels topologiques et espaces Nucleairies, Memoirs of A.M.S. **16** (1955).

[10] A. Grothendieck, Sur les applications lineaires faiblement compactes d'espaces du type $C(K)$, *Canad J. Math.* **5**, 129–173 (1955).

[11] A. Grothendieck, Resume de la theorie metrique des produits tensoriels topologiques, *Bol. Soc. Mat. Sao Pauli* **8**, 1–79 (1956)

[12] A. Grothendieck, Letter refusing Crafoord Prize (his share), Notices of A.M.S., 1958.

[13] G. H. Hardy, Ramanujan [Lectures delivered at Harvard Univ. 1940], Chelsea Publishers, New York, 1970.

[14] R. Hartshorne, Alebraic Geometry, Springer Verlag, 1979.

[15] L. Ilusie, A. Beileison, S. Bloch and V. Drinfeld, *et al.*, Reminiscences of Grothendieck and his School, Notices of A.M.S. (October) 2010.

[16] J. Lindenstrauss and A. Pelczynski, Absolutely Summing operators in L_p spaces & applications, *Studia Math.* **29**, 275–326 (1968).

[17] J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces I, II, Springer-Verlag, 1977, 1979.

[18] A. Pelczynski, Banach spaces of Analytic Functions & Absolutely Summing operators, CBMS Series #30, A.M.S., 1976.

[19] G. Pisier, Factorisation of Linear Operators and Geometry of Banach spaces, CBMS Series #60, A.M.S., 1986.

[20] W. Rudin, Functional Analysis, Tata McGraw Hill, New York/New Delhi, 1974.

Error in "Wiener's Theorem, Infinite Matrices and Banach Algebras"

S. H. Kulkarni

Department of Mathematics, Indian Institute of Technology, Madras, Chennai 600 036

E-mail: shk@iitm.ac.in

There is an error in Remark 3.7 of the article [2] "Wiener's Theorem, Infinite Matrices and Banach Algebras" that appeared in the last issue (Vol. 19, No. 2) of the Mathematics Newsletter of the Ramanujan Mathematical Society. The Remark deals with the algebras A and B, where $A = C(\Gamma)$, is the set of all continuous complex valued functions on the unit circle Γ with the supremum norm and B is the set of functions in A with absolutely convergent Fourier series. It is then said that B is a closed subalgebra of A. This is wrong, B is not a closed subalgebra of A (In fact, if this were the case, then B would coincide with A by the famous Stone-Weierstrass Theorem). Thus the inverse-closedness of B does not follow from Theorem 3.6 and this is the error.

However, the inverse-closedness of B follows from Wiener's theorem. As mentioned in the Introduction of [2], Gelfand gave an elegant proof of Wiener's theorem using the techniques from Banach Algebras. Gelfand's proof can be found in many books, for example, [1], [6]. In the expository article [3], this proof is presented along with some basic minimum facts from the theory of Banach algebras required to understand this proof. Several proofs of Wiener's theorem are available in the literature. Some of these can be found in [5]. Among these there is a very simple, short and elementary proof due to Newman [4].

Acknowledgement

I thank Prof. Subhash Bhatt for pointing out the error. It is always nice to know that someone has read your work carefully.

References

[1] F. Frank, Bonsall and John Duncan, Complete normed algebras, Springer-Verlag, New York, 1973. MR 0423029 (54 #11013). *aré Anal. Non Linéaire* **7**, No. 5, 461–476 (1990), MR1138533 (93h:47035).

[2] S. H. Kulkarni, Wiener's theorem, infinite matrices and Banach algebras, RMS Mathematics Newsletter **19**, No. 2, 1–4 (2010).

[3] S. H. Kulkarni, Gelfand's proof of Wiener's theorem. http://mat.iitm.ac.in/home/shk/public_html/wiener1.pdf

[4] D. J. Newman, A simple proof of Wiener's $1/f$ theorem, *Proc. Amer. Math. Soc.* **48**, 264–265 (1975).

[5] N. Nikolsky, Operators, Functions and Systems: An Easy Reading, vol. 1, Mathematical Surveys and Monographs 92, *Amer. Math. Soc.* (2002).

[6] W. Rudin, Functional analysis, Second edition, McGraw-Hill, New York, 1991. MR1157815 (92k:46001).

Advanced Training in Mathematics Schools

Supported by **National Board for Higher Mathematics**

Workshop on Group Theory

16–21 May 2011

Venue: Indian Institute of Science Education and Research, Mohali.

Contact Address:
Amit Kulshrestha, Convener, ATMW in Group Theory
Indian Institute of Science Education and Research
MGSIPA Complex, Sector Â-26, Chandigarh 160 019
Phone: +91 172 2791025 Extn. 2314 (Off),
+91 9988084466 (Mob)
Fax: +91 172 2790188,
E-mail: ibspassi@yahoo.co.in;
amitk@iisermohali.ac.in
For details visit: http://www.bprim.org/atm

National Conference on Applied and Engineering Mathematics (NCAEM–2011)

28–30 July, 2011

Venue: Department of Mathematics, RNS Institute of Technology.

For Further Information Contact:

Dr. S. Padmanabhan
RNS Institute of Technology
Channasandra, Uttarahalli, Kengeri Main Road
Bengaluru 560 061, Karnataka, India
Phone: +91(0)-9341248904
E-mail: ncaem2011@gmail.com;
mailto:ncaem2011@gmail.com

Indo-European Study Group Meeting on Industrial Problems (IESGMIP 2011)

2–6 May, 2011

Venue: Department of Applied Mathematics, Faculty of Technology & Engineering, The M. S. University of Baroda.

For Further Details Contact/Visit:

Dr. Dhanesh Patel
Department of Applied Mathematics
Faculty of Technology and Enggineering
The Maharaja Sayajirao University of Baroda
Vadodara, Gujarat State, India
Phone No: +91(0)-98793 35246
E-mail: indoscsgmip@yahoo.co.in;
pdhanesh@yahoo.com
Web: http://www.msubaroda.ac.in/iesgmip2011/

Developing Research Aptitude in Selected Areas in Mathematics for College Teachers

3–4 May, 2011

Venue: Department of Mathematics, Periyar Maniammai University.

For Further Information Contact:

Prof. A. Sasikala, Convenor
Periyar Maniammai University
Vallam, Thanjavur
Tamilnadu State, India
Phone: +91(0)-9790035784, 9790035790
E-mail: headmaths@pmu.edu

The 5th International Conference on Research and Education in Mathematics (ICREM5)

22–24 October, 2011

ITB Bandung, Indonesia

This conference is jointly organized by:

Faculty of Mathematics and Natural Sciences, Institute Teknologi Bandung (ITB),
Institute for Mathematical Research, Universiti Putra Malaysia (INSPEM),
Institute of Mathematics, Vietnam Academy of Science & Technology (IMVAST),
Indonesian Mathematical Society (IndoMS), and
Indonesian Combinatorial Society (InaCombS).

Keynote Speakers:

– Cedric Villani (Institute Henri Poincare, France) Fields Medalist 2010

Topics: All aspects/fields in Mathematics, Statistics and the applications as well as Mathematical Education, including, but not limited to:

Conference Program: Keynote talks, plenary & invited talks, contributed presentations and poster sessions.

Proceeding Conference:

– All the extended abstracts will be included in the Conference Abstract.
– The selected (refereed) papers from the conference will be published in the American Institute of Physics (AIP) Conference Proceedings Series.

Important Dates:

- Extended Abstract (max 2 pages) Submission deadline: 3 June 2011
- Notification of the Abstract Acceptance: 15 July 2011
- Registration Deadline: 12 August 2011
- Early Bird Registration Payment deadline: 15 August 2011
- Full papers submission deadline: 20 October 2011
- Notification of the paper Acceptance: 20 December 2011
- Proceeding Publication in the AIP Conference Proceedings Series: 20 April 2012

Organizing Committee:

Co-Chairs:

- Edy Tri Baskoro (ebaskoro@math.itb.ac.id)
- Roberd Saragih (roberd@math.itb.ac.id)

National Institute of Technology Calicut, Kerala

Department of Mathematics
In Association with
Academy of Discrete Mathematics and Applications (ADMA)

Seventh Annual Conference of Academy of Discrete Mathematics and Applications (ADMA) and Graph Theory Day VII

9–11 June, 2011

Introduction: As part of the Golden Jubilee celebrations of the National Institute of Technology Calicut, The Department of Mathematics, National Institute of Technology Calicut is hosting the Seventh Annual Conference of Academy of Discrete Mathematics and Applications (ADMA) and Graph Theory Day VII, 9–11 June 2011

Graph Theory Day VII and Professor Frank Harary Memorial Endowment Lecture: Academy of Discrete Mathematics and Applications (ADMA) is organizing its annual conferences with a celebration of Graph theory day on 10th June of every year as announced by Department of Science and Technology, Govt. of India. Seventh GRAPH THEORY DAY on 10th June 2011 will be observed with special activities such as expression of appreciation to the Invited Speakers and 'Professor Frank Harary Memorial Endowment Lecture' the First in the series, will be delivered by Prof. Tuza Zsolt, Hungarian Academy of Sciences, Budapest.

Call for Papers: Papers are invited for presentation in the ADMA-2011 on all aspects of Discrete Mathematics such as Graph Theory, Coding Theory, Cryptography, Design Theory, Combinatorics, etc. **Interested participants may send the abstracts not exceeding 200 words with a copy of full paper, as a pdf file to Ambat Vijayakumar, Academic Secretary, ADMA** latest by **1st May 2011** by E-mail: vambat@gmail.com.

Registration Fee Details:

With accommodation	: Rs. 1000/- for Life Members of ADMA and Rs. 1200/- for other participants
Without accommodation	: Rs. 500/- for Life Members of ADMA and Rs. 700/- for other participants

Important Dates:

Last date for receipt of applications : 01-05-2011
Intimation of selection: on or before : 10-05-2011

Registration Form

Name (in BLOCK LETTERS)	:	____________
Designation Address for Communication	:	____________
E-mail	:	____________
Telephone	:	____________
Accommodation required	:	Yes: ☐ No: ☐
Sex	:	Male: ☐ Female: ☐
Are you a member of ADMA?	:	Yes: ☐ No: ☐
If Yes, Membership Number	:	____________
Nature of Participation Invited speaker/Office bearer/ Participant	:	
Presenting paper	:	Yes: ☐ No: ☐

Demand Draft No : ______________________________

Date : ______________________________

Drawn on Bank : ______________________________

Amount : ______________________________

Note: Conference registration fee is to be paid by demand draft only, in favour of **Co-ordinator**, **ADMA 2011** payable at SBI, NITC (code 2207).

The filled up registration form along with the DD for registration should be send to the following address.

The Coordinator, ADMA-2011
Department of Mathematics
National Institute of Technology Calicut
NITC P.O., Kerala 673 601, India

Application forms can also be downloaded from NITC website (`www..nitc.ac.in`) and may be send to the coordinators by E-mail `sunithanitc@rediffmail.com`

Visit: `http://www.nitc.ac.in`

Indian National Science Academy
Bahadur Shah Zafar Marg,
New Delhi 110 002

Information for the applicants seeking partial assistance for participation in ICSU sponsored International Conferences abroad

International Council for Science (ICSU) is a non-governmental organization formed during 1931 to promote scientific activities in Science and their applications for benefits of humanities. Being an adhering organization to ICSU, Academy provide partial financial support for attending International scientific conferences sponsored by the International Council for Science (ICSU) and its affiliated bodies.

Scientists who has specific role in the International Conference i.e. invited to deliver plenary lecture/preside over session or whose paper has been accepted for presentation, and who will also be provided maintenance allowance during his/her stay abroad and partial travel by some agency, will be given preference over others. INSA's financial support, in case of selection is limited to a maximum of half International travel, half maintenance allowance for the duration of the conference and registration fee, wherever necessary.

Application should be sent to the Academy three months before the commencement of the conference. Applications received without sufficient notice may not be considered. Please also note that no assistance is provided for training programmes/courses/joining post doctoral Fellowships and for higher studies abroad.

Bodies Affiliated to ICSU:

International Unions of: (1) Astronomy (IAU), (2) Biochemistry and Molecular Biology IUBMB), (3) Biological Sciences (IUBS), (4) Pure & Applied Chemistry (IUPAC), (5) Crystallogaphy (IUCr), (6) Geodesy & Geophysics (IUGG), (7) Geological Sciences (IUGS), (8) History & Philososophy of Science (IUHPS), (9) Theoretical & Applied Mechanics (IUTAM), (10) Nutritional Sciences (IUNS), (11) Pune & Applied Physics (IUPAP), (12) Pure & Applied Biophysics (IUPAB) (13) Microbiological Societies (IUMS), (14) Pharmacology (IUPHAR), (15) Radio Science (URSI), (16) Physiological Sciences (IUPS), (17) Geography (IGU), (18) Mathematical Science (IMU), Scientific/Special Committees on: (19) Lithosphere (SCL), (20) Oceanic Research (SCOR), (21) Solar-Terrestrial Physics (SCOSTEP), (22) Space Research (COSPAR), (23) Data for Science and Technology (CODATA), (24) World Climate Research Programme (WCRP), (25) Antarctic Research SCAR), (26) International Geosphere Biosphere Programme (IGBP), (27) Quaternary Research (INQUA).

For International Conferences sponsored by other agencies, the applicant may send their application to:

Dr. B. Babuji Scientific Officer
Centre for International Co-operation in Science (CICS)
#2 Gandhi Mandapam Road, Guindy, Chennai 600 025
Phone: 24419466/24430228/24901367
Fax: 044-24914543
E-mail: `cics2010@vsnl.net; ccstds@vsnl.net.`

Application form for seeking financial assistance for participation in ICSU Sponsored international conferences/ Meetings aborad cane be obtained form the home page INSA.

Special Lecture Series on 60th Birthday of Prof. R. Balasubramanian

On the occasion of Prof. R. Balasubramanian's 60th birthday, number theorists from Institute of Mathematical Sciences (IMSc), Chennai Mathematical Institute (CMI) and the University of Lille, France, are arranging year long program in number theory at IMSc. The goal of this number theory year is to acquaint researchers to major developments currently underway. The details of lectures and speakers can be found at the web page:

www.imsc.res.in/~ntychennai

The organizers plan to publish notes based on these lectures which will be both affordable and accessible to a larger scientific community.

Conference Announcement and Invitation for Papers

We cordially invite you to submit a paper to the upcoming 4th International Congress on Image and Signal Processing (CISP 2011) and the 4th International Conference on BioMedical Engineering and Informatics (BMEI 2011), to be jointly held from 15–17 October 2011, in Shanghai, China.

Shanghai is the largest city in China, with famous historical and cultural heritage. Attractions include Yuyuan Garden ("Happy Garden" built in Ming Dynasty), Shanghai Museum with 120,000 pieces of rare relics, Shanghai World Financial Center, Jade Buddha Temple (Song Dynasty), Oriental Pearl TV Tower, Zhujiajiao Water Town, and Expo 2010 site.

All papers in conference proceedings will be indexed by both EI Compendex and ISTP, as well as included in the IEEE Xplore (IEEE Conference Record Number for CISP'11: 18205; IEEE Conference Record Number for BMEI'11: 18206. CISP-BMEI 2008–2010 papers have already been indexed in EI Compendex). Substantially extended versions of best papers will be considered for publication in a CISP'11-BMEI'11 special issue of the Computers and Electrical Engineering journal (SCI-indexed).

CISP-BMEI is a premier international forum for scientists and researchers to present the state of the art of multimedia, signal processing, biomedical engineering and informatics. The previous CISP-BMEI each attracted over 3000 submissions from all over the world, with acceptance rate around 50%. The registration fee of US$ 390 includes proceedings, lunches, dinners, banquet, coffee breaks, and all technical sessions. CISP'11-BMEI'11 is technically co-sponsored by the IEEE Engineering in Medicine and Biology Society.

To promote international participation of researchers from outside the country/region where the conference is held (i.e., China's mainland), researchers outside of China's mainland are encouraged to propose invited sessions. The first author of each paper in an invited session must not be affiliated with an organization in China's mainland. All papers in the invited sessions can be marked as "Invited Paper". The organizer(s) for each invited session with at least 6 registered papers will (jointly) enjoy an honorarium of US$ 400. Invited session organizers will solicit submissions, conduct reviews and recommend accept/reject decisions on the submitted papers. Invited session organizers will be able to set their own submission and review schedules, as long as a list of recommended papers is determined by 30 May 2011. Each invited session proposal should include:

(1) the name, bio, and contact information of each organizer of the invited session;
(2) the title and a short synopsis of the invited session.

Please send your proposal to CISP-BMEI@dhu.edu.cn

For more information, visit the conference web page:

http://cisp-bmei.dhu.edu.cn

If you have any questions after visiting the conference web page, please E-mail the secretariat at

CISP-BMEI@dhu.edu.cn

Join us at this major event in exciting Shanghai!!!

Contact Information:

School of Information Science and Technology
Donghua University, No. 2999, North Renmin Road,
Songjiang District, Shanghai, China 201620
Phone: +86-21-67792323
E-mail: CISP-BMEI@dhu.edu.cn

The readers may download the Mathematics Newsletter from the RMS website at www.ramanujanmathsociety.org