A Simple Proof of Gödel's Incompleteness Theorems

Arindama Singh
Department of Mathematics, IIT Madras, Chennai 600 036, India
E-mail: asingh@iitm.ac.in

1. Introduction

Gödel's incompleteness theorems are considered as achievements of twentieth century mathematics. The theorems say that the natural number system, or *arithmetic*, has a true sentence which cannot be proved and the consistency of arithmetic cannot be proved by using its own proof system; see [1]. Though the ideas involved in their proofs are very complex, they can be presented in a simple and comprehensible way.

2. Background

We assume a theory of arithmetic, say $\mathcal{N} = (\mathbb{N}, +, \times)$ to be consistent. Write $\vdash X$ for "X is a theorem in $\mathcal{N}$." The usual theorems or laws of logic hold true in this theory. We will be using explicitly the laws of Double Negation, Contradiction, Distribution of implication, Contraposition, Modus Ponens and Hypothetical Syllogism, as spelled out below.

$$\vdash \neg\neg X \leftrightarrow X. \tag{1}$$

$$\vdash X \to (\neg X \to Y). \tag{2}$$

$$\vdash (X \to (Y \to Z)) \to ((X \to Y) \to (X \to Z)). \tag{3}$$

$$\text{If } \vdash X \to \neg Y, \text{ then } \vdash Y \to \neg X. \tag{4}$$

$$\text{If } \vdash X \text{ and } \vdash X \to Y, \text{ then } \vdash Y. \tag{5}$$

$$\text{If } \vdash X \to Y \text{ and } \vdash Y \to Z, \text{ then } \vdash X \to Z. \tag{6}$$

Besides the logical laws, there are some more theorems specific to arithmetic, which are obtained by encoding formulas as natural numbers. The encoding is the so called Gödel numbers. We define Gödel number $g(\cdot)$ of symbols, formulas (in general, strings), and proofs as follows.

Enumerate the symbols such as connectives, quantifiers, punctuation marks, predicates, function symbols, and variables as:

$$\top, \bot, \neg, \wedge, \vee, \to, \leftrightarrow, (,), P_1, f_1, x_1, P_2, f_2, x_2, \ldots$$

Define $g(\sigma) = n$, where the symbol σ comes as the n-th in the above list. Extend g to strings of these symbols by

$$g(\sigma_1\sigma_2\cdots\sigma_m) = 2^{g(\sigma_1)} \times 3^{g(\sigma_2)} \times \cdots \times p_m^{g(\sigma_m)}$$

where p_i is the i-th prime number. This defines g of terms and formulas. Next, extend g to proofs of formulas by

$$g(X_1X_2\cdots X_m) = 2^{g(X_1)} \times 3^{g(X_2)} \times \cdots \times p_m^{g(X_m)}.$$

where again p_i is the i-th prime number.

Due to prime factorization theorem in $\mathcal{N}$, the function g has the following properties:

(a) g is a computable function.
(b) $g(uv)$ can be computed from those of $g(u)$ and $g(v)$.
(c) Given $n \in \mathbb{N}$, if $n = g(X)$ and X is known to be a symbol, or a formula, or a proof, then X can be computed from n.

Let $Proof(x, y)$ be a binary predicate that translates "x is the Gödel number of a proof of a formula whose Gödel number is y. Write

$$Pr(y) = \exists x \, Proof(x, y).$$

That is, we interpret Pr as the subset (unary relation) of $\mathbb{N}$ which is the set of Gödel numbers of all provable (in $\mathcal{N}$) formulas. $Pr(g(X))$ means that there is a natural number which is the Gödel number of a proof of a formula whose Gödel number is $g(X)$. Which, in turn, means that there is a natural number which is the Gödel number of a proof of X. We thus use a further abbreviation such as $P(X) = Pr(y) = Pr(g(X))$. The predicate Pr is the provability predicate. We may also say that P is a predicate whose arguments are formulas, and that $P(X)$ means that X is provable in $\mathcal{N}$. $P(X)$ is a formula in the theory $\mathcal{N}$. We thus loosely call P as the provability predicate.

3. Provability Predicate

The provability predicate P has the following properties:

$$\text{If } \vdash X, \text{then } \vdash P(X). \tag{7}$$

$$\vdash P(X \to Y) \to (P(X) \to P(Y)). \tag{8}$$

$$\vdash P(X) \to P(P(X)). \tag{9}$$

Since $0 \neq 1$ in $\mathcal{N}$, $P(0=1)$ expresses inconsistency of $\mathcal{N}$. Therefore, consistency of $\mathcal{N}$ may be formulated by asserting that the sentence $P(0 = 1)$ is not a theorem of $\mathcal{N}$. Our assumption of consistency of $\mathcal{N}$ thus gives

$$\nvdash P(0=1). \tag{10}$$

Let $B_1(n), B_2(n), \ldots$ be an enumeration of all formulas in $\mathcal{N}$ having exactly one free variable. Consider the formula $\neg P(B_n(n))$. This is one in the above list, say $B_k(n)$. Since $\vdash p \leftrightarrow p$, we have $\vdash B_k(n) \leftrightarrow \neg P(B_n(n))$. Then, $\vdash \forall n(B_k(n) \leftrightarrow \neg P(B_n(n)))$, by universal generalization and

$$\vdash B_k(k) \leftrightarrow \neg P(B_k(k)),$$

by universal specification with $[n/k]$. Abbreviating the sentence $B_k(k)$ to A, we obtain:

$$\vdash A \leftrightarrow \neg P(A). \tag{11}$$

The statement (11) says that the sentence "This sentence is not provable" is expressible and is a theorem in $\mathcal{N}$; ingenuity of Gödel.

For a formal proof of (11), start with the formula $B(x)$, having exactly one free variable. Let the *diagonalization* of $B(x)$ be the expression

$$\exists x(B(x) \wedge (x = g(B(x)))).$$

Since g a computable function, the relation

> diag(m, n) : n is the Gödel number of the diagonalization of the formula having exactly one free variable with Gödel number m

is recursive and hence representable in $\mathcal{N}$ by some binary predicate, say, $C(x, y)$. Next, define

> the formula $F(x)$ as $\exists y(C(x, y) \wedge B(y))$
> the sentence G as $\exists x(\exists y(C(x, y) \wedge B(y)) \wedge (x = g(F(x))))$.

Finally, show that $\vdash G \leftrightarrow B(g(G))$.

You have thus proved the Diagonalization Lemma:

> *for each formula $B(y)$ with exactly one free variable there exists a sentence G such that $\vdash G \leftrightarrow B(g(G))$.*

Next, take $B(y)$ as $\neg Pr(y)$ to obtain (11); see for example, [2].

Here are some more properties of this special sentence A.

$$\vdash P(A) \to P(\neg A) \tag{12}$$

Proof.

$$\vdash A \leftrightarrow \neg P(A) \tag{11}$$
$$\vdash P(A) \to \neg A \tag{4}$$
$$\vdash P(P(A) \to \neg A) \tag{7}$$
$$\vdash P(P(A) \to \neg A) \to (P(P(A)) \to P(\neg A)) \tag{8}$$
$$\vdash P(P(A)) \to P(\neg A) \tag{5}$$
$$\vdash P(A) \to P(P(A)) \tag{9}$$
$$\vdash P(A) \to P(\neg A) \tag{6}$$

The sentence A is connected to inconsistency as the following property shows:

$$\vdash P(A) \to P(0=1). \tag{13}$$

Proof.

$$\vdash A \to (\neg A \to (0=1)) \tag{2}$$
$$\vdash P(A \to (\neg A \to (0=1))) \tag{7}$$
$$\vdash P(A \to (\neg A \to (0=1))) \to (P(A) \to P(\neg A \to (0=1))) \tag{8}$$
$$\vdash P(A) \to P(\neg A \to (0=1))) \tag{5}$$
$$\vdash P(\neg A \to (0=1)) \to (P(\neg A) \to P(0=1)) \tag{8}$$
$$\vdash P(A) \to (P(\neg A) \to P(0=1)) \tag{6}$$
$$\vdash (P(A) \to (P(\neg A) \to P(0=1))) \to ((P(A) \to P(\neg A)) \to (P(A) \to P(0=1))) \tag{3}$$
$$\vdash (P(A) \to P(\neg A)) \to (P(A) \to P(0=1)) \tag{5}$$
$$\vdash P(A) \to P(\neg A) \tag{12}$$
$$\vdash P(A) \to P(0=1) \tag{5}$$

4. Incompleteness Theorems

Theorem 1. *There exists a sentence C in $\mathcal{N}$ such that $\nvdash C$ and $\nvdash \neg C$.*

Proof. Take $C = A$, the sentence used in (11). If $\vdash A$, then by (7), $\vdash P(A)$. But by (11), $\vdash A$ implies that $\vdash \neg P(A)$. This is a contradiction. On the other hand, if $\vdash \neg A$, then by (11), $\vdash P(A)$. By (13) and (5), $\vdash P(0=1)$. This contradicts (10). Therefore, neither $\vdash A$ nor $\vdash \neg A$. $\square$

Theorem 2. *There exists a true sentence in $\mathcal{N}$ which is not provable in $\mathcal{N}$.*

Proof. Consider the sentence A used in (11). Either A is true or $\neg A$ is true. But neither is provable by Theorem 1. Whichever of A or $\neg A$ is true serves as the sentence asked in the theorem. $\square$

Theorem 3. $\nvdash \neg P(0 = 1)$.

Proof. Suppose $\vdash \neg P(0 = 1)$. Due to (1), (4) and (13), $\vdash \neg P(A)$. By (11), $\vdash\ A$. By (7), $\vdash\ P(A)$. By (13), $\vdash P(0 = 1)$. This contradicts (10). $\square$

Notice that the sentence $\neg P(0 = 1)$ can also serve the purpose of the sentence asked in Theorems 1–2.

Theorems 1–2 are called as Gödel's First Incompleteness theorem; they are, in fact one theorem. Theorem 1 shows that Arithmetic is negation incomplete. Its other form, Theorem 2 shows that no axiomatic system for Arithmetic can be complete. Since axiomatization of Arithmetic is truly done in second order logic, it shows also that any axiomatic system such as Hilbert's calculus for second order logic will remain incomplete.

In (9–12), we could have used any other inconsistent sentence instead of $(0 = 1)$. Since $\neg P(0 = 1)$ expresses consistency of Arithmetic, its unprovability in Theorem 3 proves that consistency of Arithmetic cannot be proved using the proof mechanism of Arithmetic. It shattered Hilbert's program for proving the consistency of Arithmetic. Herman Wyel thus said:

God exists since mathematics is consistent,
and the Devil exists since its consistency cannot be proved.

References

[1] K. Gödel, Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I, *Monatshefte für Mathematik und Physik* **38** (1931) 173–98. Trans: On formally undecidable propositions of Principia Mathematica and related systems I, In: Solomon Feferman, ed., 1986. Kurt Gdel Collected works, Vol. I. Oxford University Press: 144–195.

[2] A. Singh and C. Goswami, Fundamentals of Logic, Indian Council of Philosophical Research, New Delhi (1998).

Intersection Properties of Balls and Projections of Norm One

T. S. S. R. K. Rao

Indian Statistical Institute, Bangalore centre, R. V. College Post, Bangalore 560 059, India

E-mail: tss@isibang.ac.in; srin@fulbrightmail.org

Abstract. Let X be a real Banach space. An important question in Banach space theory is to find conditions on a closed subspace $Y \subset X$ that ensure the existence of a linear projection $P : X \to Y$ such that $\|P\| = 1$. In this expository article we consider an intersection property of closed balls, which under some additional conditions on Y, ensures the existence of a unique such projection.

The study of intersection properties of balls and existence of linear projections dates back to the seminal work of Lindenstrauss from his 1964 AMS Memoir [9], Extension of compact operators.

In this article we consider an intersection property of balls related to the existence of projections of norm one. After developing the relevant material, we give a new proof of a result from [4] that says that any M-ideal with this intersection property is a M-summand. We also show that $Y \subset X$ is a M-ideal if and only if the bi-annihilator $Y^{\perp\perp}$ is a M-ideal in X^{**}. We show that, under the canonical embedding of a Banach space X, in its bidual, X^{**}, a M-ideal Y of X continues to be a M-ideal of the bidual X^{**} if and only if Y is a M-ideal in its bidual.

The author was a Fulbright-Nehru Senior Research Fellow at the Department of Mathematics and Statistics at the Southern Illinois University, Edwardsville, during the writing of this article.

For $x \in X$ and $r > 0$, let $B(x, r)$ denote the closed ball with center at x and radius r. For a subspace $Y \subset X$ and $y \in Y$, we use the same notation $B(y, r)$, when it is considered as a closed ball in X or in Y.

Let us start by recalling a result from [9].

Theorem 1. *Let X be a real Banach space and let $Y \subset X$ be a closed subspace. For every $x \notin Y$ there exists a linear projection $P : span\{x, Y\} \to Y$ with $\|P\| = 1$ if and only if for any family of closed balls $\{B(y_i, r_i)\}_{i\in I}$ of closed balls in X with centers from Y that intersect in X, also intersect in Y.*

Proof. Suppose for every $x \notin Y$ there is projection $P : \text{span}\{x, Y\} \to Y$ such that $\|P\| = 1$. Let $\{B(y_i, r_i)\}_{i\in I}$ be a family of closed balls such that $\|x - y_i\| \leq r_i$ for all $i \in I$. If $x \in Y$, the balls intersect in Y, so we are done. So let $x \notin Y$. By hypothesis, there is a projection $P : \text{span}\{x, Y\} \to Y$ with $\|P\| = 1$. Let $y_0 = P(x)$. Now

$$\begin{aligned}\|y_i - y_0\| &= \|P(y_i) - P(x)\| \\ &= \|P(y_i - x)\| \leq \|y_i - x\| \leq r_i\end{aligned}$$

for all $i \in I$.

Conversely suppose that any family of closed balls $\{B(y_i, r_i)\}_{i\in I}$ of closed balls in X with centers from Y that intersect in X, also intersect in Y. Let $x \notin Y$. Consider the family of closed balls, $\{B(y, \|x - y\|)\}_{y\in Y}$. Clearly x is in all of them. By hypothesis, there exists a $y_0 \in Y$ such that $\|y - y_0\| \leq \|y - x\|$ for all $y \in Y$. Define $P : \text{span}\{x, Y\} \to Y$ by $P(\alpha x + y) = \alpha y_0 + y$. It is easy to see that P is a linear projection. For $\alpha \neq 0$,

$$\begin{aligned}\|P(\alpha x + y)\| &= |\alpha|\|y_0 + \frac{1}{|\alpha|}y\| \\ &\leq |\alpha|\|x + \frac{1}{|\alpha|}y\| = \|\alpha x + y\|.\end{aligned}$$

Therefore $\|P\| = 1$. □

Motivated by this Bandyopadhyay and Dutta, [4], call a closed subspace $Y \subset X$, almost constrained (AC, for short), if every family of closed balls in X with centers from Y, that intersect in X, also intersect in Y. It follows from the above arguments, if Y is the range of a contractive projection (such a Y is also called a constrained subspace), then Y is a AC subspace. An interesting question is to find conditions under which if a subspace has AC, then it is the range of a projection of norm one.

Remak 2. *Let X be a Banach space. Consider X as canonically embedded in its bidual, X^{**}. By the same token X^* is canonically embedded in its bidual, X^{***}. An important contractive projection is $Q : X^{***} \to X^*$ defined by $Q(\Lambda) = \Lambda|X$. In particular, if $M \subset X^*$ is a weak*-closed subspace. Then $M = (X|N)^*$ for some closed subspace, $N \subset X$ and hence is a dual space. Now using the canonical identification of $M^{\perp\perp}$ with M^{**}, we see that M is the range of a projection of norm one on $M^{\perp\perp}$.*

Example 3. Consider $c_0 \subset \ell^\infty$. For the constant sequence $1 \in \ell^\infty$, we have $c = \text{span}\{1, c_0\}$. The sequence of closed balls $\{B(e_n, \frac{1}{2})\}_{n\geq 1}$, have the constant sequence $\frac{1}{2}$ in all of them. For $x \in \ell^\infty$, if $\|x - e_n\| \leq \frac{1}{2}$ for all n, then $|x(n)| \geq \frac{1}{2}$ for all n, so that $x \notin c_0$.

A famous example of Lindenstrauss, [10] is a Banach space $Y \subset X$ of codimension 2, where for any $x \notin Y$, there is a contractive projection $P : \text{span}\{x, Y\} \to Y$ but there is no contractive projection from X onto Y.

We next want to consider projections that satisfy stronger geometric properties.

Definition 4. *A projection $P : X \to X$ is said to be a M (L) projection, if $\|x\| = max\{\|P(x)\|, \|x - P(x)\|\}$ ($\|x\| = \|P(x)\| + \|x - P(x)\|$), for all $x \in X$. The range of a $M(L)$-projection is called a $M(L)$-summand.*

If $M \subset X$ is a M (L)-summand, we also write $X = M \bigoplus_\infty N$ $(M \bigoplus_1 N)$ for some closed subspace N. Here the index ∞ (1) stands for the max (sum) norm of the components.

It is easy to see that if $P : X \to X$ is a M-projection then $P^* : X^* \to X^*$ is a L-projection and vice versa. These concepts are very important from the approximation theoretic point of view, as the next two propositions illustrate. We recall that $Y \subset X$ is said to be a proximinal subspace, if for every $x \notin Y$, there is a $y_0 \in Y$ such that $d(x, Y) = \|x - y_0\|$. If for every x there is a unique y_0 with this property, then we say that Y is a Chebyshev subspace. For a $x \in X$, the set of best approximations to x in Y is denoted by, $P_Y(x) = \{y \in Y : d(x, Y) = \|x - y\|\}$. Understanding the structure of this closed set is an important question in approximation theory. Here we consider conditions under which it is a closed ball. The contents of the next set of propositions is from Chapters I, II and III of [6].

Proposition 5. *Let $Y \subset X$ be a L-summand. Then Y is a Chebyshev subspace of X.*

Proof. Let $Y \subset X$ be a L-summand, with a corresponding L-projection P. Let $x \in X$. For any $y \in Y$, since $\|I - P\| = 1$, $\|y - x\| \geq \|x - P(x)\|$. So $P(x) \in Y$ is a best approximation. If $y_0 \in Y$ is another best approximation to x, then

$$\begin{aligned}\|x\| &= \|x - P(x)\| + \|P(x)\| = \|x - y_0\| + \|P(x)\| \\ &= \|P(x) - y_0\| + \|x - P(x)\| + \|P(x)\| \\ &= \|P(x) - y_0\| + \|x\|.\end{aligned}$$

Thus $P(x) = y_0$ so that Y is a Chebyshev subspace. $\square$

Proposition 6. *Let $Y \subset X$ be a M-summand. It is a proximinal subspace and for $x \notin Y$, the set of best approximation projections, $P_Y(x) = \{y \in Y : d(x, Y) = \|x - y\|\}$ is a closed ball with radius, $d(x, Y)$.*

Proof. Let $P : X \to Y$ be the M-projection. Since $I - P$ is a contractive projection, for $x \notin Y$, $P(x)$ is clearly a best approximation. We claim that $P_Y(x) = B(P(x), d(x, Y))$. Suppose $\|x - y\| = d(x, Y)$. $\|y - P(x)\| = \|P(x - y)\| \leq \|x - y\| = d(x, Y)$, thus $y \in B(P(x), d(x, Y))$. On the other hand if $y_0 \in B(P(x), d(x, Y))$, then

$$\begin{aligned}\|x - y_0\| &= \max\{\|P(x) - y_0\|, \|x - P(x)\|\} \\ &\leq d(x, Y) \leq \|x - y_0\|.\end{aligned}$$

Thus $y_0 \in P_Y(x)$. $\square$

The following simple lemma emphasizes the strength of a M-summand.

Lemma 7. *Let $M, N \subset X$ be two closed subspaces such that $X = M \bigoplus_\infty N$. Let $Z \subset M$ be a proximinal subspace. Then Z is a proximinal subspace of X.*

Proof. Let $x = m + n \in X$ for $m \in M$, $n \in N$. Suppose $x \notin Z$ and also $m \notin Z$. By proximinality of Z in M, there exists a $z_0 \in Z$ such that $d(m, Z) = \|m - z_0\|$. We claim that z_0 is the required best approximation to $x = m + n$.

$$\begin{aligned}\|z - (m + n)\| &= \max\{\|z - m\|, \|n\|\} \\ &\geq \max\{\|z_0 - m\|, \|n\|\} = \|(m + n) - z_0\|.\end{aligned}$$

If $m \in Z$, then m is a best approximation to $x = m+n$ in Z, as,

$$\begin{aligned}\|z - (m + n)\| &= \max\{\|z - m\|, \|n\|\} \\ &\geq \|n\| = \|m - (m + n)\|.\end{aligned}$$

$\square$

An important notion based on these concepts, due to E M Alfsen and E G Effors, [1], is that of a M-ideal. A closed subspace $Y \subset X$ is said to be a M-ideal, if $Y^\perp$ is the kernel of a L-projection in X^*.

It is easy to see that any M-summand is a M-ideal. Alfsen and Effros ([1]) have also shown that any M-ideal $Y \subset X$ is a proximinal subspace. And in this case $P_Y(x)$ need not always be a ball, but its weak*-closure in X^{**} is a ball with radius, $d(x, Y)$. The monograph by Harmand, Werner and Werner, 'M-ideals in Banach spaces and Banach algebras' [6] is a good reference for these matters.

Example 8. Let K be a compact Hausdorff space, let $E \subset K$ be a closed set. Let $Y = \{f \in C(K) : f(E) = 0\}$. Now $P : \mu \to \mu|K - E$ is a L projection in $C(K)^*$ with $\ker(P) = Y^\perp$, we see that Y is a M-ideal. Y is a M-summand precisely when E is a clopen set. More generally Alfsen and Effros showed that in any C^*-algebra, M-ideals are precisely closed,two sided ideals.

We are now ready to prove the uniqueness of projections of norm one.

Proposition 9. *Let $P : X \to X$ be a L-projection and let $Q : X \to X$ be a projection of norm one such that* $\ker(Q) = \ker(P)$. *Then $P = Q$. In particular if $Y \subset X$ is a M-ideal and $Q : X \to Y$ is an onto projection, then $Q^* = P$, where P is the L-projection associated with Y. Therefore Q is the unique projection of norm one with range Y. Q is a M-projection and hence Y is a M-summand.*

Proof. Let P be a L-projection and suppose $\ker(P) = \ker(Q)$ for some contractive projection Q. Now for $x \in X$, $x - P(x)$ is the best approximation in $\ker(P)$. But as $\|Q\| = 1$, $x - Q(x)$ is a best approximation to x, in $\ker(Q) = \ker(P)$. Thus $x - P(x) = x - Q(x)$ so that $P = Q$.

Now suppose Y is a M-ideal and Q is a contractive projection onto Y. Let P be the L-projection in X^* with $\ker(P) = Y^\perp$. Now Q^* is a contractive projection with $\ker(Q^*) = Y^\perp = \ker(P)$. Thus $Q^* = P$. So that Q^* is a L-projection and hence Q is a M-projection. Also if Q_1 is another contractive projection onto Y, then $Q_1^* = Q^*$ so that $Q = Q_1$. □

Remak 10. *An interesting consequence of the above Proposition is that, if X is a M-ideal in X^{**} under the canonical embedding, then since the canonical projection $Q : X^{***} \to X^{***}$ is a contractive projection with* $\ker(Q) = X^{\perp}$, *we can conclude that Q is a L-projection.*

An almost immediate consequence is the following interesting result. For a Banach space X, its fourth dual is denoted by $X^{(IV)}$.

Proposition 11. *A closed subspace $Y \subset X$ is a M-ideal if and only if $Y^{**} \subset X^{**}$ is a M-ideal.*

Proof. Suppose $Y \subset X$ is a M-ideal. Since $Y^{\perp\perp}$ is a M-summand, it is clearly a M-ideal in X^{**}.

Conversely suppose that $Y^{\perp\perp}$ is a M-ideal in X^{**}. Let $P^* : X^{(IV)} \to X^{(IV)}$ be the M-projection with range $Y^{\perp\perp\perp\perp}$. As $Y^{\perp\perp}$ is a dual space, by Remark 2 there is an onto contractive projection $Q : Y^{\perp\perp\perp\perp} \to Y^{\perp\perp}$. Now $Q \circ P^*|X^{**} \to Y^{\perp\perp}$ is an onto contractive projection. Therefore by Proposition 9 we have that $Y^{\perp\perp}$ is a M-summand, but this means that Y is a M-ideal. □

Turning back to intersection properties again, we have the following internal characterization of M-ideals due to Alfsen and Effros, which we state without proof.

Theorem 12. *A closed subspace $Y \subset X$ is a M-ideal if and only if for any* 3 *closed balls $B(x_1, r_1)$, $B(x_2, r_2)$, $B(x_3, r_3)$ in X such that $\cap_1^3 B(x_i, r_i) \neq \emptyset$, $Y \cap B(x_i, r_i) \neq \emptyset$ for $i = 1, 2, 3$, for any $\epsilon > 0$, $Y \cap_1^3 B(x_i, r_i + \epsilon) \neq \emptyset$. In this case the conclusion also holds for any finite collection of closed balls in X.*

Remak 13. *Contained in the above theorem is the interesting observation that if Z is a closed subspace of X such that $Y \subset Z \subset X$, then Y is a M-ideal in Z. Also for $Y \subset Z \subset X$ if Y is a M-ideal in Z and Z is a M-ideal in X, then Y is a M-ideal in X.*

Here is an interesting consequence of this remark.

Proposition 14. *Let $Y \subset X$ be a M-ideal. Suppose $y_0 \in Y$ is such that span$\{y_0\}$ is a M-summand in Y. Then span$\{y_0\}$ is a M-summand in X.*

Proof. Since any M-summand is a M-ideal, by the above remark we have that span$\{y_0\}$ is a M-ideal in X. Now since span$\{y_0\}$ is the range of a contractive projection on X, we have by Proposition 9 that, span$\{y_0\}$ is a M-summand in X. □

Remak 15. *For $1 < p < \infty$, the space of compact operators, $\mathcal{K}(\ell^p) \subset \mathcal{L}(\ell^p)$ is a M-ideal. In this case $\mathcal{L}(\ell^p)$ is the bidual of $\mathcal{K}(\ell^p)$. See [6] Chapter VI, for several examples of classical spaces X, Y for which $\mathcal{K}(X, Y)$ is a M-ideal in $\mathcal{L}(X, Y)$. It is known that $\mathcal{K}(X)$ is a M-ideal in $\mathcal{L}(X)$ if and only if $\mathcal{K}(X)$ is a M-ideal in span$\{I, \mathcal{K}(X)\}$.*

Remak 16. *It is easy to note that any M-summand has this intersection property of balls with $\epsilon = 0$. Suppose $Y \subset X$ be a M-summand with Q as the corresponding M-projection. Suppose $\|x - x_i\| \leq r_i$ and there exists $y_i \in Y$ with $\|x_i - y_i\| \leq r_i$. Now*

$$r_i \geq \|x_i - y_i\| = \max\{\|Q(x_i) - y_i\|, \|x_i - Q(x_i)\|\}$$
$$\geq \|x_i - Q(x_i)\|.$$

Thus

$$\|Q(x) - x_i\| = \max\{\|Q(x) - Q(x_i)\|, \|x_i - Q(x_i)\|\}$$
$$\leq \max\{\|x - x_i\|, \|x_i - Q(x_i)\|\} \leq r_i.$$

Here x_i's can clearly be a finite or infinite collection. Thus if $Y \subset X$ is a M-summand then for any family $\{B(x_i, r_i)\}_{i \in I}$ of closed balls having non-empty intersection, each intersecting Y, then all of them intersect Y. An interesting theorem of R. Evans ([5]) is that this ball intersection property characterizes M-summands.

The following proposition and remarks, illustrates the abstract ideas contained in [11].

Proposition 17. *Let $Y \subset X$ be a M-ideal. Under the canonical embedding, $Y \subset X^{**}$ is a M-ideal if and only if Y is a M-ideal in Y^{**}*

Proof. We recall the canonical identification of Y^{**} with $Y^{\perp\perp} \subset X^{**}$. Thus if Y is a M-ideal in X^{**}, then by above remarks, $Y \subset Y^{\perp\perp}$ is a M-ideal.

Conversely suppose that $Y \subset Y^{**}$ is a M-ideal. Under the canonical identifications, we have, $Y \subset Y^{**} = Y^{\perp\perp} \subset X^{**}$. Now since Y is a M-ideal in X, $Y^{\perp\perp}$ is a M-ideal in X^{**}.

Thus by the transitivity of this property, Y is a M-ideal in X^{**}. $\square$

An interesting feature of the above result is the following. Let $Y \subset Y^{**}$ be a M-ideal. Let X be a Banach space such that $Y \subset X$, is a M-ideal. Now consider $Y \subset X \subset X^{**} \subset X^{(IV)}$. By above, we have that Y is a M-ideal in X^{**}. Applying this result once more, we have that Y is a M-ideal in $X^{(IV)}$. Hence under the canonical embedding, Y is a M-ideal of all duals of even order of X.

A very remarkable feature of being a M-ideal in the bidual is that, it can be described internally. We quote the following result from [8]. For a set $A \subset X$, by $CO(A)$ we denote the convex hull.

Proposition 18. *A Banach space Y is a M-ideal in its bidual if and only if for all y and sequences $\{y_n\}_{n\geq 1}$ such that $\|y\| \leq 1$, $\|y_n\| \leq 1$ for all n,and for any $\epsilon > 0$, there is some positive integer n and a $u \in CO\{y_1, \ldots, y_n\}$, $t \in CO\{y_i\}_{i\geq n+1}$ such that $\|y + t - u\| \leq 1 + \epsilon$.*

Similar to the question of M-ideals, one can also ask, if $Y \subset X$ is a proximinal subspace, when is it a proximinal subspace of X^{**}? This is a difficult question in approximation theory. We present below a partial positive answer.

Proposition 19. *Let $Y \subset X$ be a M-ideal. Suppose under the canonical embedding, Y is a proximinal subspace of $Y^{**} = Y^{\perp\perp}$. Then Y is a proximinal subspace of X^{**}.*

Proof. Consider $Y \subset Y^{\perp\perp}$. Since $Y \subset X$ is a M-ideal, we have, $Y^{\perp\perp}$ is a M-summand of X^{**}. Thus by Lemma 7, Y is a proximinal subspace of X^{**}. $\square$

We next need a Lemma that is the converse of Proposition 6 for M-ideals.

Lemma 20. *Let $Y \subset X$, be a M-ideal. If $P_Y(x)$ is a ball with radius $d(x, Y)$ for all $x \notin Y$, then Y is a M-summand.*

Proof. Let $P : X^* \to X^*$ be the L-projection with $\ker(P) = Y^\perp$. Then P^* is a M-projection with range $Y^{\perp\perp}$. Let $x \notin Y$. We ignore the canonical embedding map and consider $X \subset X^{**}$. It is easy to see that $X \cap Y^{\perp\perp} = Y$. Thus $x \notin Y^{\perp\perp}$. Also we recall the canonical identification of the quotient space, $(X|Y)^{**} = X^{**}|Y^{\perp\perp}$. Thus $d(x, Y) = d(x, Y^{\perp\perp})$ which we assume as 1. Thus $P_Y(x) \subset P_{Y^{\perp\perp}}(x)$. Now the former set, by hypothesis, is a ball with center say y_0 and radius 1. We also note that for a closed ball $B(x_0, r)$ in X, its weak*-closure is the ball $B(x_0, r)$ now considered in X^{**}. As $Y^{\perp\perp}$ is a M-summand, $P_{Y^{\perp\perp}}(x) = B(P^*(x), 1)$. Thus $B(y_0, 1) \subset B(P^*(x), 1)$. Therefore $B(y_0, 1) = B(P^*(x), 1)$ so that $P^*(x) = y_0 \in Y$. Thus $P^*|X \to Y$ is an onto M-projection and hence Y is a M-summand. $\square$

Lemma 21. *Let $Y \subset X$ be a M-ideal and also a AC-subspace. For any $x \notin Y$, Y is a M-summand in span$\{x, Y\}$.*

Proof. Suppose $Y \subset X$ is a M-ideal as well as AC-subspace. Let $x \notin Y$. By Theorem 1 there is a contractive projection, $P : \text{span}\{x, Y\} \to Y$. By Remark 13, Y is a M-ideal in span$\{x, Y\}$. Therefore by Proposition 9, P is a M-projection and hence Y is a M-summand in span$\{x, Y\}$. $\square$

Corollary 22. *Let X be an infinite dimensional Banach space. Suppose $\mathcal{K}(X)$ is a M-ideal in span$\{I, \mathcal{K}(X)\}$. Then $\mathcal{K}(X)$ fails to be AC in span$\{I, \mathcal{K}(X)\}$.*

Proof. Suppose $\mathcal{K}(X)$ is a M-ideal and is AC subspace of span$\{I, \mathcal{K}(X)\}$. Then by the above Lemma, $\mathcal{K}(X)$ is a M-summand in span$\{I, \mathcal{K}(X)\}$. In particular for any $T \in \mathcal{K}(X)$, $\|I + T\| = \max\{1, \|T\|\}$. Let $x_0 \in X$ and $x_0^* \in X^*$ be such that $x_0^*(x_0) = 1 = \|x_0\| = \|x_0^*\|$. Consider the rank one operator $x_0^* \otimes x_0 \in \mathcal{K}(X)$. Since $\|x_0^* \otimes x_0\| = 1$, we have $\|I + x_0^* \otimes x_0\| = 1$. But $\|(I + x_0^* \otimes x_0)(x_0)\| = \|x_0 + x_0^*(x_0)x_0\| = 2$ which is a contradiction. Therefore $\mathcal{K}(X)$ is not a AC subspace of span$\{I, \mathcal{K}(X)\}$. $\square$

We are now ready to prove the main result.

Theorem 23. *Let $Y \subset X$ be a M-ideal such that for any $x \notin Y$, Y is a M-summand in span$\{x, Y\}$. Then Y is a M-summand in X.*

Proof. It is enough to show that for $d(x, Y) = 1$, $P_Y(x)$ is a ball of radius 1. By hypothesis $Y \subset \text{span}\{x, Y\}$ is a M-summand with the corresponding M-projection, P. Thus by Proposition 5, $P_Y(x) = B(P(x), 1)$. Therefore Y is a M-summand in X. $\square$

The following corollary now follows from the above Theorem and Lemma. It was first proved by different methods in [4].

Corollary 24. *Let $Y \subset X$ be a closed subspace that is a M-ideal and AC-subspace. Then Y is a M-summand and hence is the range of a unique contractive projection.*

See the papers [3] and [12] for more recent information on these topics.

References

[1] E. M. Alfsen and E. G. Effros, Structure in Real Banach spaces, I, II, *Ann. of Math.*, **96** (1972) 98–173.

[2] P. Bandyopadhyay and T. S. S. R. K. Rao, Central subspaces of Banach spaces, *J. Approx. Theory*, **103** (2000) 206–222.

[3] P. Bandyopadhyay and S. Dutta, Almost constrained subspaces of Banach spaces. II, *Houston J. Math.*, **35** (2009) 945–957.

[4] P. Bandyopadhyay and S. Dutta, Almost constrained subspaces of Banach spaces, *Proc. Amer. Math. Soc.*, **132** (2004) 107-–115.

[5] R. Evans, A Characterization of M-summands, *Proc. Cambridge. Phil. Soc.*, **76** (1974) 157–159.

[6] P. Harmand, D. Werner and W. Werner, M-ideals in Banach spaces and Banach algebras, *Lecture Notes in Mathematics*, 1547. Springer-Verlag, Berlin, 1993. viii+387 pp.

[7] H. E. Lacey, The isometric theory of classical Banach spaces, Die Grundlehren der mathematischen Wissenschaften, Band 208. Springer-Verlag, New York-Heidelberg, 1974. x+270 pp.

[8] A. Lima, E. Oja, T. S. S. R. K. Rao and D. Werner, Geometry of operator spaces, *Michigan Math. J.*, **41** (1994) 473–490.

[9] J. Lindenstrauss, Extension of compact operators, *Mem. Amer. Math. Soc.*, **48** (1964) 112 pp.

[10] J. Lindenstrauss, On projections wirh norm 1-an example, *Proc. Amer. Math. Soc.*, **15** (1964) 403–406.

[11] T. S. S. R. K. Rao, On the geometry of higher duals of a Banach space, *Illinois J. Math.*, **45** (2001) 1389–1392.

[12] T. S. S. R. K. Rao Central and almost constrained subspaces of Banach spaces, *Operators and Matrices*, **6** (2012) 371–383.

Geometric Properties of Partial Sums of Univalent Functions

V. Ravichandran

Department of Mathematics, University of Delhi, Delhi 110 007, India

School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM, Penang, Malaysia

E-mail: vravi@maths.du.ac.in

Abstract. The nth partial sum of an analytic function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is the polynomial $f_n(z) := z + \sum_{k=2}^{n} a_k z^k$. A survey of the univalence and other geometric properties of the nth partial sum of univalent functions as well as other related functions including those of starlike, convex and close-to-convex functions are presented.

Keywords. Univalent function, starlike function, convex function, sections, partial sums.

2010 Mathematics Subject Classification. 30C45, 30C80

1. Introduction

For $r > 0$, let $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ be the open disk of radius r centered at $z = 0$ and $\mathbb{D} := \mathbb{D}_1$ be the open unit disk. An analytic function f is *univalent* in the unit disk $\mathbb{D}$ if it maps different points to different points. Denote the class of all (normalized) univalent functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

by $\mathcal{S}$. Denote by $\mathcal{A}$, the class of all analytic functions of the form (1.1). The Koebe function k defined by

$$k(z) = \frac{z}{(1-z)^2} = z + \sum_{k=2}^{\infty} k a_k \quad (z \in \mathbb{D})$$

is univalent and it is also extremal for many problems in geometric function theory of univalent functions. A domain D is starlike with respect to a point $a \in D$ if every line segment joining the point a to any other point in D lies completely inside D. A domain starlike with respect to the origin is simply called starlike. A domain D is convex if every line segment joining any two points in D lies completely inside D; in other words, the domain D is convex if and only if it is starlike with respect to every point in D. A function $f \in \mathcal{S}$ is *starlike* if $f(\mathbb{D})$ is starlike (with respect to the origin) while it is *convex* if $f(\mathbb{D})$ is convex. The classes of all starlike and convex functions are respectively denoted by $\mathcal{S}^*$ and $\mathcal{C}$. Analytically, these classes are characterized by the equivalence

$$f \in \mathcal{S}^* \Leftrightarrow \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,$$

and

$$f \in \mathcal{C} \Leftrightarrow \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0.$$

More generally, for $0 \leq \alpha < 1$, let $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ be the subclasses of $\mathcal{S}$ consisting of respectively starlike functions of order α, and convex functions of order α. These classes are defined analytically by the equivalence

$$f \in \mathcal{S}^*(\alpha) \Leftrightarrow \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha,$$

and

$$f \in \mathcal{C}(\alpha) \Leftrightarrow \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha.$$

Another related class is the class of close-to-convex functions. A function $f \in \mathcal{A}$ satisfying the condition

$$\operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right) > \alpha \quad (0 \leq \alpha < 1)$$

for some (not necessarily normalized) convex univalent function g, is called *close-to-convex of order* α. The class of all such functions is denoted by $\mathcal{K}(\alpha)$. Close-to-convex functions of order 0 are simply called close-to-convex functions. Using the fact that a function $f \in \mathcal{A}$ with

$$\operatorname{Re}(f'(z)) > 0$$

is in $\mathcal{S}$, close-to-convex functions can be shown to be univalent.

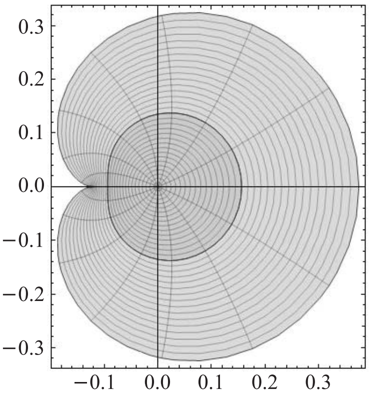

Figure 1. Images of $\mathbb{D}_{1/8}$ and $\mathbb{D}_{1/4}$ under the mapping $w = z + 2z^2$.

The nth *partial sum* (or nth *section*) of the function f, denoted by f_n, is the polynomial defined by

$$f_n(z) := z + \sum_{k=2}^{n} a_k z^k.$$

The second partial sum f_2 of the Koebe function k is given by

$$f_2(z) = z + 2z^2 \quad (z \in \mathbb{D}).$$

It is easy to check directly (or by using the fact that $|f_2'(z) - 1| < 1$ for $|z| < 1/4$) that this function f_2 is univalent in the disk $\mathbb{D}_{1/4}$ but, as $f_2'(-1/4) = 0$, not in any larger disk. This simple example shows that the partial sums of univalent functions need not be univalent in $\mathbb{D}$.

The second partial sum of the function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is the function $f_2(z) = z + a_2 z^2$. If $a_2 = 0$, then $f_2(z) = z$ and its properties are clear. Assume that $a_2 \neq 0$. Then the function f_2 satisfies, for $|z| \leq r$, the inequality

$$\operatorname{Re}\left(\frac{zf_2'(z)}{f_2(z)}\right) = \operatorname{Re}\left(1 + \frac{a_2 z}{1 + a_2 z}\right) \geq 1 - \frac{|a_2|r}{1 - |a_2|r} > 0$$

provided $r < 1/(2|a_2|)$. Thus the radius of starlikeness of f_2 is $1/(2|a_2|)$. Since f_2 is convex in $|z| < r$ if and only if zf_2' is starlike in $|z| < r$, it follows that the radius of convexity of f_2 is $1/(4|a_2|)$. If f is univalent or starlike univalent, then $|a_2| \leq 2$ and therefore the radius of univalence of f_2 is 1/4 and the radius of convexity of f_2 is 1/8. (See Fig. 4 for the image of $\mathbb{D}_{1/4}$ and $\mathbb{D}_{1/8}$ under the function $z+2z^2$.) For a convex function f as well as for functions f whose derivative has positive real part in $\mathbb{D}$, $|a_2| \leq 1$ and so the radius of univalence for the second partial sum f_2 of these functions is 1/2 and the radius of convexity is 1/4. In [26], the starlikeness and convexity of the initial partial sums of the Koebe function $k(z) = z/(1-z)^2$ and the function $l(z) = z/(1-z)$ are investigated.

It is therefore of interest to determine the largest disk $\mathbb{D}_\rho$ in which the partial sums of the univalent functions are univalent. Szegö also wrote a survey [51] on partial sums in 1936. In the present survey, the various results on the partial sums of functions belonging to the subclasses of univalent functions are given. However, Cesàro means and other polynomials approximation of univalent functions are not considered in this survey.

2. Partial Sums of Univalent Functions

The second partial sum of the Koebe function indicates that the partial sums of univalent functions cannot be univalent in a disk of radius larger than 1/4. Indeed, by making use of Koebe's distortion theorem and Löwner's theory of univalent functions, Szegö [50] in 1928 proved the following theorem.

Theorem 2.1 (Szegö Theorem). *The partial sums of univalent functions $f \in \mathcal{S}$ are univalent in the disk $\mathbb{D}_{1/4}$ and the number $1/4$ cannot be replaced by a larger one.*

Using an inequality of Goluzin, Jenkins [14] (as well as Ilieff [12], see Duren [4, §8.2, pp. 241–246]) found a simple proof of this result and also shown that the partial sums of odd univalent functions are univalent in $\mathbb{D}_{1/\sqrt{3}}$. The number $1/\sqrt{3}$ is shown to be the radius of starlikeness of the partial sums of the odd univalent functions by He and Pan [10]. Iliev [13] investigated the radius of univalence for the partial sums $\sigma_n^{(k)}(z) = z + c_1^{(k)} z^{k+1} + \cdots + c_n^{(k)} z^{nk+1}$, $n = 1, 2, \ldots$, of univalent function of the form $f_k(z) = z + c_1^{(k)} z^{k+1} + \cdots$. For example, it is shown that $\sigma_n^{(2)}$ is univalent in $|z| < 1/\sqrt{3}$, and $\sigma_n^{(3)}$ is univalent in $|z| < \sqrt[3]{3}/2$, for all $n = 1, 2, \ldots$. He has also shown that $\sigma_n^{(1)}(z)$ is univalent in $|z| < 1 - 4(\ln n)/n$ for $n \geq 15$. Radii of univalence are also determined for $\sigma_n^{(2)}$ and $\sigma_n^{(3)}$, as functions of n, and for $\sigma_1^{(k)}$ as a function of k.

Szegö's theorem states that the partial sums of univalent functions are univalent in $\mathbb{D}_{1/4}$ was strengthened to starlikeness by Hu and Pan [11]. Ye [54] has shown that the partial sums of univalent functions are convex in $\mathbb{D}_{1/8}$ and that the number 1/8 is sharp. Ye [54] has proved the following result.

Theorem 2.2. *Let $f \in \mathcal{S}$ and*

$$f^{1/k}(z^k) = \sum_{\nu=0}^{\infty} b_\nu^{(k)} z^{\nu k+1}, \quad (k = 2, 3, \ldots, b_0^{(k)} = 1).$$

Then $\sum_{\nu=0}^{n} b_\nu^{(k)} z^{\nu k+1}$ are convex in $\mathbb{D}_{\sqrt[k]{k/(2(k+1)^2)}}$. The radii of convexity are sharp.

Ruscheweyh gave an extension of Szegö's theorem that the nth partial sums f_n are starlike in $\mathbb{D}_{1/4}$ for functions belonging not only to $\mathcal{S}$ but also to the closed convex hull of $\mathcal{S}$.

Let $\mathcal{F} = \operatorname{clco}\left\{\sum_{k=1}^{n} x^{k-1} z^k : |x| \leq 1\right\}$ where clco stands for the closed convex hull. Convolution of two analytic functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ is the function $f * g$ defined by

$$(f * g)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

Ruscheweyh [34] proved the following theorem.

Theorem 2.3. *If $f \in \operatorname{clco} \mathcal{S}$ and $g \in \mathcal{F}$, then $f * g$ is starlike in $\mathbb{D}_{1/4}$. The constant 1/4 is best possible.*

In particular, for $g(z) = z + z^2 + \cdots + z^n$, Theorem 2.3 reduces to the following result.

Corollary 2.1. *If f belongs to* clco $\mathcal{S}$ *or, in particular, to the class of the normalized typically real functions, then the nth partial sum f_n is starlike in $\mathbb{D}_{1/4}$. The constant 1/4 is best possible.*

The class $\mathcal{F}$ contains the following two subsets:

$$\mathcal{R}_{1/2} := \left\{f \in \mathcal{A} : \operatorname{Re}(f(z)/z) > 1/2, z \in \mathbb{D}\right\} \subset \mathcal{F}$$

and

$$\mathcal{D} := \left\{\sum_{k=1}^{n} a_k z^k \in \mathcal{A} : 0 \leq a_{k+1} \leq a_k\right\} \subset \mathcal{F}.$$

Since the class $\mathcal{C}$ of convex functions is a subset of $\mathcal{R}_{1/2}$, it is clear that $\mathcal{C} \subset \mathcal{F}$. For $g(z) = z/(1-z)^2 \in \mathcal{S}^*$, Theorem 2.3 reduces to the following:

Corollary 2.2. *If f belongs to $\mathcal{F}$, then the function f and, in particular, the nth partial sum f_n, is convex in $\mathbb{D}_{1/4}$. The constant 1/4 is best possible.*

We remark that Suffridge [49] has shown that the partial sums of the function e^{1+z} are all convex. More generally, Ruscheweyh and Salinas [36] have shown that the functions of the form $\sum_{k=0}^{\infty} a_k (1+z)^k / k!$, $a_0 \geq a_1 \geq \cdots \geq 0$ are either constant or convex univalent in the unit disk $\mathbb{D}$. Let $F(z) = z + \sum_1^{\infty} a_k z^{-k}$ be analytic $|z| > 1$. Reade [28] obtained the radius of univalence for the partial sums $F_n(z) = z + \sum_1^n a_k z^{-k}$ when F is univalent or when $\operatorname{Re} F'(z) > 0$ in $|z| > 1$.

3. Partial Sums of Starlike Functions

Szegö [50] showed that the partial sums of starlike (respectively convex) functions are starlike (respectively convex) in the disk $\mathbb{D}_{1/4}$ and the number 1/4 cannot be replaced by a larger one. If n is fixed, then the radius of starlikeness of f_n can be shown to depend on n. Motivated by a result of Von Victor Levin that the nth partial sum of univalent functions is univalent in D_ρ where $\rho = 1 - 6(\ln n)/n$ for $n \geq 17$, Robertson [29] determined R_n such that the nth partial sum f_n to have certain property P in $\mathbb{D}_{R_n}$ when the function f has the property P in $\mathbb{D}$. He considered the function has one of the following properties: f is starlike, f/z has positive real part, f is convex, f is typically-real or f is convex in the direction of the imaginary axis and is real on the real axis. An error in the expression for R_n was later corrected in his paper [30] where he has extended his results to multivalent starlike functions.

The radius of starlikeness of the nth partial sum of starlike function is given in the following theorem.

Theorem 3.1 [30] (see [39, Theorem 2, p. 1193]). *If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is either starlike, or convex, or typically-real, or convex in the direction of imaginary axis, then there is n_0 such that, for $n \geq n_0$, the partial sum $f_n(z) := z + \sum_{k=2}^{n} a_k z^k$ has the same property in $\mathbb{D}_\rho$ where $\rho \geq 1 - 3\log n/n$.*

An analytic function $f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$ is p-valently starlike [30, p. 830] if f assumes no value more than p times, at least one value p times and

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathbb{D}).$$

For p-valently starlike functions, Robertson [30, Theorem A, p. 830] proved that the radius of p-valently starlikeness of the nth partial sum $f_n(z) = z^p + \sum_{k=1}^{n} a_{p+k} z^{p+k}$ is at least $1 - (2p+2)\log n/n$. Ruscheweyh [33] has given a simple proof that the partial sums f_n of p-valently starlike (or close-to-convex) function is p-valently starlike (or respectively close-to-convex) in $|z| < 1/(2p+2)$.

Kobori [16] proved the following theorem and Ogawa [24] gave another proof of this result.

Theorem 3.2. *If f is a starlike function, then every partial sum f_n of f is convex in $|z| < 1/8$ and the number 1/8 cannot be increased.*

In view of the above theorem, the nth partial sum of Koebe function $z/(1-z)^2$ is convex in $|z| < 1/8$. A verification of this fact directly can be used to give another proof of this theorem by using the fact [35] that the convolution of two convex function is again convex. It is also known [35] that $\operatorname{Re}(f(z)/f_n(z)) > 1/2$ for a function f starlike of order 1/2. This result was extended by Singh and Paul [46] in the following theorem.

Theorem 3.3. *If $f \in \mathcal{S}^*(1/2)$, then*

$$\operatorname{Re}\left(\lambda\frac{zf'(z)}{f(z)} + \mu\frac{f_n(z)}{f(z)}\right) > 0 \quad (z \in \mathbb{D})$$

provided that λ and μ are both nonnegative with at least one of them nonzero or provided that μ is a complex number with $|\lambda| > 4|\mu|$. The result is sharp in the sense that the ranges of λ and μ cannot be increased.

A locally univalent normalized analytic function f satisfying the condition

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{3}{2}$$

is starlike univalent and, for such functions, Obradović and Ponnusamy [23] have shown that each partial sum $s_n(z)$ is starlike in the disk $\overline{\mathbb{D}}_{1/2}$ for $n \geq 12$ and also $\operatorname{Re}(s_n'(z)) > 0$ in $\overline{\mathbb{D}}_{1/2}$ for $n \geq 13$.

4. Partial Sums of Convex Functions

For a convex function $f \in \mathcal{C}$, it is well-known that $\operatorname{Re}(f(z)/z) > 1/2$. Extending this result, Sheil-Small [37] proved the following theorem.

Theorem 4.1. *If $f \in \mathcal{C}$, then the nth partial sum f_n of f satisfies*

$$\left|1 - \frac{f_n(z)}{f(z)}\right| \leq |z|^n < 1 \quad (z \in \mathbb{D}, n \geq 1)$$

and hence

$$\operatorname{Re}\frac{f(z)}{f_n(z)} > \frac{1}{2} \quad (z \in \mathbb{D}, n \geq 1). \tag{4.1}$$

As a consequence of this theorem, he has shown that the function Q_n given by $Q_n(z) = \int_0^z (f_n(\eta)/\eta)d\eta$ is a close-to-convex univalent function. In fact, the inequality (4.1) holds for $f \in \mathcal{S}^*(1/2)$ as shown in [35]. The inequality (4.1) also holds for odd starlike functions as well as for functions whose derivative has real part greater than 1/2 [47].

Recall once again that [50] the partial sums of convex functions are convex in the disk $\mathbb{D}_{1/4}$ and the number $1/4$ cannot be replaced by a larger one. A different proof of this result can be given by using results about convolutions of convex functions. In 1973, Ruscheweyh and Shiel-Small proved the Polya-Schoenberg conjecture (of 1958) that the convolution of two convex univalent functions is again convex univalent. Using this result, Goodman and Schoenberg gave another proof of the following result of Szegö [50].

Theorem 4.2. *If f is convex function, then every partial sum f_n of f is convex in $|z| < 1/4$.*

Proof. The convex function $l(z) = z/(1-z)$ is extremal for many problems associated with the class of convex functions. By Szegö's result the partial sum $l_n(z) = z + z^2 + \cdots + z^n$ of l is convex in $\mathbb{D}_{1/4}$ and therefore $4l_n(z/4)$ is a convex univalent function. If f is also convex, then its convolution with the convex function $4l_n(z/4)$ is convex and so $4(f * l_n)(z/4) = f(z)*(4l_n(z/4))$ is convex. Therefore, the partial sum f_n of the convex function f, as $f * l_n = f_n$, is convex in $\mathbb{D}_{1/4}$. In view of this discussion, another proof of Szegö result comes if we can directly show that $l_n(z)$ is convex in $\mathbb{D}_{1/4}$. This will be done below.

A calculation shows that

$$\begin{aligned} 1 + \frac{zl_n''(z)}{l_n'(z)} &= \frac{n(n+1)z^n(z-1)}{1-(n+1)z^n + nz^{n+1}} + \frac{1+z}{1-z} \\ &= \frac{N(z)}{D(z)} + M(z) \end{aligned}$$

where $N(z) = n(n+1)z^n(z-1)$, $D(z) = 1-(n+1)z^n+nz^{n+1}$ and $M(z) = (1+z)/(1-z)$. The bilinear transformation $w = M(z)$ maps $|z| < 1/4$ onto the disk $|w - 17/15| < 8/15$ and hence

$$\mathrm{Re} M(z) > 3/5.$$

It is therefore enough to show that

$$\left|\frac{N(z)}{D(z)}\right| < \frac{3}{5}$$

as this inequality together with $\mathrm{Re} M(z) > 3/5$ yield

$$\mathrm{Re}\left(\frac{N(z)}{D(z)} + M(z)\right) \geq \mathrm{Re} M(z) - \left|\frac{N(z)}{D(z)}\right| > \frac{3}{5} - \frac{3}{5} = 0.$$

Now, for $|z| < 1/4$, we have

$$|N(z)| < \frac{5n(n+1)}{4^{1+n}},$$

$$\begin{aligned} |D(z) - 1| &= |(n+1)z^n - nz^{n+1}| \\ &< \frac{1}{4^n}(n+1+n/4) \\ &= \frac{5n+4}{4^{n+1}} < 1 \end{aligned}$$

and so

$$|D(z)| \geq 1 - |D(z) - 1| > 1 - \frac{5n+1}{4^{n+1}}.$$

Therefore, it follows that

$$\left|\frac{N(z)}{D(z)}\right| < \frac{3}{5}$$

holds if

$$\frac{5n(n+1)}{4^{n+1}} < \frac{3}{5}\left(1 - \frac{5n+4}{4^{n+1}}\right)$$

or equivalently

$$\frac{25}{12} \leq \frac{4^n}{n(n+1)} - \frac{1}{n} - \frac{1}{4(n+1)}.$$

The last inequality becomes an equality for $n = 2$ and the right hand side expression is an increasing function of n. $\square$

Let $P_{\alpha,n}$ denote the class of functions $p(z) = 1 + c_n z^n + \cdots$ $(n \geq 1)$ analytic and satisfying the condition $\mathrm{Re}\, p(z) > \alpha$ $(0 \leq \alpha < 1)$ for $z \in \mathbb{D}$. Bernardi [3] proved that the sharp inequality

$$\frac{|zp'(z)|}{\mathrm{Re}(p(z) - \alpha)} \leq \frac{2nr^n}{1 - r^{2n}}$$

holds for $p(z) \in P_{\alpha,n}$, $|z| = r < 1$, and $n = 1, 2, 3, \ldots$. He has also shown that, for any complex μ, $\mathrm{Re}\mu = \beta > 0$,

$$\left|\frac{zp'(z)}{p(z) - \alpha + (1-\alpha)\mu}\right| \leq \frac{2nr^n}{(1-r^n)(1+\beta+(1-\beta)r^n)}.$$

For a convex function f, he deduced the sharp inequality

$$\left|\frac{zf'(z)}{f(z)} - \frac{zf_n'(z)}{f_n(z)}\right| \leq \frac{nr^n}{1-r^n}.$$

Making use of this inequality, he proved the following theorem.

Theorem 4.3 [3, Theorem 4, p. 117]. *If f is convex, then the nth partial sum f_n is starlike in $|z| < r_n$ where r_n is the positive root of the equation $1 - (n+1)r^n - nr^{n+1} = 0$. The result is sharp for each even n for $f(z) = z/(1-z)$.*

The above theorem with a weaker conclusion that f_n is univalent was obtained earlier by Ruscheweyh [32]. Singh [45] proved that the nth partial sum f_n of a starlike function of order 1/2 is starlike in $|z| < r_n$ where r_n is given in Theorem 4.3. He has also shown that the conclusion of Theorem 4.3 can be strengthened to convexity if one assumes that f is a convex function of order 1/2. In addition, for a convex function f of order 1/2, he has shown that $\operatorname{Re}(f_n(z)/z) > 1/2$ and 1/2 is sharp. It is also known that all the partial sums of a convex function f of order 1/2 are close-to-convex with respect to f itself and that there are convex functions of order $\alpha < 1/2$ whose partial sums are not univalent [31]. Singh and Puri [47] have however shown that each of the partial sums of an odd convex function f is close-to-convex with respect to f.

Silverman [39] also proved Theorem 4.3 by finding the radius of starlikeness (of order α) of the nth partial sums of the function $z/(1-z)$. The result then follows from the fact that the classes of convex and starlike functions are closed under convolution with convex functions.

Lemma 4.1. *The function $g_n(z) = \frac{z(1-z^n)}{1-z}$ is starlike of order α in $|z| < r_n$ where r_n is the smallest positive root of the equation*

$$1-\alpha-\alpha r+(\alpha-1-n)r^n+(\alpha-n)r^{n+1}=0.$$

The result is sharp for even n.

Proof. The bilinear transformations $w = 1/(1-z)$ maps the circular region $|z| \le r$ onto the circle

$$\left|\frac{1}{1-z}-\frac{1}{1-r^2}\right| \le \frac{r}{1-r^2}.$$

Similarly, the bilinear transformations $w = z/(1-z)$ maps the circular region $|z| \le r$ on to the circle

$$\left|\frac{z}{1-z}-\frac{r^2}{1-r^2}\right| \le \frac{r}{1-r^2}.$$

Since

$$\frac{zg_n'(z)}{g_n(z)} = \frac{1}{1-z}-\frac{nz^n}{1-z^n},$$

it follows that, for $|z| \le r < 1$,

$$\left|\frac{zg_n'(z)}{g_n(z)}-\frac{1}{1-r^2}+\frac{nr^{2n}}{1-r^{2n}}\right| \le \frac{r}{1-r^2}+\frac{nr^n}{1-r^{2n}}.$$

The above inequality shows that

$$\operatorname{Re}\frac{zg_n'(z)}{g_n(z)} \ge \frac{1}{1+r}-\frac{nr^n}{1-r^n} \ge \alpha$$

provided

$$1-\alpha-\alpha r+(\alpha-1-n)r^n+(\alpha-n)r^{n+1} \ge 0.$$

The sharpness follows easily. □

Theorem 4.4 [39, Theorem 1, p. 1192]. *If f is convex, then the nth partial sum f_n is starlike in $|z| < (1/(2n))^{1/n}$ for all n. In particular, f_n is starlike in $|z| < 1/2$ and the radius $1/2$ is sharp.*

Proof. In view of the previous lemma, it is enough to show that

$$1-(n+1)r^n-nr^{n+1} \ge 0$$

for $0 \le r \le (1/(2n))^{1/n}$. For $0 \le r \le (1/(2n))^{1/n}$, the above inequality is equivalent to

$$\frac{1}{n}+\frac{1}{(2n)^{1/n}} \le 1,$$

which holds for $n \ge 2$.

The second result follows as $1/(2n)^{1/n}$ is an increasing function of n and from the fact that, for $g_2(z) = z+z^2$, $g_2'(-1/2) = 0$. □

Silverman [39, Corollary 2, pp. 1192] also proved that the nth partial sum f_n of a convex function f is starlike in $|z| < \sqrt{23/71}$ for $n \ge 3$ and the radius $\sqrt{23/71}$ is sharp. For a convex function f, its nth partial sum f_n is shown to be starlike of order α in $|z| < (1-\alpha)/(2-\alpha)$, convex of order α in $|z| < (1-\alpha)/(2(2-\alpha))$ and the radii are sharp.

A function $f \in \mathcal{S}$ is *uniformly convex*, written $f \in \mathcal{UCV}$, if f maps every circular arc γ contained in $\mathbb{D}$ with center $\zeta \in \mathbb{D}$ onto a convex arc. The class $\mathcal{S}_P$ of *parabolic starlike functions* consists of functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)}-1\right|, \quad z \in \mathbb{D}.$$

In other words, the class $\mathcal{S}_P$ consists of functions $f = zF'$ where $F \in \mathcal{UCV}$. A survey of these classes can be found in [1].

Lemma 4.2. *The function $g_n(z) = \frac{z(1-z^n)}{1-z}$ is in $\mathcal{S}_P$ for $|z| < r_n$ where r_n is the smallest positive root of the equation*

$$1-r = (1+2n)r^n+(2n-1)r^{n+1}.$$

The result is sharp for even n.

Proof. By Lemma 4.1, the function g_n is starlike of order 1/2 in $|z| < r_n$ where r_n is as given in the Lemma 4.2. From the proof of Lemma 4.1, it follows that, for $|z| = r$, the values of $zg_n'(z)/g_n(z)$ is in the disk with diametric end points at

$$x_1 = \frac{1}{1+r} - \frac{nr^n}{1-r^n} \quad \text{and} \quad x_2 = \frac{1}{1-r} + \frac{nr^n}{1+r^n}.$$

For $r = r_n$, one has $x_1 = 1/2$ and the disk is completely inside the parabolic region $\mathrm{Re}\,w > |w-1|$. □

Theorem 4.5. *If $f(z)$ is convex of order 1/2, then the partial sums f_n are uniformly convex for $|z| < r_n$ where r_n is the smallest positive root of*

$$1 - r = (2n+1)r^n + (2n-1)r^{n+1}.$$

Proof. For the function $f(z)$ convex of order 1/2, the function $zf'(z)$ is starlike of order 1/2. Since $\int_0^z g_n(t)/t dt$ is uniformly convex in $|z| < r_n$,

$$f_n(z; f) = f(z) * g_n(z) = zf'(z) * \left(\int_0^z g_n(t)/t dt\right)$$

is uniformly convex in $|z| < r_n$. □

5. Functions whose Derivative is Bounded or has Positive Real Part

Theorem 5.1 [21]. *If the analytic function f given by* (1.1) *satisfies the inequality $|f'(z)| \le M$, $M > 1$, then the radius of starlikeness of f_n is $1/M$.*

Proof. The function f is starlike in $\mathbb{D}_r$ if

$$\sum_{k=2}^{\infty} k|a_k|r^{k-1} \le 1.$$

This sufficient condition is now well-known (Alexandar II) and it was also proved by Noshiro [21]. The Parseval–Gutzmer formula for a function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ analytic in $\overline{\mathbb{D}}_r$ is

$$\int_0^{2\pi} |f(re^{i\vartheta})|^2 \mathrm{d}\vartheta = 2\pi \sum_{k=0}^{\infty} |a_k|^2 r^{2k}.$$

Using this formula for f' and noting that $|f'(z)| \le M$, it follows that

$$1 + \sum_{k=2}^{\infty} k^2|a_k|^2 = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\vartheta})|^2 \mathrm{d}\vartheta \le M^2.$$

Now, by using the Cauchy–Schwarz inequality, it readily follows that, for $r < 1/M$,

$$\begin{aligned}\sum_{k=2}^{\infty} k|a_k|r^{k-1} &\le \sqrt{\sum_{k=2}^{\infty} k^2|a_k|^2}\sqrt{\sum_{k=2}^{\infty} r^{2k-2}} \\ &\le \sqrt{M^2-1}\sqrt{\frac{r^2}{1-r^2}} \\ &< 1.\end{aligned}$$

The sharpness follows from the function f_0 given by

$$\begin{aligned}f_0(z) &= M \int_0^z \frac{1-Mz}{M-z} dz \\ &= M\left(Mz + (M^2-1)\log\left(1 - \frac{z}{M}\right)\right);\end{aligned}$$

its derivative vanishes at $z = 1/M$. □

Nabetani [20] noted that Theorem 5.1 holds even if the inequality $|f'(z)| \le M$ is replaced by the inequality

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |f'(re^{i\vartheta})|^2 \mathrm{d}\vartheta\right)^{1/2} \le M.$$

He has shown that the radii of starlikeness and convexity of functions f satisfying the inequality

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\vartheta})|^2 \mathrm{d}\vartheta\right)^{1/2} \le M$$

are respectively the positive root of the equations

$$(M^2-1)R^2 = (1-R^2)^3$$

and

$$(M^2-1)(1+11R^2+11R^4+R^6) = M^2(1-R^2)^5.$$

For functions whose derivative has positive real part, MacGregor [18] proved the following result.

Theorem 5.2. *If the analytic function f given by* (1.1) *satisfies the inequality* $\mathrm{Re}\,f'(z) > 0$, *then f_n is univalent in* $|z| < 1/2$.

Proof. Since $\mathrm{Re}\,f'(z) > 0$, $|a_k| \le 2/k$, $(k \ge 2)$, and so, with $|z| = r$,

$$|f'(z) - f_n'(z)| \le \sum_{k=n+1}^{\infty} |ka_k z^{k-1}| \le \sum_{k=n+1}^{\infty} 2r^{k-1} = \frac{2r^n}{1-r}.$$

This together with the estimate $\operatorname{Re} f'(z) > (1-r)/(1+r)$ shows that

$$\operatorname{Re} f_n'(z) \geq \frac{1-r}{1+r} - \frac{2r^n}{1-r}.$$

The result follows from this for $n \geq 4$. For $n = 2, 3$, a different analysis is needed, see [18]. Compare Theorem 5.3. □

Theorem 5.3 [29]. *If $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic and has positive real part in $\mathbb{D}$, then, for $n \geq 2$, $p_n(z) = 1 + c_1 z + c_2 z^2 + \cdots + c_n z^n$ has positive real part in $\mathbb{D}_\rho$ where ρ is the root $R_n \geq 1 - 2\log n/n$ in $(0, 1)$ of the equation*

$$(1-r)^2 = 2r^{n+1}(1+r).$$

Singh [44] investigated the radius of convexity for functions whose derivative has positive real part and proved the following result.

Theorem 5.4. *If the analytic function f given by (1.1) satisfies the inequality $\operatorname{Re} f'(z) > 0$, then f_n is convex in $|z| < 1/4$. The number 1/4 cannot be replaced by a greater one.*

Extending Theorem 5.2 of MacGregor, Silverman [43] has shown that, whenever $\operatorname{Re} f'(z) > 0$, f_n is univalent in $\{z : |z| < r_n\}$, where r_n is the smallest positive root of the equation $1 - r - 2r^n = 0$, and the result is sharp for n even. He also shown that $r_n > (1/2n)^{1/n}$ and $r_n > 1 - \log n/n$ for $n \geq 5$. Also he proved that the sharp radius of univalence of f_3 is $\sqrt{2}/2$. Yamaguchi [52] has shown that f_n is univalent in $|z| < 1/4$ if the analytic function f given by (1.1) satisfies the inequality $\operatorname{Re}(f(z)/z) > 0$.

Let $0 \leq \alpha < 1$ and denote by R_α the class of functions $f(z) = z + a_2 z^2 + \cdots$ that are regular and univalent in the unit disk and satisfy $\operatorname{Re} f'(z) > \alpha$. Let $f_n(z) = z + a_2 z^2 + \cdots + a_n z^n$. Kudelski [17] proved the following results. The corresponding results for $f \in R_0$ were proved by Aksentév [2].

Theorem 5.5. *Let $f \in R_\alpha$. Then $\operatorname{Re} {f_n}'(z) > 0$ in the disk $|z| < r_n(\alpha)$, where $r_n(\alpha)$ is the least positive root of the equation $2r^n + r - 1 + 4\alpha r/((1-\alpha)(1+r)) = 0$. Also f_n is univalent for $|z| < R_n(\alpha)$, where $R_n(\alpha)$ is the least positive root of $2r^n + r - 1 - \alpha(1-r)^2/((1-\alpha)(1+r)) = 0$.*

6. Close-to-convex Functions

Recall that a function $f \in \mathcal{A}$ satisfying the condition

$$\operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right) > 0$$

for some (not necessarily normalized) convex univalent function g, is called *close-to-convex*. In this section, some results related to close-to-convex functions are presented.

Theorem 6.1 [19]. *Let the analytic function f be given by (1.1). Let $g(z) = z + b_2 z^2 + \cdots$ be convex. If $\operatorname{Re}(f'(z)/g'(z)) > 0$ for $z \in \mathbb{D}$, then $\operatorname{Re}(f_n'(z)/g_n'(z)) > 0$ for $|z| < 1/4$ and $1/4$ is the best possible constant.*

The function f satisfying the hypothesis of the above theorem is clearly close-to-convex. This theorem implies that f_n is also close-to-convex for $|z| < 1/4$ and therefore it is a generalization of Szegö result. The result applies only to a subclass of the class of close-to-convex functions as g is assumed to be normalized. Ogawa [25] proved the following two theorems.

Theorem 6.2. *If $f(z) = z + \sum_2^\infty a_\nu z^\nu$ is analytic and satisfy*

$$\operatorname{Re} \frac{zf'(z)}{\phi(z)} > 0,$$

where $\phi(z) = z + \sum_2^\infty b_\nu z^\nu$ is starlike univalent, then, for each $n > 1$,

$$\operatorname{Re} \frac{(zf_n'(z))'}{\phi_n'(z)} > 0 \quad \left(|z| < \frac{1}{8}\right),$$

and the constant $1/8$ cannot be replaced by any greater one.

Theorem 6.3. *Let $f(z) = z + \sum_2^\infty a_\nu z^\nu$, be analytic and satisfy*

$$\operatorname{Re} \frac{(zf'(z))'}{\phi'(z)} > 0,$$

where $\phi(z) = z + \sum_2^\infty b_\nu z^\nu$ is schlicht and convex in $|z| < 1$. Then, for each $n > 1$,

$$\operatorname{Re} \frac{zf_n'(z)}{\phi_n(z)} > 0 \quad \left(|z| < \frac{1}{2}\right).$$

The constant $1/2$ cannot be replaced by any greater one.

Theorem 6.4 [9]. *Let $f(z) = z + \sum_2^\infty a_\nu z^\nu$, be analytic and satisfy*

$$\operatorname{Re} \frac{(zf'(z))'}{\phi'(z)} > 0,$$

where $\phi(z) = z + \sum_2^\infty b_\nu z^\nu$ is starlike in $|z| < 1$. Then, for each $n > 1$,

$$\operatorname{Re} \frac{(zf_n'(z))'}{\phi_n'(z)} > 0 \quad \left(|z| < \frac{1}{6}\right).$$

The constant $1/6$ cannot be replaced by any greater one.

A domain D is said to be linearly accessible if the complement of D can be written as the union of half-lines. Such a domain is simply connected and therefore, if it is not the whole plane, the domain is the image of the unit disk $\mathbb{D}$ under a conformal mapping. Such conformal mappings are called linearly accessible. For linearly accessible functions, Sheil-Small [38] proved the following theorem.

Theorem 6.5. *If $f(z) = \sum_{k=1}^{\infty} a_k z^k$ is linearly accessible in $\mathbb{D}$, then*

$$\left|1 - \frac{f_n(z)}{f(z)}\right| \leq (2n+1)|z|^n \quad (z \in \mathbb{D}).$$

7. Partial Sums of Functions Satisfying Some Coefficient Inequalities

Silverman [42] considered functions f of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ satisfying one of the inequalities

$$\sum_{k=2}^{\infty}(k-\alpha)|a_k| \leq (1-\alpha) \quad \text{or} \quad \sum_{k=2}^{\infty} k(k-\alpha)|a_k| \leq (1-\alpha),$$

where $0 \leq \alpha < 1$. These coefficient conditions are sufficient for f to be starlike of order α and convex of order α, respectively. If f satisfies either of the inequalities above, the partial sums f_n also satisfy the same inequality.

Silverman [42] obtained the sharp lower bounds on $\operatorname{Re}\{f(z)/f_n(z)\}$, $\operatorname{Re}\{f_n(z)/f(z)\}$, $\operatorname{Re}\{f'(z)/s_n'(z)\}$, and $\operatorname{Re}\{s_n'(z)/f'(z)\}$ for functions f satisfying either one of the inequalities above. In fact, he proved the following theorem.

Theorem 7.1. *If the analytic function f satisfies*

$$\sum_{k=2}^{\infty}(k-\alpha)|a_k| \leq (1-\alpha)$$

for some $0 \leq \alpha < 1$, then

$$\operatorname{Re}\frac{f(z)}{f_n(z)} \geq \frac{n}{n+1-\alpha},$$

$$\operatorname{Re}\frac{f_n(z)}{f(z)} \geq \frac{n+1-\alpha}{n+2-2\alpha},$$

$$\operatorname{Re}\frac{f'(z)}{f_n'(z)} \geq \frac{\alpha n}{n+1-\alpha},$$

$$\operatorname{Re}\frac{f_n'(z)}{f'(z)} \geq \frac{n+1-\alpha}{(n+1)(2-\alpha)-\alpha}.$$

The inequalities are sharp for the function

$$f(z) = z + \frac{1-\alpha}{n+1-\alpha} z^{n+1}.$$

Silverman [42] also proved a similar result for function satisfying the inequality $\sum_{k=2}^{\infty} k(k-\alpha)|a_k| \leq (1-\alpha)$. These results were extended in [7] for classes of functions satisfying an inequality of the form $\sum c_k|a_k| \leq \delta$.

For functions belonging to the subclass $\mathcal{S}$, it is well-known that $|a_n| \leq n$ for $n \geq 2$. A function f whose coefficients satisfy the inequality $|a_n| \leq n$ for $n \geq 2$ are analytic in $\mathbb{D}$ (by the usual comparison test) and hence they are members of $\mathcal{A}$. However, they need not be univalent. For example, the function

$$f(z) = z - 2z^2 - 3z^3 - 4z^4 - \cdots = 2z - \frac{z}{(1-z)^2}$$

satisfies the inequality $|a_n| \leq n$ but its derivative vanishes inside $\mathbb{D}$ and therefore the function f is not univalent in $\mathbb{D}$. For the function f satisfying the inequality $|a_n| \leq n$, Gavrilov [8] showed that the radius of univalence of f and its partial sums f_n is the real root of the equation $2(1-r)^3 - (1+r) = 0$ while, for the functions whose coefficients satisfy $|a_n| \leq M$, the radius of univalence is $1 - \sqrt{M/(1+M)}$. Later, in 1982, Yamashita [53] showed that the radius of univalence obtained by Gavrilov is also the same as the radius of starlikeness of the corresponding functions. He also found lower bounds for the radii of convexity for these functions. Kalaj, Ponnusamy, and Vuorinen [15] have investigated related problems for harmonic functions. For functions of the form $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ whose Taylor coefficients a_n satisfy the conditions $|a_2| = 2b$, $0 \leq b \leq 1$, and $|a_n| \leq n$, M or M/n $(M > 0)$ for $n \geq 3$, the sharp radii of starlikeness and convexity of order α, $0 \leq \alpha < 1$, are obtained in [27]. Several other related results can also be found.

Theorem 7.2. *Let $f \in \mathcal{A}$, $|a_2| = 2b$, $0 \leq b \leq 1$ and $|a_n| \leq n$ for $n \geq 3$. Then f satisfies the inequality*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1 - \alpha \quad (|z| \leq r_0)$$

where $r_0 = r_0(\alpha)$ is the real root in $(0,1)$ of the equation

$$1 - \alpha + (1+\alpha)r = 2\big(1 - \alpha + (2-\alpha)(1-b)r\big)(1-r)^3.$$

The number $r_0(\alpha)$ is also the radius of starlikeness of order α. The number $r_0(1/2)$ is the radius of parabolic starlikeness of the given functions. The results are all sharp.

8. Partial Sums of Rational Functions

Define $\mathcal{U}$ to be the class of all analytic functions $f \in \mathcal{A}$ satisfying the condition

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^2 - 1\right| < 1 \quad (z \in \mathbb{D}).$$

It is well-known that $\mathcal{U}$ consists of only univalent functions. In this section, we consider the partial sums of functions belonging to $\mathcal{U}$. All the results in this section are proved by Obradović and Ponnusamy [22].

Theorem 8.1. *If $f \in \mathcal{S}$ has the form*

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots \tag{8.1}$$

such that b_k is real and non-negative for each $k \geq 2$, then for each $n \geq 2$

$$\left|\frac{f_n(z)}{f(z)} - 1\right| < |z|^n(n + 1 - n\log(1 - |z|)) \quad (z \in \mathbb{D})$$

In particular,

$$\left|\frac{f_n(z)}{f(z)} - 1\right| < 1$$

in the disk $|z| < r$, where r is the unique positive root of the equation:

$$1 - r^n(n + 1 - n\log(1 - r)) = 0 \tag{8.2}$$

and, for $n \geq 3$, we also have $r \geq r_n = 1 - \frac{2\log n}{n}$.

The values of r corresponding to $n = 2, 3, 4, 5$ from (8.2) are $r = 0.481484$, $r = 0.540505$, $r = 0.585302$, $r = 0.620769$ respectively.

Theorem 8.2. *If $f \in \mathcal{U}$ has the form* (8.1)*, then*

$$\sum_{n=2}^{\infty}(n-1)^2|b_n|^2 \leq 1. \tag{8.3}$$

In particular, we have $|b_1| \leq 2$ and $|b_n| \leq \frac{1}{n-1}$ for $n \geq 2$. The results are sharp.

Theorem 8.3. *Suppose that $f \in \mathcal{U}$ and $f_n(z)$ is its partial sum. Then for each $n \geq 2$*

$$\left|\frac{f_n(z)}{f(z)} - 1\right| < |z|^n(n+1)\left(1 + \frac{\pi}{\sqrt{6}}\frac{|z|}{1-|z|}\right) \quad (z \in \mathbb{D}).$$

Proof. Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ so that

$$f_n(z) = z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n$$

is its n-th partial sum. Also, let

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots .$$

Then

$$\frac{f_n(z)}{f(z)} = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots$$

where $c_n = -a_{n+1}$ and

$$c_m = b_{m-n+1}a_n + b_{m-n+2}a_{n-1} + \cdots + b_m a_1,$$

for $m = n+1, n+2, \ldots$. By de Branges theorem, $|a_n| \leq n$ for all $n \geq 2$, and therefore, we obtain that

$$|c_n| = |-a_{n+1}| \leq n + 1$$

and that for $m \geq n + 1$ (by the Cauchy-Schwarz inequality)

$$|c_m|^2 \leq \left(\sum_{k=1}^{n}\frac{(n+1-k)^2}{(m-(n+1-k))^2}\right)$$
$$\times\left(\sum_{k=1}^{n}(m-n+k-1)^2|b_{m-n+k}|^2\right) := AB \text{ (say)}.$$

From Theorem 8.2, we deduce that $B \leq 1$, while for $m \geq n+1$ we have

$$A = \sum_{k=1}^{n}\frac{(n+1-k)^2}{(m-(n+1-k))^2}$$
$$\leq \sum_{k=1}^{n}\frac{(n+1-k)^2}{k^2}$$
$$= (n+1)^2\sum_{k=1}^{n}\frac{1}{k^2} - 2(n+1)\sum_{k=1}^{n}\frac{1}{k} + \sum_{k=1}^{n}1.$$

in view of the inequalities

$$\sum_{k=1}^{n}\frac{1}{k} > \log(n+1), \quad \text{and} \quad \log(n+1) > 1 \quad \text{for } n \geq 3,$$

it follows easily that

$$A < \frac{\pi^2}{6}(n+1)^2 - 2(n+1)\log(n+1)$$
$$+ n < \frac{\pi^2}{6}(n+1)^2 - (n+2),$$

which, in particular, implies that

$$|c_m| < \frac{\pi}{\sqrt{6}}(n+1) \quad \text{for } m \geq n+1 \quad \text{and} \quad n \geq 3.$$

This inequality, together with the fact that $|c_n| = |a_{n+1}| \leq n+1$, gives that

$$\begin{aligned}
&\left|\frac{f_n(z)}{f(z)} - 1\right| \\
&\quad \leq |c_n|\,|z|^n + |c_{n+1}|\,|z|^{n+1} + \cdots \\
&\quad \leq (n+1)|z|^n + \frac{\pi}{\sqrt{6}}(n+1)\left(|z|^{n+1} + |z|^{n+2} + \cdots\right) \\
&\quad = (n+1)|z|^n \left(1 + \frac{\pi}{\sqrt{6}}\frac{|z|}{1-|z|}\right)
\end{aligned}$$

for $n \geq 3$. The proof is complete. □

As a corollary, the following result holds.

Corollary 8.1. *Suppose that $f \in \mathcal{U}$. Then for $n \geq 3$ one has*

$$\left|\frac{f_n(z)}{f(z)} - 1\right| < \frac{1}{2} \quad \text{for } |z| < r_n := 1 - \frac{2\log n}{n}$$

or equivalently

$$\left|\frac{f(z)}{f_n(z)} - \frac{4}{3}\right| < \frac{2}{3} \quad \text{for } |z| < r_n.$$

In particular, Corollary 8.1 shows that for $f \in \mathcal{U}$, we have

$$\mathrm{Re}\frac{f_n(z)}{f(z)} > \frac{1}{2} \quad \text{for } |z| < r_n \quad \text{and} \quad n \geq 3$$

and

$$\mathrm{Re}\frac{f(z)}{f_n(z)} > \frac{2}{3} \quad \text{for } |z| < r_n \quad \text{and} \quad n \geq 3.$$

When the second Taylor coefficient of the function f vanish, the following results hold.

Theorem 8.4. *If $f(z) = z + \sum_{k=3}^{\infty} a_k z^k$ (i.e. $a_2 = 0$) belongs to the class $\mathcal{U}$, then the n-th partial sum f_n is in the class $\mathcal{U}$ in the disk $|z| < r$, where r is the unique positive root of the equation*

$$(1-r)^3(1+r)^2 - r^n(1+r^2)^2[5 + r + n(1-r^2)] = 0.$$

In particular, for $n \geq 5$, we have

$$r \geq r_n = 1 - \frac{3\log n - \log(\log n)}{n}.$$

For $n = 3, 4, 5$, one has

$$r = 0.361697, \ r = 0.423274, \ r = 0.470298,$$

respectively.

Theorem 8.5. *Let $f(z) = z + \sum_{k=3}^{\infty} a_k z^k$ (i.e. $a_2 = 0$) belong to the class $\mathcal{U}$. Then for each integer $n \geq 2$, we have*

$$\mathrm{Re}\left(\frac{f(z)}{f_n(z)}\right) > \frac{1}{2}$$

in the disk $|z| < \sqrt{\sqrt{5} - 2}$.

9. Generalized Partial Sum

By making use of the fact that the convolution of starlike function with a convex function is again a starlike function, Silverman [39] has proved the following result.

Theorem 9.1. *If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is convex, then $F_k(z) = z + \sum_{j=1}^{\infty} a_{jk+1} z^{jk+1}$, $(k = 2, 3, \ldots)$, is starlike in $|z| < (1/(k-1))^{1/k}$. The bound is sharp for every k.*

The proof follows from the following inequality satisfied by $G_k(z) = z/(1-z^k)$:

$$\mathrm{Re}\frac{zG_k'(z)}{G_k(z)} \geq \frac{1 - (k-2)r^k - (k-1)r^{2k}}{|1 - z^k|^2} \qquad (|z| = r < 1).$$

Since $(1/(k-1))^{1/k}$ attains its minimum when $k = 5$, it follows that, for a convex function f, the F_k is starlike in $|z| < (1/4)^{1/5}$. Since the radius of convexity of G_2 is $\sqrt{2} - 1$, it follows that F_2 is convex in $|z| < \sqrt{2} - 1$ whenever f is convex.

To an arbitrary increasing sequence (finite or not) $\{n_k\}_{k=2}^{\infty}$ of integers with $n_k \geq k$ and a function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, the function $\tilde{f}(z) = z + \sum_{k=2}^{\infty} a_{n_k} z^{n_k} = f(z) * (z + \sum_{k=2}^{\infty} z^{n_k})$ is called the generalized partial sum of the function f. For the generalized partial sum of convex mappings, Fournier and Silverman [5] proved the following results.

Theorem 9.2. *If f is convex, then the generalized partial sum $\tilde{f}$ of the function f is*

(1) *convex univalent in $|z| < c$ where $c(\approx 0.20936)$ is the unique root in $(0, 1)$ of the equation*

$$x(1+x^2)/(1-x^2)^3 = 1/4.$$

(2) *starlike univalent in $|z| < b$ where b (≈ 0.3715) is the unique root in $(0, 1)$ of the equation*

$$x/(1-x^2)^2 = 1/2.$$

The function $z+\sum_{k=1}^{\infty} z^{2k} = z + z^2/(1-z^2)$ associated with the convex function $z+\sum_{k=2}^{\infty} z^k = z/(1-z)$ is extremal for the radii of convexity and starlikeness.

These results are proved by using the information about neighborhoods of convex functions. They [5] also proved that, for a starlike function f, the generalized partial sum $\tilde{f}$ is starlike in $|z| < c$ where c is as above or in other words,

$$f \in H \Rightarrow \tilde{f}(cz)/c \in H \tag{9.1}$$

where H is the class of starlike univalent functions. The above implication in (9.1) is also valid for the classes of convex univalent functions and close-to-convex functions and the class M consisting of functions f for which $(f * g)(z)/z \neq 0$ for all starlike univalent functions $g \in \mathcal{S}^*$. They [6] later showed that the implication in (9.1) is also valid for the class $\mathcal{S}$ of univalent functions by proving the following theorem.

Theorem 9.3. *If $f \in \mathcal{S}$, then the generalized partial sum $\tilde{f}$ of the function f satisfies $\operatorname{Re}\tilde{f}'(cz) > 0$ for all $z \in \mathbb{D}$, where c is as in Theorem 9.2. The function $f(z) = z/(1-z)^2$ and $\{n_k\}_{k=2}^{\infty} = \{2k-2\}_{k=2}^{\infty}$ show that the result is sharp.*

They [6] have also proved that if f is analytic and $\operatorname{Re}\{f(z)/z\} > \frac{1}{2}$, then

$$|z\tilde{f}''(z)| \leq \operatorname{Re}\tilde{f}'(z) \quad (|z| < c)$$

for any choice of $\{n_k\}_{k=2}^{\infty}$.

For the class $\mathcal{R}$ of functions f in $\mathcal{A}$ for which $\operatorname{Re}(f'(z) + zf''(z)) > 0$, $z \in \mathbb{D}$, Silverman [40] proved the following result and some related results can be found in [41].

Theorem 9.4. *Let r_0 denote the positive root of the equation $r+\log(1-r^2) = 0$. If $f \in \mathcal{R}$, then $\operatorname{Re}\tilde{f}'(z) \geq 0$ for $|z| \leq r_0 \approx 0.71455$. The result is sharp, with extremal function $\tilde{f}(z) = z + 2\sum_{n=1}^{\infty} z^{2n}/(2n)^2$.*

For functions $f \in \mathcal{R}$, it is also known [48] that the nth partial sum f_n of f satisfies $\operatorname{Re} f_n'(z) > 0$ and hence f_n is univalent. Also $\operatorname{Re}(f_n(z)/z) > 1/3$.

Acknowledgement

The author is thankful to Sumit Nagpal for carefully reading this manuscript.

References

[1] R. M. Ali and V. Ravichandran, Uniformly convex and uniformly starlike functions, *Ramanujan Math. Newsletter*, **21** no. 1 (2011) 16–30.

[2] L. A. Aksentév, On the univalence of partial sums of power series, *Izv. Vysš. Učebn. Zaved. Matematika*, no. 5(18) (1960) 12–15. MR0130970 (24 #A824)

[3] S. D. Bernardi, New distortion theorems for functions of positive real part and applications to the partial sums of univalent convex functions, *Proc. Amer. Math. Soc.*, **45** (1974) 113–118. MR0357755 (50 #10223)

[4] P. L. Duren, Univalent functions, Grundlehren der Mathematischen Wissenschaften, 259, Springer, New York (1983). MR0708494 (85j:30034)

[5] R. Fournier and H. Silverman, Radii problems for generalized sections of convex functions, *Proc. Amer. Math. Soc.*, **112** no. 1 (1991) 101–107. MR1047000 (91h:30016)

[6] R. Fournier and H. Silverman, On generalized sections of univalent functions, *Complex Variables Theory Appl.*, **17** no. 3–4 (1992) 141–147. MR1147044 (93d:30013)

[7] B. A. Frasin, Generalization of partial sums of certain analytic and univalent functions, *Appl. Math. Lett.*, **21** no. 7 (2008) 735–741. MR2423054 (2009d:30022)

[8] V. I. Gavrilov, Remarks on the radius of univalence of holomorphic functions, *Mat. Zametki*, **7** (1970) 295–298. MR0260989 (41 #5609)

[9] R. M. Goel, On the partial sums of a class of univalent functions, *Ann. Univ. Mariae Curie-Sklodowska Sect. A*, **19** (1965) 17–23 (1970). MR0259093 (41 #3735)

[10] K. He and Y. F. Pan, The sections of odd univalent functions, *Chinese Ann. Math. Ser. B*, **5** no. 4 (1984) 727–730. MR0795480 (86m:30013)

[11] K. Hu and Y. F. Pan, On a theorem of Szegő, *J. Math. Res. Exposition*, **4** no. 1 (1984) 41–44. MR0767005 (86j:30025)

[12] L. Ilieff, Über die Abschnitte der schlichten Funktionen, *Acta Math. Acad. Sci. Hungar.*, **2** (1951) 109–112. MR0045819 (13,640d)

[13] L. Iliev, Classical extremal problems for univalent functions, in *Complex analysis (Warsaw, 1979)*, 89–110,

Banach Center Publ., 11 PWN, Warsaw. MR0737754 (86b:30012)

[14] J. A. Jenkins, On an inequality of Golusin, *Amer. J. Math.*, **73** (1951) 181–185. MR0041230 (12,816e)

[15] D. Kalaj, S. Ponnusamy and M. Vuorinen, Radius of close-to-convexity of harmonic functions, arXiv:1107.0610v1 [math.CV]

[16] A. Kobori, Zwei sätze über die abschnitte der schlichten potenzreihen, *Mem. Coll. Sci. Kyoto Imp. Univ. Ser. A.*, **17** (1934) 171–186.

[17] F. Kudelski, On the univalence of Taylor sums for a class of univalent functions, *Ann. Univ. Mariae Curie-Sklodowska Sect. A*, **17** (1963) 65–67 (1965). MR0197701 (33 #5864)

[18] T. H. MacGregor, Functions whose derivative has a positive real part, *Trans. Amer. Math. Soc.*, **104** (1962) 532–537. MR0140674 (25 #4090)

[19] Y. Miki, A note on close-to-convex functions, *J. Math. Soc. Japan*, **8** (1956) 256–268. MR0091343 (19,951a)

[20] K. Nabetani, Some remarks on a theorem concerning star-shaped representation of an analytic function, *Proc. Imp. Acad.*, **10** no. 9 (1934) 537–540. MR1568384

[21] K. Noshiro, On the starshaped mapping by an analytic function, *Proc. Imp. Acad.*, **8** no. 7 (1932) 275–277. MR1568331

[22] M. Obradović and S. Ponnusamy, Partial sums and radius problem for certain class of conformal mappings, *Siberian Math. J.*, **52** no. 2 (2011) 291–302.

[23] M. Obradović and S. Ponnusamy, Injectivity and starlikeness of sections of a class of univalent functions, in: Complex Analysis and Dynamical Systems V, Contemporary Mathematics, Amer. Math. Soc., to appear.

[24] S. Ogawa, A note on close-to-convex functions. II, *J. Nara Gakugei Univ.*, **8** no. 2 (1959) 11–17. MR0179341 (31 #3589)

[25] S. Ogawa, A note on close-to-convex functions. III, *J. Nara Gakugei Univ.*, **9** no. 2 (1960) 11–17. MR0138741 (25 #2184)

[26] S. Owa, H. M. Srivastava and N. Saito, Partial sums of certain classes of analytic functions, *Int. J. Comput. Math.*, **81** no. 10 (2004) 1239–1256. MR2173456 (2006e:30013)

[27] V. Ravichandran, Radii of starlikeness and convexity of analytic functions satisfying certain coefficient inequalities, preprint.

[28] M. O. Reade, On the partial sums of certain Laurent expansions, *Colloq. Math.*, **11** (1963/1964) 173–179. MR0164024 (29 #1323)

[29] M. I. S. Robertson, On the theory of univalent functions, *Ann. of Math. (2)*, **37** no. 2 (1936) 374–408. MR1503286

[30] M. S. Robertson, The partial sums of multivalently starlike functions, *Ann. of Math. (2)*, **42** (1941) 829–838. MR0004905 (3,79b)

[31] M. S. Robertson, Univalent functions starlike with respect to a boundary point, *J. Math. Anal. Appl.*, **81** no. 2 (1981) 327–345. MR0622822 (82i:30017)

[32] S. Ruscheweyh, On the radius of univalence of the partial sums of convex functions, *Bull. London Math. Soc.*, **4** (1972) 367–369. MR0316697 (47 #5244)

[33] S. Ruscheweyh, On the partial sums of prestarlike and related functions, *Indian J. Pure Appl. Math.*, **11** no. 12 (1980) 1587–1589. MR0617836 (82i:30018)

[34] S. Ruscheweyh, Extension of Szegő's theorem on the sections of univalent functions, *SIAM J. Math. Anal.*, **19** no. 6 (1988) 1442–1449. MR0965264 (89m:30032)

[35] St. Ruscheweyh and T. Sheil-Small, Hadamard products of Schlicht functions and the Pólya-Schoenberg conjecture, *Comment. Math. Helv.*, **48** (1973) 119–135. MR0328051 (48 #6393)

[36] S. Ruscheweyh and L. Salinas, On convex univalent functions with convex univalent derivatives, *Rocky Mountain J. Math.*, **35** no. 3 (2005) 1017–1027. MR2150321 (2006a:30017)

[37] T. Sheil-Small, A note on the partial sums of convex schlicht functions, *Bull. London Math. Soc.*, **2** (1970) 165–168. MR0265576 (42 #485)

[38] T. Sheil-Small, On linearly accessible univalent functions, *J. London Math. Soc. (2)*, **6** (1973) 385–398. MR0325953 (48 #4299)

[39] H. Silverman, Radii problems for sections of convex functions, *Proc. Amer. Math. Soc.*, **104** no. 4 (1988) 1191–1196. MR0942638 (89e:30028)

[40] H. Silverman, Generalized sequences for a subfamily of univalent functions, *J. Math. Anal. Appl.*, **183** no. 2 (1994) 326–334. MR1274143 (95c:30020)

[41] H. Silverman, Convolution properties for generalized partial sums, *J. Korean Math. Soc.*, **33** no. 3 (1996) 601–607. MR1419755 (97m:30015)

[42] H. Silverman, Partial sums of starlike and convex functions, *J. Math. Anal. Appl.*, **209** no. 1 (1997) 221–227. MR1444523 (98j:30012)

[43] H. Silverman, Partial sums of a class of univalent functions, *Tamkang J. Math.*, **29** no. 3 (1998) 171–174. MR1651551 (99m:30028)

[44] R. Singh, Radius of convexity of partial sums of a certain power series, *J. Austral. Math. Soc.*, **11** (1970) 407–410. MR0281902 (43 #7616)

[45] R. Singh, On the partial sums of convex functions of order $1/2$, *Proc. Amer. Math. Soc.*, **102** no. 3 (1988) 541–545. MR0928976 (89e:30029)

[46] R. Singh and S. Paul, Linear sums of certain analytic functions, *Proc. Amer. Math. Soc.*, **99** no. 4 (1987) 719–725. MR0877046 (88b:30023)

[47] R. Singh and S. Puri, Odd starlike functions, *Proc. Amer. Math. Soc.*, **94** no. 1 (1985) 77–80. MR0781060 (86f:30014)

[48] R. Singh and S. Singh, Convolution properties of a class of starlike functions, *Proc. Amer. Math. Soc.*, **106** no. 1 (1989) 145–152. MR0994388 (90d:30037)

[49] T. J. Suffridge, On a family of convex polynomials, *Rocky Mountain J. Math.*, **22** no. 1 (1992) 387–391. MR1159965 (93c:30009)

[50] G. Szegö, Zur Theorie der schlichten Abbildungen, *Math. Ann.*, **100** no. 1 (1928) 188–211. MR1512482

[51] G. Szegö, Some recent investigations concerning the sections of power series and related developments, *Bull. Amer. Math. Soc.*, **42** no. 8 (1936) 505–522. MR1563346

[52] K. Yamaguchi, On functions satisfying $R\{f(z)/z\} > 0$, *Proc. Amer. Math. Soc.*, **17** (1966) 588–591. MR0192041 (33 #268)

[53] S. Yamashita, Radii of univalence, starlikeness, and convexity, *Bull. Austral. Math. Soc.*, **25** no. 3 (1982) 453–457. MR0671491 (84e:30021)

[54] Z. Ye, The sections of univalent functions, *Bull. Inst. Math. Acad. Sin. (N.S.)*, **2** no. 1 (2007) 83–90. MR2294110 (2008c:30024)

Integers without Large Prime Factors: From Ramanujan to de Bruijn

Pieter Moree

Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany

E-mail: moree@mpim-bonn.mpg.de

In memoriam: Nicolaas Govert (‘Dick’) de Bruijn (1918–2012)

Abstract. A small survey of work done on estimating the number of integers without large prime factors up to around the year 1950 is provided. Around that time N. G. de Bruijn published results that dramatically advanced the subject and started a new era in this topic.

Keywords. Friable integers, Dickman-de Bruijn function.

2010 Mathematics Subject Classification. Primary 11N25; Secondary 34K25.

1. Introduction

Let $P(n)$ denote the largest prime divisor of n. We set $P(1) = 1$. A number n is said to be *y-friable*[1] if $P(n) \le y$. We let $S(x, y)$ denote the set of integers $1 \le n \le x$ such that $P(n) \le y$. The cardinality of $S(x, y)$ is denoted by $\Psi(x, y)$. We write $y = x^{1/u}$, that is $u = \log x / \log y$.

Fix $u > 0$. In 1930, Dickman [13] proved that

$$\lim_{x\to\infty} \frac{\Psi(x, x^{1/u})}{x} = \rho(u), \tag{1}$$

[1] In the older literature one usually finds y-smooth. Friable is an adjective meaning easily crumbled or broken.

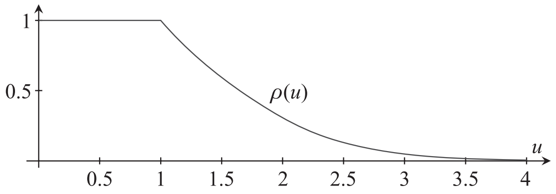

Figure 1. The Dickman-de Bruijn function $\rho(u)$.

with

$$\rho(u) = \rho(N) - \int_N^u \frac{\rho(v-1)}{v} dv,$$
$$(N < u \leq N+1, N = 1, 2, 3, \ldots),$$

and $\rho(u) = 1$ for $0 < u \leq 1$ (see Figure 1). It is left to the reader to show that we have

$$\rho(u) = \begin{cases} 1 & \text{for } 0 \leq u \leq 1; \\ \frac{1}{u}\int_0^1 \rho(u-t)dt & \text{for } u > 1. \end{cases} \tag{2}$$

The function $\rho(u)$ in the literature is either called the *Dickman function* or the *Dickman-de Bruijn function*.

In this note I will briefly discuss the work done on friable integers up to the papers of de Bruijn [7,8] that appeared around 1950 and dramatically advanced the subject. A lot of the early work was carried out by number theorists from India.

De Bruijn [7] improved on (1) by establishing a result that together with the best currently known estimate for the prime counting function (due to I. M. Vinogradov and Korobov in 1958) yields the following result.

Theorem 1. *The estimate*[2]

$$\Psi(x, y) = x\rho(u)\left\{1 + O^2\left(\frac{\log(u+1)}{\log y}\right)\right\}, \tag{3}$$

holds for $1 \leq u \leq \log^{3/5-\epsilon} y$, *that is*, $y > \exp(\log^{5/8+\epsilon} x)$.

De Bruijn's most important tool in his proof of this result is the *Buchstab equation* [9],

$$\Psi(x, y) = \Psi(x, z) - \sum_{y<p\leq z} \Psi\left(\frac{x}{p}, p\right), \tag{4}$$

where $1 \leq y < z \leq x$. The Buchstab equation is easily proved on noting that the number of integers $n \leq x$ with $P(n) = p$ equals $\Psi(x/p, p)$. Given a good estimate for $\Psi(x, y)$ for $u \leq h$, it allows one to obtain a good estimate for $u \leq h+1$.

[2]The reader not familiar with the Landau-Bachmann O-notation we refer to wikipedia or any introductory text on analytic number theory, e.g., Tenenbaum [35]. Instead of $\log\log x$ we sometimes write $\log_2 x$, instead of $(\log x)^A$, $\log^A x$.

De Bruijn [8] complemented Theorem 1 by an asymptotic estimate for $\rho(u)$. That result has as a corollary that, for $u \geq 3$,

$$\rho(u) = \exp\left\{-u\left\{\log u + \log_2 u - 1 + \frac{\log_2 u - 1}{\log u} + O\left(\left(\frac{\log_2 u}{\log u}\right)^2\right)\right\}\right\}, \tag{5}$$

which will suffice for our purposes. Note that (5) implies that, as $u \to \infty$,

$$\rho(u) = \frac{1}{u^{u+o(u)}}, \quad \rho(u) = \left(\frac{e+o(1)}{u\log u}\right)^u,$$

formulas that suffice for most purposes and are easier to remember. For a more detailed description of this and other work of de Bruijn in analytic number theory, we refer to Moree [20].

2. Results on $\rho(u)$

Note that $\rho(u) > 0$, for if not, then because of the continuity of $\rho(u)$ there is a smallest zero $u_0 > 1$ and then substituting u_0 in (2) we easily arrive at a contradiction. Note that for $u > 1$ we have

$$\rho'(u) = -\frac{\rho(u-1)}{u} \tag{6}$$

It follows that $\rho(u) = 1 - \log u$ for $1 \leq u \leq 2$. For $2 \leq u \leq 3$, $\rho(u)$ can be expressed in terms of the dilogarithm. However, with increasing u one has to resort to estimating $\rho(u)$ or finding a numerical approximation.

Since $\rho(u) > 0$ we see from (6) that $\rho(u)$ is strictly decreasing for $u > 1$. From this and (2) we then find that $u\rho(u) \leq \rho(u-1)$, which on using induction leads to $\rho(u) \leq 1/[u]!$ for $u \geq 0$. It follows that $\rho(u)$ quickly tends to zero as u tends to infinity.

Ramaswami [29] proved that

$$\rho(u) > \frac{C}{u4^u\Gamma(u)^2}, u \geq 1,$$

for a suitable constant C, with Γ the Gamma function. By Stirling's formula we have $\log\Gamma(u) \sim u\log u$ and hence the latter inequality is for u large enough improved on by the following inequality due to Buchstab [9]:

$$\rho(u) > \exp\left\{-u\left\{\log u + \log_2 u + 6\frac{\log_2 u}{\log u}\right\}\right\}, \quad (u \geq 6). \tag{7}$$

Note that on its turn de Bruijn's result (5) considerably improves on the latter inequality.

3. S. Ramanujan (1887–1920) and the Friables

In his first letter (January 16th, 1913) to Hardy (see, e.g. [3]), one of the most famous letters in all of mathematics, Ramanujan claims that

$$\Psi(n,3)=\frac{1}{2}\frac{\log(2n)\log(3n)}{\log 2\log 3}. \tag{8}$$

The formula is of course intended as an approximation, and there is no evidence to show how accurate Ramanujan supposed it to be. Hardy [17, pp. 69–81] in his lectures on Ramanujan's work gave an account of an interesting analysis that can be made to hang upon the above assertion. I return to this result in the section on the $\Psi(x,y)$ work of Pillai.

In the so-called Lost Notebook [27] we find at the bottom half of page 337:

$\phi(x)$ is the no. of nos of the form

$$2^{a_2}\cdot 3^{a_3}\cdot 5^{a_5}\cdots p^{a_p}\quad p\le x^{\epsilon}$$

not exceeding x.

$\frac{1}{2}\le\epsilon\le 1$,

$$\phi(x)\sim x\left\{1-\int_{\epsilon}^{1}\frac{d\lambda_0}{\lambda_0}\right\}$$

$\frac{1}{3}\le\epsilon\le\frac{1}{2}$,

$$\phi(x)\sim x\left\{1-\int_{\epsilon}^{1}\frac{d\lambda_0}{\lambda_0}+\int_{\epsilon}^{\frac{1}{2}}\frac{d\lambda_1}{\lambda_1}\int_{\lambda_1}^{1-\lambda_1}\frac{d\lambda_0}{\lambda_0}\right\}$$

$\frac{1}{4}\le\epsilon\le\frac{1}{3}$,

$$\phi(x)\sim x\left\{1-\int_{\epsilon}^{1}\frac{d\lambda_0}{\lambda_0}+\int_{\epsilon}^{\frac{1}{2}}\frac{d\lambda_1}{\lambda_1}\int_{\lambda_1}^{1-\lambda_1}\frac{d\lambda_0}{\lambda_0}\right.$$
$$\left.-\int_{\epsilon}^{\frac{1}{3}}\frac{d\lambda_2}{\lambda_2}\int_{\lambda_2}^{\frac{1-\lambda_2}{2}}\frac{d\lambda_1}{\lambda_1}\int_{\lambda_1}^{1-\lambda_1}\frac{d\lambda_0}{\lambda_0}\right\}$$

$\frac{1}{5}\le\epsilon\le\frac{1}{4}$,

$$\phi(x)\sim x\left\{1-\int_{\epsilon}^{1}\frac{d\lambda_0}{\lambda_0}+\int_{\epsilon}^{\frac{1}{2}}\frac{d\lambda_1}{\lambda_1}\int_{\lambda_1}^{1-\lambda_1}\frac{d\lambda_0}{\lambda_0}\right.$$
$$-\int_{\epsilon}^{\frac{1}{3}}\frac{d\lambda_2}{\lambda_2}\int_{\lambda_2}^{\frac{1-\lambda_2}{2}}\frac{d\lambda_1}{\lambda_1}\int_{\lambda_1}^{1-\lambda_1}\frac{d\lambda_0}{\lambda_0}$$
$$\left.+\int_{\epsilon}^{\frac{1}{4}}\frac{d\lambda_3}{\lambda_3}\int_{\lambda_3}^{\frac{1-\lambda_3}{3}}\frac{d\lambda_2}{\lambda_2}\int_{\lambda_2}^{\frac{1-\lambda_2}{2}}\frac{d\lambda_1}{\lambda_1}\int_{\lambda_1}^{1-\lambda_1}\frac{d\lambda_0}{\lambda_0}\right\}$$

and so on.

In the book by Andrews and Berndt [1, §8.2] it is shown that Ramanujan's assertion is equivalent with (1) with

$$\rho(u)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}I_k(u),$$

where

$$I_k(u)=\int_{\substack{t_1,\ldots t_k\ge 1\\ t_1+\cdots+t_k\le u}}\frac{dt_1}{t_1}\cdots\frac{dt_k}{t_k}.$$

This is one of many examples where Ramanujan reached with his hand from his grave to snatch a theorem, in this case from Dickman who was at least 10 years later than Ramanujan, cf. Berndt [2]. Chowla and Vijayaragahavan [12] seemed to have been the first to rigorously prove (1) with $\rho(u)$ expressed as a sum of iterated integrals (cf. the section on Buchstab). The asymptotic behaviour of the integrals $I_k(u)$ has been studied by Soundararajan [32].

Ramanujan's claim reminds me of the following result of Chamayou [10]: If $x_1,x_2,x_3,\ldots$ are independent random variables uniformly distributed in $(0,1)$, and $u_n=x_1+x_1x_2+\cdots+x_1x_2\cdots x_n$, then u_n converges in probability to a limit u_∞ and u_∞ has a probability distribution with density function $\rho(t)e^{-\gamma}$, where γ denotes Euler's constant.

4. I. M. Vinogradov (1891–1983) and the Friables

The first person to have an application for $\Psi(x,y)$ estimates seems to have been Ivan Matveyevich Vinogradov [36] in 1927. Let $k\ge 2$ be a prescribed integer and $p\equiv 1(\mathrm{mod}\ k)$ a prime. The k-th powers in $(\mathbb{Z}/p\mathbb{Z})^*$ form a subgroup of order $(p-1)/k$ and so the existence follows of $g_1(p,k)$, the least k-th power non-residue modulo a prime p. Suppose that $y<g_1(p,k)$, then $S(x,y)$ consists of k-th power residues only. It follows that

$$\Psi(x,y)\le\#\{n\le x: n\equiv a^k(\mathrm{mod}\ p)\text{ for some }a\}.$$

The idea is now to use good estimates for the quantities on both sides of the inequality sign in order to deduce an upper bound for $g_1(p,k)$.

Vinogradov [36] showed that $\Psi(x,x^{1/u})\ge\delta(u)x$ for $x\ge 1$, $u>0$, where $\delta(u)$ depends only on u and is positive. He applied this to show that if $m\ge 8$, $k>m^m$, and $p\equiv 1(\mathrm{mod}\ k)$ is sufficiently large, then

$$g_1(p,k)<p^{1/m}. \tag{9}$$

See Norton [21] for a historical account of the problem of determining $g_1(p,k)$ and original results.

5. K. Dickman (1861–1947) and the Friables

Karl Dickman was active in the Swedish insurance business in the end of the 19th century and the beginning of the 20th century. Probably, he studied mathematics in the 1880's at Stockholm University, where the legendary Mittag-Leffler was professor[3].

As already mentioned Dickman proved (1) and in the same paper[4] gave an heuristic argument to the effect that

$$\lim_{x\to\infty} \frac{1}{x} \sum_{2\le n\le x} \frac{\log P(n)}{\log n} = \int_0^\infty \frac{\rho(u)}{(1+u)^2}du. \tag{10}$$

Denote the integral above by λ. Dickman argued that $\lambda \approx 0.62433$. Mitchell [19] in 1968 computed that $\lambda = 0.62432998854\ldots$. The interpretation of Dickman's heuristic is that for an average integer with m digits, its greatest prime factor has about λm digits. The constant λ is now known as the *Golomb-Dickman constant*, as it arose independently in research of Golomb and others involving the largest cycle in a random permutation.

De Bruijn [7] in 1951 was the first to prove (10). He did this using his $\Lambda(x, y)$-function, an approximation of $\Psi(x, y)$, that he introduced in the same paper.

6. S.S. Pillai (1901–1950) and the Friables

Subbayya Sivasankaranarayana Pillai (1901–1950) was a number theorist who worked on problems in classical number theory (Diophantine equations, Waring's problem, etc.). Indeed, he clearly was very much inspired by the work of Ramanujan. He tragically died in a plane crash near Cairo while on his way to the International Congress of Mathematicians (ICM) 1950, which was held at Harvard University.

Pillai wrote two manuscripts on friable integers, [23,24], of which [23] was accepted for publication in the Journal of the London Mathematical Society, but did not appear in print. Also [24] was never published in a journal.

In [23], see also [26, pp. 481–483], Pillai investigates $\Psi(x, y)$ for y fixed. Let $p_1, p_2, \ldots, p_k$ denote all the different primes $\le y$. Notice that $\Psi(x, y)$ equals the cardinality of the set

$$\left\{(e_1,\ldots,e_k)\in\mathbb{Z}^k : e_i\ge 0, \sum_{i=1}^k e_i\log p_i \le x\right\}.$$

Thus $\Psi(x, y)$ equals the number of lattice points in a k-dimensional tetrahedron with sides of length $\log x/\log 2, \ldots, \log x/\log p_k$. This tetrahedron has volume

$$\frac{1}{k!}\prod_{p\le y}\left(\frac{\log x}{\log p}\right).$$

Pillai shows that

$$\Psi(x, y) = \frac{1}{k!}\prod_{p\le y}\left(\frac{\log x}{\log p}\right) \times\left(1+(1+o(1))\frac{k\log(p_1p_2\ldots p_k)}{2\log x}\right).$$

If $\rho_1, \ldots, \rho_k$ are positive real numbers and ρ_1/ρ_2 is irrational, then the same estimate with $\log p_i$ replaced by ρ_i holds for

$$\left\{(e_1,\ldots,e_k)\in\mathbb{Z}^k : e_i\ge 0, \sum_{i=1}^k e_i\rho_i \le x\right\}.$$

This was proved by Specht [33] (after whom the Specht modules are named), see also Beukers [4]. A much sharper result than that of Pillai/Specht was obtained in 1969 by Ennola [15] (see also Norton [21, pp. 24–26]). In this result Bernoulli numbers make their appearance.

Note that Pillai's result implies that

$$\Psi(x, 3) = \frac{1}{2}\frac{\log(2x)\log(3x)}{\log 2\log 3} + o(\log x), \tag{11}$$

and that the estimate

$$\Psi(x, 3) = \frac{\log^2 x}{2\log 2\log 3} + o(\log x)$$

is false. Thus Ramanujan's estimate (8) is more precise than the trivial estimate $\log^2 x/(2\log 2\log 3)$. Hardy [17, §5.13] showed that the error term $o(\log x)$ in (11) can be replaced by $o(\log x/\log_2 x)$. In the proof of this he uses a result of Pillai [22], see also [25, pp. 53–61], saying that given $0<\delta<1$, one has $|2^x - 3^y| > 2^{(1-\delta)x}$ for all integers x and y with $x > x_0(\delta)$ sufficiently large.

In [24], see also [26, pp. 515–517], Pillai claims that, for $u \ge 6$, $B/u < \rho(u) < A/u$, with $0 < B < A$ constants. He proves this result by induction assuming a certain estimate for $\rho(6)$ holds. However, this estimate for $\rho(6)$ does not hold. Indeed, the claim contradicts (5) and is false.

[3]I have this information from Lars Holst.

[4]Several sources falsely claim that Dickman wrote only one mathematical paper. He also wrote [14].

Since Pillai reported on his work on the friables at conferences in India and stated open problems there, his influence on the early development of the topic was considerable. E.g., one of the questions he raised was whether $\Psi(x, x^{1/u}) = O(x^{1/u})$ uniformly for $u \le (\log x)/\log 2$. This question was answered in the affirmative by Ramaswami [29].

7. R. A. Rankin (1915–2001) and the Friables

In his work on the size of gaps between consecutive primes Robert Alexander Rankin [31] in 1938 introduced a simple idea to estimate $\Psi(x, y)$ which turns out to be remarkably effective and can be used in similar situations. This idea is now called 'Rankin's method' or 'Rankin's trick'. Starting point is the observation that for any $\sigma > 0$

$$\Psi(x, y) \le \sum_{n \in S(x,y)} \left(\frac{x}{n}\right)^{\sigma} \le x^{\sigma} \sum_{P(n) \le y} \frac{1}{n^{\sigma}} = x^{\sigma} \zeta(\sigma, y), \tag{12}$$

where

$$\zeta(s, y) = \prod_{p \le y} (1 - p^{-s})^{-1},$$

is the partial Euler product up to y for the Riemann zeta function $\zeta(s)$. Recall that, for $\Re s > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}.$$

By making an appropriate choice for σ and estimating $\zeta(\sigma, y)$ using analytic prime number theory, a good upper bound for $\Psi(x, y)$ can be found. E.g., the choice $\sigma = 1 - 1/(2 \log y)$ leads to

$$\begin{aligned} \zeta(\sigma, y) &\ll \exp\left\{\sum_{p \le y} \frac{1}{p^{\sigma}}\right\} \\ &\le \exp\left\{\sum_{p \le y} \frac{1}{p} + O\left((1-\sigma) \sum_{p \le y} \frac{\log p}{p}\right)\right\} \ll \log y, \end{aligned}$$

which gives rise to

$$\Psi(x, y) \ll x \mathrm{e}^{-u/2} \log y. \tag{13}$$

8. A. A. Bukhshtab (1905–1990) and the Friables

Aleksandr Adol'fovich Bukhshtab's[5] most important contribution is the equation (4) now named after him. A generalization of it plays an important role in sieve theory. Buchstab [9] in 1949 proved (1) and gave both Dickman's differential-difference equation as well as the result

$$\begin{aligned} \rho(u) = 1 + \sum_{n=1}^{N} (-1)^n \int_n^u \int_{n-1}^{t_1 - 1} \\ \times \int_{n-2}^{t_2 - 1} \cdots \int_1^{t_{n-1} - 1} \frac{dt_n dt_{n-1} \cdots dt_1}{t_1 t_2 \cdots t_n}, \end{aligned} \tag{14}$$

for $N \le u \le N + 1$ and $N \ge 1$ an integer, simplifying Chowla and Vijayaragahavan's expression (they erroneously omitted the term $n = N$). Further, Buchstab established inequality (7) and applied his results to show that the exponent in Vinogradov's result (9) can be roughly divided by two.

9. V. Ramaswami and the Friables

V. Ramaswami[6] [28] showed that

$$\Psi(x, x^{1/u}) = \rho(u)x + O_U\left(\frac{x}{\log x}\right),$$

for $x > 1$, $1 < u \le U$, and remarked that the error term is best possible. He sharpened this result in [29] and showed there that, for $u > 2$,

$$\Psi(x, x^{1/u}) = \rho(u)x + \sigma(u)\frac{x}{\log x} + O\left(\frac{x}{\log^{3/2} x}\right), \tag{15}$$

with $\sigma(u)$ defined similarly to $\rho(u)$. Indeed, it turns out that

$$\sigma(u) = (1 - \gamma)\rho(u - 1),$$

but this was not noticed by Ramaswami. In [30] Ramaswami generalized his results to $B_l(m, x, y)$ which counts the number of integers $n \le x$ with $P(n) \le y$ and $n \equiv l (\mathrm{mod}\ m)$ [7]. Norton [21, pp. 12–13] points out some deficits of this paper and gives a reproof [21, §4] of Ramaswami's result on $B_l(m, x, x^{1/u})$ generalizing (15).

From de Bruijn's paper [7, Eqs. (5.3), (4.6)] one easily derives the following generalization of Ramaswami's results[8]:

Theorem 2. *Let $m \ge 0$, $x > 1$, and suppose $m + 1 < u < \sqrt{\log x}$. Then*

$$\Psi(x, y) = x \sum_{r=0}^{m} a_r \frac{\rho^{(r)}(u)}{\log^r y} + O_m\left(\frac{x}{\log^{m+1} y}\right),$$

[5] Buchstab in the German spelling.

[6] He worked at Andhra University until his death in 1961. I will be grateful for further biographical information.

[7] Buchstab [9] was the first to investigate $B_l(m, x, y)$.

[8] The notation O_m indicates that the implied constant might depend on m.

with $\rho^{(r)}(u)$ the r-th derivative of $\rho(u)$ and $a_0, a_1, \ldots$ are the coefficients in the power series expansion

$$\frac{z}{1+z}\zeta(1+z) = a_0 + a_1 z + a_2 z^2 + \cdots,$$

with $|z| < 1$.

It is well-known (see, e.g., Briggs and Chowla [5]) that around $s = 1$ the Riemann zeta function has the Laurent series expansion

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\gamma_k (s-1)^k,$$

with γ_k the k-th Stieltjes constant and with $\gamma_0 = \gamma$ Euler's constant. Using this we find that $a_0 = 1$ and $a_1 = \gamma - 1$. Thus Theorem 2 yields (15) with $\sigma(u) = (1-\gamma)\rho(u-1)$ for the range $2 < u < \sqrt{\log x}$. For $u > \sqrt{\log x}$ the estimate (15) in view of (5) reduces to

$$\Psi(x, x^{1/u}) \ll x \log^{-3/2} x,$$

which easily follows from (13).

10. S. Chowla (1907–1995) and the Friables

The two most prominent number theorists in the period following Ramanujan were S. S. Pillai and Sarvadaman Chowla. They kept in contact through an intense correspondence [34]. Chowla in his long career published hundreds of research papers.

Chowla and Vijayaragahavan [12] expressed $\rho(u)$ as an iterated integral and gave a formula akin to (14). De Bruijn [6] established some results implying that $\Psi(x, \log^h x) = O(x^{1-1/h+\epsilon})$ for $h > 2$. An easier reproof of the latter result was given by Chowla and Briggs [11].

11. Summary

It seems that the first person to look at friable integers was Ramanujan, starting with his first letter to Hardy (1913), also Ramanujan seems to have been the first person to arrive at the Dickman-de Bruijn function $\rho(u)$. Pillai generalized some of Ramanujan's work and spoke about it on conferences in India, which likely induced a small group of Indian number theorists to work on friable integers. Elsewhere in the same period (1930–1950) only incidental work was done on the topic. Around 1950 N. G. de Bruijn published his groundbreaking papers [7,8]. Soon afterwards the Indian number theorists stopped publishing on friable integers.

It should also be said that the work on friable integers up to 1950 seems to contain more mistakes than more recent work. Norton [21] points out and corrects many of these mistakes.

Further Reading

As a first introduction to friable numbers I can highly recommend Granville's 2008 survey [16]. It has a special emphasis on friable numbers and their role in algorithms in computational number theory. Mathematically more demanding is the 1993 survey by Hildebrand and Tenenbaum [18]. Chapter III. 5 in Tenenbaum's book [35] deals with $\rho(u)$ and approximations to $\Psi(x, y)$ by the saddle point method.

Acknowledgement

I thank R. Thangadurai for helpful correspondence on S. S. Pillai and the friables and providing me with a PDF file of Pillai's collected works. B. C. Berndt kindly sent me a copy of [1]. K. K. Norton provided helpful comments on an earlier version. His research monograph [21], which is the most extensive source available on the early history of friable integer counting, was quite helpful to me. In [21], by the way, new results (at the time) on $g_1(p, k)$ and $\Psi_m(x, y)$, the number of y-friable integers $1 \le n \le x$ coprime with m are established. Figure 1 was kindly created for me by Jon Sorenson and Alex Weisse (head of the MPIM computer group).

References

[1] G. E. Andrews and B. C. Berndt, Ramanujan's Lost Notebook, Part IV, Springer-Verlag, preprint, to appear.

[2] B. C. Berndt, Ramanujan reaches his hand from his grave to snatch your theorems from you, *Asia Pac. Math. Newsl.*, **1(2)** (2011) 8–13.

[3] B. C. Berndt and R. A. Rankin, *Ramanujan. Letters and commentary*, AMS, Providence, RI; London Mathematical Society, London, 1995.

[4] F. Beukers, The lattice-points of n-dimensional tetrahedra, *Indag. Math.*, **37** (1975) 365–372.

[5] W. E. Briggs and S. Chowla, The power series coefficients of $\zeta(s)$, *Amer. Math. Monthly*, **62** (1955) 323–325.

[6] N. G. de Bruijn, On the number of uncancelled elements in the sieve of Eratosthenes, *Indag. Math.*, **12** (1950) 247–256.

[7] N. G. de Bruijn, On the number of positive integers $\leq x$ and free of prime factors $> y$, *Indag. Math.*, **13** (1951) 50–60.

[8] N. G. de Bruijn, The asymptotic behaviour of a function occurring in the theory of primes, *J. Indian Math. Soc. (N.S.)*, **15** (1951). 25–32.

[9] A. A. Buchstab, On those numbers in arithmetical progression all of whose prime factors are small in order of magnitude (Russian), *Dokl. Akad. Nauk. SSSR*, **67** (1949) 5–8.

[10] J.-M.-F. Chamayou, A probabilistic approach to a differential-difference equation arising in analytic number theory, *Math. Comp.*, **27** (1973) 197–203.

[11] S. D. Chowla and W. E. Briggs, On the number of positive integers $\leq x$ all of whose prime factors are $\leq y$, *Proc. Amer. Math. Soc.*, **6** (1955) 558–562.

[12] S. D. Chowla and T. Vijayaraghavan, On the largest prime divisors of numbers, *J. Indian Math. Soc. (N.S.)*, **11** (1947) 31–37.

[13] K. Dickman, On the frequency of numbers containing prime factors of a certain relative magnitude, *Ark. Mat. Astr. Fys.*, **22** (1930) 1–14.

[14] K. Dickman, Om maximiantalet konsekutiva summander till ett helt tal (Swedish), *8. Skand. Mat. Kongr., Stockholm*, (1934) 385–388.

[15] V. Ennola, On numbers with small prime divisors, *Ann. Acad. Sci. Fenn. Ser.*, A I **440** (1969) 16 pp.

[16] A. Granville, Smooth numbers: Computational number theory and beyond, Algorithmic number theory: lattices, number fields, curves and cryptography, *Math. Sci. Res. Inst. Publ.*, **44**, Cambridge Univ. Press, Cambridge, (2008) 267–323.

[17] G. H. Hardy, Ramanujan: twelve lectures on subjects suggested by his life and work, Chelsea Publishing Company, New York, (1959).

[18] A. Hildebrand and G. Tenenbaum, Integers without large prime factors, *J. Théor. Nombres Bordeaux*, **5** (1993), 411–484.

[19] W. C. Mitchell, An evaluation of Golomb's constant, *Math. Comp.*, **22** (1968) 411–415.

[20] P. Moree, Nicolaas Govert de Bruijn, the enchanter of friable integers, preprint, arXiv:1212.1579.

[21] K. K. Norton, Numbers with small prime factors, and the least kth power non-residue, *Memoirs of the American Mathematical Society*, **106**, *American Mathematical Society, Providence, R.I.* (1971).

[22] S. S. Pillai, On the inequality $0 < a^x - b^y \leq n$, *J. Indian Math. Soc.*, **19** (1931) 1–11.

[23] S. S. Pillai, Concerning a statement of Ramanujan, unpublished manuscript, written around (1931).

[24] S. S. Pillai, The number of positive integers $\leq x$ and free of prime divisors $> x^{1/r}$, unpublished manuscript, written around (1947).

[25] S.S. Pillai, Collected works of S. Sivasankaranarayana Pillai **I**, (Eds.) R. Balasubramanian and R. Thangadurai, *Ramanujan Mathematical Society Collected Works Series*, **1**, *Ramanujan Mathematical Society*, Mysore (2010).

[26] S. S. Pillai, Collected works of S. Sivasankaranarayana Pillai **II**, Published/unpublished papers and letters, (Eds.) R. Balasubramanian and R. Thangadurai, *Ramanujan Mathematical Society Collected Works Series*, **1**, *Ramanujan Mathematical Society*, Mysore (2010).

[27] S. Ramanujan, The lost notebook and other unpublished papers, Narosa, New Delhi, (1988).

[28] V. Ramaswami, On the number of positive integers less than x and free of prime divisors greater than x^c, *Bull. Amer. Math. Soc.*, **55** (1949) 1122–1127.

[29] V. Ramaswami, The number of positive integers $\leq x$ and free of prime divisors $> x^c$ and a problem of S. S. Pillai, *Duke Math. J.* **16** (1949) 99–109.

[30] V. Ramaswami, Number of integers in an assigned A. P., $\leq x$ and prime to primes greater than x^c, *Proc. Amer. Math. Soc.*, **2** (1951) 318–319.

[31] R. A. Rankin, The difference between consecutive prime numbers, *J. London Math. Soc.*, **13** (1938) 242–247.

[32] K. Soundararajan, An asymptotic expression related to the Dickman function, *Ramanujan J.* **29** (2012) 25–30.

[33] W. Specht, Zahlenfolgen mit endliche viele Primteilern, *Bayer. Akad. Wiss. Math.-Natur. Abt. S.-B.*, **1948** (1949) 149–169.

[34] B. Sury and R. Thangadurai, S. Chowla and S. S. Pillai, *Resonance*, **17(9)** (2012) 855–883.

[35] G. Tenenbaum, Introduction to analytic and probabilistic number theory, *Cambridge Studies in Advanced Mathematics*, **46**, Cambridge University Press, Cambridge, (1995).

[36] I. M. Vinogradov, On a bound for the least nth power non-residue, *Trans. Amer. Math. Soc.*, **29** (1927) 218–226.

Problems and Solutions

Edited by Amritanshu Prasad

The section of the Newsletter contains problems contributed by the mathematical community. These problems range from mathematical brain-teasers through instructive exercises to challenging mathematical problems. Contributions are welcome from everyone, students, teachers, scientists and other maths enthusiasts. We are open to problems of all types. We only ask that they be reasonably original, and not more advanced than the MSc level. A cash prize of Rs. 500 will be given to those whose contributions are selected for publication. Please send solutions along with the problems. The solutions will be published in the next issue. Please send your contribution to `problems@imsc.res.in` with the word "**problems**" somewhere in the subject line. Please also send solutions to these problems (with the word "**solutions**" somewhere in the subject line). Selected solutions will be featured in the next issue of this Newsletter.

1. **B. Sury, ISI Bangalore**. For each positive integer n, the value $\phi(n)$ of the Euler totient function is the number of integers $1 < k < n$ that are coprime to n. Show that every factorial is a value of the Euler totient function: for every positive integer m, there exists a positive integer n such that $\phi(n) = m!$.
2. **B. Sury, ISI Bangalore**. Consider the $n \times n$ matrix A whose (i, j)th entry is $a_{ij} = gcd(i, j)$. Determine det A.
3. **Amritanshu Prasad, IMSc**. Show that
$$\frac{q^{3n+3} + q^{3n-1} - q^{2n+2} - q^{2n+1} - q^{2n} - q^{2n-1} + 2q^n}{(q^2-1)(q-1)}$$
is a polynomial in q with non-negative integer coefficients for all positive integers n.
4. **Priyamvad Srivastav, IMSc**. Let k be a positive integer. Suppose that $\{a_n\}_{n\geq 1}$ is a sequence of positive integers satisfying $a_m^k + a_n^k | m^k + n^k$ whenever $gcd(m, n) = 1$. Show that $a_n = n$ for all n.
5. **S. Viswanath, IMSc**. Let $P_n(x)$ denote the nth Legendre polynomial given by
$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n].$$
Evaluate the determinant
$$\begin{vmatrix} P_0(x_1) & P_1(x_1) & \cdots & P_{n-1}(x_1) \\ P_0(x_2) & P_1(x_2) & \cdots & P_{n-1}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ P_0(x_n) & P_1(x_n) & \cdots & P_{n-1}(x_n) \end{vmatrix}$$
6. **S. Kesavan, IMSc**. Let $u_1, u_2, \ldots, u_N$ be a set of n linearly independent unit vectors in R^N. The $N \times N$ matrix $G = (g_{ij})$, where $g_{ij} = u_i \cdot u_j$ (the usual dot product of vectors) is called the Gram matrix associated to $u_1, \ldots, u_N$. Show that λ_1, the smallest eigenvalue of G satisfies
$$\lambda_1 > \frac{\det G}{e}.$$
7. **S. Kesavan, IMSc**. Let $\Omega \subset \mathbf{R}^N$ be a bounded and connected open set with continuous boundary $\partial\Omega$. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be such that $u > 0$ in Ω and $u = 0$ in $\partial\Omega$. If there exists a constant λ such that, for all $x \in \Omega$ and all $1 \leq i, j \leq N$,
$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \lambda\delta_{ij},$$
show that Ω is a ball.

Solutions to Problems from the September Issue

1. **Apoorva Khare, IMSc**. Find the remainder when $1^{16} + 2^{16} + \cdots + 22^{16}$ is divided by 23.

   Solution. The non-zero elements of the finite field $\mathbf{Z}/23\mathbf{Z}$ form a cyclic group of order 22 under multiplication[1]. Let g be a generator for this group. Then $1, g, g^2, \ldots, g^{21}$ gives all the non-zero elements of this field, and $g^{22} = 1$. Thus the desired sum is a geometric series $1^{16} + g^{16} + (g^2)^{16} + \cdots + (g^{21})^{16} = (1 - (g^{16})^{22})/(1 - g^{16}) = 0$. Thus the remainder when $1^{16} + \cdots + 22^{16}$ is divided by 23 is 0.

   Solution received. A correct solution to this problem was received from Hari Kishan, D. N. College, Meerut, India.

2. **Anuj V.T., IMSc**. Consider a decomposition $n = a_1 + \cdots + a_l$ of a positive integer as a sum of distinct positive integers $a_1 < \cdots < a_l$ such that the product $a_1 \ldots a_l$ is maximized. If $8n + 25$ is a perfect square then $a_i - a_{i-1} = 1$ for each $i \in \{2, \ldots, l\}$.

   Solution. If $8n + 25$ is a perfect square, then it must be the square of an odd number, say $2k + 1$. Thus

$$\begin{aligned} 8n + 25 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 8(k(k + 1)/2) + 1. \end{aligned}$$

   Therefore, $n = k(k-1)/2 - 3 = 3 + 4 + \cdots + k$. We claim that this is the partition of n into distinct parts which maximizes the product of the parts.

   We begin with a few observations on product-maximizing sequences $a_1 < \cdots < a_l$ such with sum n:

   - $a_1 \neq 1$, for then replacing $a_1, \ldots, a_l$ by $a_2, \ldots, a_{l-1}, a_l + 1$ would increase the product while preserving the sum.
   - $a_1 \leq 5$, for otherwise, replacing $a_1, \ldots, a_l$ by $2, a_1 - 2, a_2, \ldots, a_l$ would increase the product while preserving the sum.
   - If $a_i > a_{i-1}+1$ for some i, then replacing a_{i-1} by $a_{i-1}+1$ and replacing a_i by $a_i - 1$ increases the product while preserving the sum.
   - If there exist $i < j$ such that $a_i - a_{i-1} = 2$ and $a_j - a_{j-1} = 2$, then replacing a_{i-1} by $a_{i-1} + 1$ and a_j by $a_j - 1$ increases the product while preserving the sum.

   Thus the sequence $a_1, \ldots, a_l$ is either consecutive, or of the form

$$a, a + 1, \ldots, m - 1, m + 1, \ldots, r \qquad (*)$$

   for some $2 \leq a \leq 4$ and $a < m < r$.

   Now assume that n is also the sum of numbers of the form $(*)$.

   If $a = 3$, $n = (3 + \cdots + r) - m = 3 + \cdots + k$, hence $r > k$. Therefore, $m = (k + 1) + \cdots + r$, contradicting the assumption that $m < r$.

   If $a = 4$, $n = (4 + \cdots + r) - m = 3 + \cdots + k$, or equivalently, $(4 + \cdots + r) = (4 + \cdots + k) + (m + 3)$, so again $r > k$. We get $m = (k + 1) + \cdots + r - 3$, which, since $k + 1 > 3$, contradicts the assumption that $m < r$.

   Finally, if $a = 2$, then $(2 + \cdots + r) - m = 3 + \cdots + k$. In other words, $(3 + \cdots + r) = (3 + \cdots + k) + m - 2$. Since $m > 3$, we still have $r > k$. Therefore, $(k + 1) + \cdots + r = m - 2$, contradicting the assumption that $m < r$.

3. **Partha Sarathi Chakraborty, IMSc**. For a maths quiz the teacher forms a random team from a class of 40 as follows:

   (1) Tickets numbered 1–40 are distributed randomly among the students.
   (2) A fair coin is tossed 6 times and the number of heads are counted, say k.
   (3) Those with tickets numbered 1,2, ... ,k are selected.
   (4) A random team of extras are selected from those holding tickets numbered 7, ... ,40.
   (5) (3) and (4) together constitute the final team (here random means that each subset of students holding tickets numbered 7 to 40 has probability 2^{-34} of being selected).

   What is the probability that the final team consists of (a) Judhisthir, Bheema, Arjuna, Nakula and Sahadeva. (b) Duryodhana, Duhshasana and Karna.

   Solution. The probability of each possible team (consisting of anything between 0 and 40 members) is $1/2^{40}$.

[1] See, for example, Chapter I, Section 1.2 of *A Course in Arithmetic* by J.-P. Serre, or Chapter I, Section 1 of *Basic Number Theory* by A. Weil.

In other words, the probability distribution is uniform on the set of all possible teams (the power set of the set of all students). To see this, first observe that the probability that a certain team is selected depends only on its cardinality. This is because the tickets were distributed randomly to all students at the beginning of the selection. Thus, we just have to show that the probability of choosing a team with n members is the probably that out of 40 tosses of a fair coin n land heads (the binomial distribution $B(40, 1/2)$).

Now observe that for each distribution of tickets, the selection of k follows the binomial distribution $B(6, 1/2)$ (because it is determined by coin tosses) and the number of holders of tickets 7 to 40 follows the binomial distribution $B(34, 1/2)$ (because this part of the team is chosen randomly). Therefore the total size of the team follows the binomial distribution $B(40, 1/2)$ as required.

4. **Anilkumar C.P., IMSc**. Given an integer n, consider the convex hull P of the set of points $(d, n/d)$, where d runs over the set $1 = d_1 < d_2 < \cdots < d_k = n$ of divisors of n (P is a polygon in the Cartesian plane). In terms of $d_1, \ldots, d_k$ find an expression for

   (1) The area of P.

   (2) The number of points on the boundary of P with integer coordinates.

   Solution. (1) The area of P is

$$\frac{n^2-1}{2} - \sum_{i=1}^{k-1} \left(\frac{n}{d_i} - \frac{n}{d_{i+1}}\right)\left(\frac{d_{i+1}+d_i}{2}\right).$$

   (2) The number of points with integer coordinates on the boundary of P is

$$(n-1) + \sum_{i=1}^{k-1} \frac{d_{i+1}-d_i}{d_i d_{i+1}} \gcd(n, d_i d_{i+1}).$$

5. **B. Sury, ISI Bengaluru**. If we take any quadrilateral and join the midpoints of the sides,we can easily see that we obtain a parallelogram. More generally, take any polygon of $2n$ sides and consider the centroids of the sets of n consecutive vertices. Prove that the polygon of $2n$ sides so formed has opposite pairs of sides equal and parallel.

   Solution. If $P_1, \ldots, P_{2n}$ are the vertices of the original polygon, then the centroid of $P_i, P_{i+1}, \ldots, P_{i+n-1}$ is $Q_i = (P_i + \cdots + P_{i+n-1})/n$ for each $1 \le i \le 2n$. Here, we read the index j of P_j modulo $2n$ (that is, P_{2n+j} means P_j etc.) and these sums could be thought of as vectors or in terms of co-ordinates. Thus,

$$Q_{i+1} - Q_i = \frac{P_{i+n} - P_i}{n}.$$

   In the polygon formed by the Q_j 's, the opposite side to the side joining Q_i and Q_{i+1} is the one joining Q_{n+i} and Q_{n+i+1}. Since

$$Q_{n+i+1} - Q_{n+i} = \frac{P_{i+2n} - P_{i+n}}{n} = \frac{P_i - P_{i+n}}{n}$$

   the two opposite sides are parallel and equal in length.

6. **B. Sury, ISI Bengaluru**. Evaluate

$$1 - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \frac{1}{23} + \frac{1}{25} - \cdots.$$

   Solution.

$$\begin{aligned} &1 - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} \\ &= \int_0^1 (1 - x^6 + x^8 - x^{14} + x^{16} - \cdots)dx \\ &= \int_0^1 (1-x^6)(1 + x^8 + x^{16} + \cdots)dx \\ &= \int_0^1 \frac{1-x^6}{1-x^8}dx \end{aligned}$$

   Break the integrand into partial fractions as

$$\int_0^1 \frac{dx}{2(1+x^2)} + \int_0^1 \frac{(1+x^2)dx}{2(1+x^4)} = I_1 + I_2, \text{ say}$$

   So $I_1 = \pi/8$.

   For I_2, write $1+x^4 = (x^2+\sqrt{2}x+1)\,(x^2 - \sqrt{2}x + 1)$ and break the integrand into partial fractions. We get

$$I_2 = \frac{1}{4}\int_0^1 \left(\frac{1}{x^2+\sqrt{2}x+1} + \frac{1}{x^2-\sqrt{2}x+1}\right)dx.$$

   On writing $x^2 \pm \sqrt{2}x + 1 = (x \pm \frac{1}{\sqrt{2}})^2 + \frac{1}{2}$, we get

$$I_2 = \frac{\sqrt{2}}{4}(\arctan(\sqrt{2}+1) + \arctan(\sqrt{2}-1)) = \frac{\pi\sqrt{2}}{8}$$

   since $\arctan x + \arctan(1/x) = \pi/2$. Hence the series has sum $\frac{\pi(1+\sqrt{2})}{8}$.

On Backward Heat Conduction Problem

M. T. Nair
Department of Mathematics, I.I.T. Madras, Chennai 600 036, India
E-mail: mtnair@iitm.ac.in

Abstract. It is well-known that backward heat conduction problem (BHCP) is a severely ill-posed problem. Hence, stability estimates for determining temperature at a time t_0 from the knowledge of the temperature at time $\tau > t_0$ is possible only by imposing certain a priori conditions on the unknown temperature. Standard results in this regard shows stability for the case of $t_0 > 0$, and such results are not valid for $t_0 = 0$. In this paper we analyze the BHCP in a general frame work of an operator equation, derive stability estimate under a general source condition, and deduce results for BHCP as a special case which are valid for any t_0 with $0 \le t_0 < \tau$. Also, we consider truncated singular value decomposition as a regularization method for obtaining a stable approximate solution, and derive error bound associated with the general source condition.

Keywords. Stability estimates, backward heat conduction problem, severely ill-posed, TSVD-regularization, general source condition.

1. Introduction

In many of the practical situations when dealing with a heat conducting body one may have to investigate the temperature distribution and heat flux history from the known data at a particular time, say $t = \tau$. From this knowledge one would like to know the temperature for $t < \tau$ as well as for $t > \tau$. It is well known that the latter is a well-posed problem. However, the former, the so called *backward heat conduction problem* is an ill-posed problem. Our consideration here is to discuss issues related to the backward heat conduction problem (BHCP).

Recall that one-dimensional heat conduction problem can be described by the equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \ell, \quad t > 0, \tag{1.1}$$

where $u(x, t)$ represents the temperature at the point x on a 'thin wire' of length ℓ at time t, and $c^2 = \kappa/(\sigma\rho)$ where κ is the thermal conductivity, σ the specific heat and ρ is the density of the material of the wire. For the sake of simplicity, let us assume that the temperature at both ends of the wire are kept zero, i.e.,

$$u(0, t) = 0 = u(\ell, t). \tag{1.2}$$

Now the two type of problems mentioned in the last paragraph can be categorised as *direct problem* and *inverse problem*.

Direct Problem. From the knowledge of the temperature at a time $t = t_0$, that is, $u(x, t_0)$, determine the temperature at $t > t_0$, that is, to determine $u(x, t)$ for $t > t_0$.

Inverse Problem. From the knowledge of the temperature at a time $t = \tau$, that is, $u(x, \tau)$, determine the temperature at time $t < \tau$, that is, to determine $u(x, t)$ for $t < \tau$.

In the early 1920's Hadamard (c.f. [3]) prescribed certain criteria for well-posedness of problems that occur in applications. According to Hadamard, a problem is *well-posed* if it satisfies all the following three requirements:

- A solution must exist.
- Solution must be unique.
- The solution must depend continuously on the data.

A problem is *ill-posed* if it violates atleast one of the above three criteria for well-posedness.

We shall see that, for the heat condtion problem, the direct problem is well-posed, where as the inverse problem above is ill-posed. In fact, the ill-posedness of the inverse problem above is too severe that a solution need not exist unless the data $g := u(x, \tau)$ is too smooth, and even in such case, the solution does not depend continuously on the data. So, in order to obtain a *stable approximate solution* for the BHCP, some regularization method has to be applied.

A *regularization method* involves a family of well-posed problems, say P_α, depending on certain parameters α belonging to certain parameter set Λ, which is either $\mathbb{R}^+$, the set of all positive real numbers, or $\{\alpha_n : n \in \mathbb{N}\}$ where (α_n) is a strictly decreasing sequence of positive real numbers. A primary requirement on the regularization method, apart from the well-posedness, is that if the exact data, say g, is known then the corresponding solution f_α must converge to the exact solution f as $\alpha \to \alpha_0$, where α_0 is a limit point of Λ. This requirement may not guarantee the convergence when one has only a noisy data. This points to the requirement that if the data is noisy, say $\tilde{g}$ with

$$\|g - \tilde{g}\| \leq \delta$$

for some nose level $\delta > 0$, then the *regularization parameter* α has to be chosen appropriately, say $\alpha := \alpha_\delta$ such that

$$\|f - f_{\alpha_\delta}\| \to 0 \quad \text{as} \quad \delta \to 0.$$

In applications, convergence of the approximation methods is not enough. One has to know the *rate of convergence* as well. Further, one must get a *best possible rate* of convergence under certain specified *source conditions* on the unknown solution. To satisfy these requirements, error estimates have to be derived and these error estimates have to the tested for their *optimality* (cf. [7]). Standard results in this regard for BHCP shows stability of $u(\cdot, t_0)$ for $t_0 > 0$, and such results are not valid for $t_0 = 0$ (See, e.g., Kirsch [5], Ames et al. [1]). In this paper we deduce stability results under a general source condition in a general frame work of an operator equation suitable for BHCP which are valid for any t_0 with $0 \leq t_0 < \tau$.

For obtaining stable approximations for ill-posed problems, many regularization methods are available in the literature (See, e.g. Engl, Hanke and Neubauer [2]). But, in the case of backward heat conduction problem, we have full knowledge of the singular values and singular vectors of the associated forward operator. Therefore, truncated singular value decomposition (TSVD) would be an easy to apply regularization method for this case. In this regard, we show that TSVD does provide order optimal error estimate under a general source condition, by an appropriate a priori choice of the level of truncation.

2. Ill-Posedness of the BHCP

For $t \geq 0$, we shall denote by $\tilde{u}(t)$ the function $u(\cdot, t)$ which is the solution of (1.1) satisfying the boundary conditions in (1.2). Also, unless it is specified otherwise, it is assumed that the variable x belongs to $[0, \ell]$ and $t > 0$.

Now, given $u(\cdot, t_0) \in L^2[0, \ell]$, by using the method of separation of variables, we have

$$u(x, t) := \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 t} \sin\left(\frac{n\pi x}{\ell}\right), \tag{2.1}$$

where

$$\lambda_n := \frac{cn\pi}{\ell} \quad \text{and} \quad a_n = \sqrt{\frac{2}{\ell}} e^{\lambda_n^2 t_0} \langle \tilde{u}(t_0), \varphi_n \rangle \quad \forall n \in \mathbb{N}, \tag{2.2}$$

with

$$\varphi_n(x) := \sqrt{2/\ell} \sin(n\pi x/\ell), \quad n \in \mathbb{N}. \tag{2.3}$$

It is seen that $\{\varphi_n : n \in \mathbb{N}\}$ with

$$\varphi_n(x) := \sqrt{2/\ell} \sin(n\pi x/\ell), \quad x \in [0, \ell], \quad n \in \mathbb{N}, \tag{2.4}$$

is an orthonormal basis of $L^2[0, \ell]$. Thus,

$$\tilde{u}(t) = \sum_{n=1}^{\infty} e^{-\lambda_n^2 (t - t_0)} \langle \tilde{u}(t_0), \varphi_n \rangle \varphi_n. \tag{2.5}$$

From this it is clear that the temperature at any time $t > t_0$ can be determined from the knowledge of the temperature at time $t = t_0$. In particular, for $\tau > t_0$,

$$\tilde{u}(\tau) = \sum_{n=1}^{\infty} e^{-\lambda_n^2 (\tau - t_0)} \langle \tilde{u}(t_0), \varphi_n \rangle \varphi_n. \tag{2.6}$$

This also implies that $\langle \tilde{u}(\tau), \varphi_n \rangle = e^{-\lambda_n^2 (\tau - t_0)} \langle \tilde{u}(t_0), \varphi_n \rangle$ for all $n \in \mathbb{N}$ so that from (2.5)

$$\tilde{u}(t_0) = \sum_{n=1}^{\infty} e^{\lambda_n^2 (\tau - t_0)} \langle \tilde{u}(\tau), \varphi_n \rangle \varphi_n. \tag{2.7}$$

Expressions (2.6) and (2.7) convey the following.

- The problem of finding $\tilde{u}(\tau) := u(\cdot, \tau)$ from the knowledge of $\tilde{u}(t_0) := u(\cdot, t_0)$ for $t_0 < \tau$ is a well posed problem. Indeed, if g is the approximation of $\tilde{u}(\tau)$ obtained by replacing $\tilde{u}(t_0)$ in the right hand side of (2.6) by an approximate data f, then we have

  $$\|\tilde{u}(\tau) - g\|^2 = \sum_{n=1}^{\infty} e^{-2\lambda_n^2 (\tau - t_0)} |\langle \tilde{u}(t_0) - f, \varphi_n \rangle|^2$$

  so that $\tilde{u}(\tau)$ depends continuously on $\tilde{u}(t_0)$.

- The BHCP of determining $\tilde{u}(t_0) := u(\cdot, t_0)$ from the knowledge of $\tilde{u}(\tau) := u(\cdot, \tau)$, $\tau > t_0$ is an ill-posed problem:

  (i) The problem has *no solution unless* $\tilde{u}(\tau)$ satisfies the *Piccard condition*

$$\sum_{n=1}^{\infty} e^{2\lambda_n^2(\tau - t_0)} |\langle \tilde{u}(\tau), \varphi_n \rangle|^2 < \infty. \tag{2.8}$$

  (ii) If $\tilde{u}_k^\delta(\tau) := \tilde{u}(\tau) + \delta\varphi_k$ for some $k \in \mathbb{N}$ and $\delta > 0$, and define

$$\tilde{u}_k^\delta(t_0) = \sum_{n=1}^{\infty} e^{\lambda_n^2(\tau - t_0)} \langle \tilde{u}_k^\delta(\tau), \varphi_n \rangle \varphi_n,$$

  then (2.7) gives

$$\tilde{u}_k^\delta(t_0) - \tilde{u}(t_0) = \delta\, e^{\lambda_k^2(\tau - t_0)} \varphi_k.$$

  Since $\lambda_n \to \infty$ as $n \to \infty$, the above observation shows that closeness of $\tilde{u}_k^\delta(\tau)$ to $\tilde{u}(\tau)$ *does not imply* closeness of $\tilde{u}_k^\delta(t_0)$ to $\tilde{u}(t_0)$.

3. BHCP as a Compact Operator Equation

For $t \in [0, \tau]$, let

$$A(t)\varphi := \sum_{n=1}^{\infty} e^{-\lambda_n^2(\tau - t)} \langle \varphi, \varphi_n \rangle \varphi_n, \quad \varphi \in L^2[0, \ell]. \tag{3.1}$$

Then it follows from (2.6) that $\tilde{u}(t_0) := u(\cdot, t_0)$ is a solution of the operator equation

$$A(t)\tilde{u}(t_0) = \tilde{u}(\tau), \quad f_\tau := u(\cdot, \tau), \tag{3.2}$$

whenever $\tilde{u}(\tau)$ satisfies (2.8). Note that $e^{-\lambda_n^2(\tau - t_0)} \to 0$ as $n \to \infty$. Thus, BHCP is a special case of the problem of solving an operator equation

$$Af = g, \tag{3.3}$$

with the operator $A : H \to H$ is defined on a Hilbert space H by

$$Af = \sum_{n=1}^{\infty} \sigma_n \langle f, v_n \rangle v_n, \quad f \in H, \tag{3.4}$$

where (σ_n) is a sequence of positive real numbers that converges to 0 and $\{v_n : n \in \mathbb{N}\}$ is an orthonormal basis of H. It can be easily seen that the operator A defined above is a bounded linear operator which is positive and self adjoint, and

$$\|A\| = \max_{n \in \mathbb{N}} \sigma_n.$$

Since $\sigma_n \to 0$ as $n \to \infty$, A is also a compact linear operator on H and (3.4) is its spectral representation (cf. [6]).

Note that for $f, g \in H$,

$$Af = g \iff \langle f, v_n \rangle = \frac{\langle g, v_n \rangle}{\sigma_n} \quad \forall n \in \mathbb{N}$$

so that (3.3) has a solution for $g \in H$ if and only if it satisfies the *Piccard condition*

$$\sum_{n=1}^{\infty} \frac{|\langle g, v_n \rangle|^2}{\sigma_n^2} < \infty, \tag{3.5}$$

and in that case the solution f of (3.3) is given by

$$f = \sum_{n=1}^{\infty} \frac{\langle g, v_n \rangle}{\sigma_n} v_n.$$

From the above observation we deduce the following.

Theorem 3.1. *Suppose* $g \in H$ *satisfies* (3.5). *For* $\delta > 0$*, let*

$$g_k = g + \delta v_k \quad \text{and} \quad f_k := \sum_{n=1}^{\infty} \frac{\langle g_k, v_n \rangle}{\sigma_n} v_n, \quad k \in \mathbb{N}.$$

Then,

$$Af_k = g_k \quad \text{and} \quad \|f - f_k\| \le \frac{\delta}{\sigma_k} \quad \forall k \in \mathbb{N}.$$

Since $\sigma_n \to 0$ as $n \to \infty$, the above theorem shows that even a small error in the data can result in large deviation in the solution.

Remark 3.2. Often, an operator equation of the form (3.3) is said to be *mildly ill-posed* or polynomially ill-posed if the sequence (σ_n) decays as in $(1/n^p)$ for some $p > 0$; otherwise, we say that (3.3) is *severely ill-posed*. The equation (3.3) is said to be *exponentially ill-posed* if (σ_n) decays as in $(1/\exp(\beta n^p))$ for some $\beta > 0$ and $p > 0$. Thus, we can infer that BHCP is an exponentially ill-posed problem, and hence it is severely ill-posed as well. For other examples of severely ill-posed problems one may refer Hohag [4] and Pereverzev and Schock [9].

4. A Stability Estimate

Now, the question is whether stability can be achieved by imposing some a priori conditions on certain unknown entities. Recall from (2.5) that for any $t \ge 0$,

$$\tilde{u}(t) = \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \langle u_0, \varphi_n \rangle \varphi_n, \quad u_0 := \tilde{u}(0).$$

In particular, we have

$$\tilde{u}(\tau) = \sum_{n=1}^{\infty} e^{-\lambda_n^2 \tau} \langle u_0, \varphi_n \rangle \varphi_n.$$

Hence, for $0 < t_0 < \tau$ and for $p > 1$, and $q = p/(p-1)$, the use of Hölder's inequality gives

$$\begin{aligned}
\|\tilde{u}(t_0)\|^2 &= \sum_{n=1}^{\infty} e^{-2\lambda_n^2 t_0} |\langle u_0, \varphi_n \rangle|^2 \\
&= \sum_{n=1}^{\infty} e^{-2\lambda_n^2 t_0} |\langle u_0, \varphi_n \rangle|^{2/p} |\langle u_0, \varphi_n \rangle|^{2/q} \\
&= \left(\sum_{n=1}^{\infty} e^{-2p\lambda_n^2 t_0} |\langle u_0, \varphi_n \rangle|^2 \right)^{1/p} \left(\sum_{n=1}^{\infty} |\langle u_0, \varphi_n \rangle|^2 \right)^{1/q} \\
&= \left(\sum_{n=1}^{\infty} e^{-2p\lambda_n^2 t_0} |\langle u_0, \varphi_n \rangle|^2 \right)^{1/p} \|u_0\|^{2/q}
\end{aligned}$$

Since $\tilde{u}(\tau) = \sum_{n=1}^{\infty} e^{-\lambda_n^2 \tau} \langle u_0, \varphi_n \rangle \varphi_n$, taking $p = \tau/t_0$ we get

$$\|\tilde{u}(t_0)\| \le \|\tilde{u}(\tau)\|^{t_0/\tau} \|u_0\|^{1-t_0/\tau}. \tag{4.1}$$

Thus, if $\|u_0\| \le \rho$, then

$$\|\tilde{u}(t_0)\| \le \rho^{1-t_0/\tau} \|\tilde{u}(\tau)\|^{t_0/\tau} = \rho \left(\frac{\|\tilde{u}(\tau)\|}{\rho} \right)^{t_0/\tau}. \tag{4.2}$$

Recall from (2.5) that

$$u_0 = \sum_{n=1}^{\infty} e^{\lambda_n^2 t_0} \langle \tilde{u}(t_0), \varphi_n \rangle \varphi_n.$$

Therefore,

$$\|u_0\| \le \rho \iff \sum_{n=1}^{\infty} e^{2\lambda_n^2 t_0} |\langle \tilde{u}(t_0), \varphi_n \rangle|^2 \le \rho^2.$$

Thus, the requirement on u_0 that $\|u_0\| \le \rho$ is same as requiring the unknown solution $\tilde{u}(t_0)$ to belong to the source set

$$M_\rho := \left\{ f \in L^2[0,\ell] : \sum_{n=1}^{\infty} e^{2\lambda_n^2 t_0} |\langle f, \varphi_n \rangle|^2 \le \rho^2 \right\}. \tag{4.3}$$

Writing

$$B_0\varphi := \sum_{n=1}^{\infty} e^{-\lambda_n^2 t_0} \langle \varphi, \varphi_n \rangle \varphi_n, \quad \varphi \in L^2[0,\ell], \tag{4.4}$$

it follows that B_0 is a compact operator on $L^2[0,\ell]$, and $\tilde{u}(t_0)$ is in the range of B_0. Note that

$$f = B_0\varphi \iff \varphi = \sum_{n=1}^{\infty} e^{\lambda_n^2 t_0} \langle f, \varphi_n \rangle \varphi_n$$

so that the source set M_ρ takes the form

$$M_\rho = \{ f := B_0\varphi : \|\varphi\| \le \rho \}.$$

In order to have stability of the solution $\tilde{u}(t_0)$ under perturbation in the data, let us look at the situation when we restrict the operator $A_0 := A(0)$ to the source set M_ρ.

Theorem 4.1. *Suppose g is a noisy data available in place of $\tilde{u}(\tau)$. Let us assume that both $\tilde{u}(\tau)$ and g are in the image of M_ρ under $A_0 := A(0)$. Then*

$$\|\tilde{u}(t_0) - f\| \le 2\rho \left(\frac{\|\tilde{u}(\tau) - g\|}{2\rho} \right)^{t_0/\tau},$$

where f satisfies $A_0 f = g$.

Proof. By hypothesis, there exist $\varphi, \tilde{\varphi}$ satisfying $\|\varphi\| \le \rho$, $\|\tilde{\varphi}\| \le \rho$ such that

$$\tilde{u}(t_0) = B_0\varphi, \quad f = B_0\tilde{\varphi},$$

where B_0 is as in (4.4). Thus,

$$\tilde{u}(t_0) - f = B_0(\varphi - \tilde{\varphi}) = \sum_{n=1}^{\infty} e^{-\lambda_n^2 t_0} \langle \varphi - \tilde{\varphi}, \varphi_n \rangle \varphi_n.$$

Now, following the computations involved in arriving at (4.1), we obtain

$$\begin{aligned}
\|\tilde{u}(t_0) - f\| &\le \|\tilde{u}(\tau) - g\|^{t_0/\tau} \|\varphi - \tilde{\varphi}\|^{1-t_0/\tau} \\
&\le \|\tilde{u}(\tau) - g\|^{t_0/\tau} (2\rho)^{1-t_0/\tau} \\
&= 2\rho \left(\frac{\|\tilde{u}(\tau) - g\|}{2\rho} \right)^{t_0/\tau}.
\end{aligned}$$

This completes the proof. □

The above theorem shows that if the solutions are allowed to vary only in the source set M_ρ, then small error in the data $\tilde{u}(\tau) := u(\cdot, \tau)$ does not lead to large deviation in the solution $\tilde{u}(t_0) := u(\cdot, t_0)$ for $t_0 < \tau$. However, in order to obtain the above stability estimate, it is assumed that the unknown solution satisfies a too severe restriction. Moreover, the stability estimate (4.2) does not help us for the case $t_0 = 0$. We consider this issue in the next section.

5. Stability Estimates Under General Source Conditions

Recall that in the case of BHCP, the assumption that an unknown function is to belongs to the source set

$$M_\rho := \left\{ \varphi \in L^2[0,\ell] : \sum_{n=1}^{\infty} e^{2\lambda_n^2 t_0} |\langle \varphi, \varphi_n \rangle|^2 \le \rho^2 \right\}$$

in (4.3) is a severe restriction on the solution. More over, this source set is not suitable for obtaining stability estimate for the problem of determining temperature at time $t = 0$ from the knowledge of the temperature at time $t = \tau > 0$. A less restrictive assumption on the unknown solution would be that it belongs to the set

$$\tilde{M}_\rho := \left\{ \varphi \in L^2[0,\ell] : \sum_{n=1}^{\infty} \lambda_n^2 |\langle \varphi, \varphi_n \rangle|^2 \le \rho^2 \right\}. \qquad (5.1)$$

Now the question is whether it is possible to get a stability estimate for $\tilde{u}(t_0)$ for $0 \le t_0 < \tau$ by assuming that it belongs to $\tilde{M}_\rho$. Taking

$$\sigma_n := e^{-\lambda_n^2(\tau - t_0)}, \quad n \in \mathbb{N},$$

the set M_ρ takes the form

$$M_\rho = \left\{ \varphi \in L^2[0,\ell] : \sum_{n=1}^{\infty} \frac{|\langle \varphi, \varphi_n \rangle|^2}{\sigma_n^{2\mu}} \le \rho^2 \right\}, \; \mu := \frac{t_0}{\tau - t_0}.$$

We may also observe that λ_n can be written in terms of σ_n as

$$\lambda_n = \left[\frac{1}{\tau - t_0} \ln\left(\frac{1}{\sigma_n}\right) \right]^{1/2}, \quad n \in \mathbb{N}.$$

Thus the set $\tilde{M}_\rho$ in (5.1) takes the form

$$\tilde{M}_\rho = \left\{ \varphi \in L^2[0,\ell] : \sum_{n=1}^{\infty} \frac{|\langle \varphi, \varphi_n \rangle|^2}{h(\sigma_n)^2} \le \rho^2 \right\}$$

with

$$h(\sigma_n) := \left[\frac{1}{\tau - t_0} \ln\left(\frac{1}{\sigma_n}\right) \right]^{-1/2}, \quad n \in \mathbb{N}.$$

Note that $(h(\sigma_n))$ converges to 0 more slowly than (σ_n^μ).

Now, let us consider the situation in the general context of the operator A considered in Section 3. Corresponding to the spectral representation (3.4) of A, let

$$M_{h,\rho} := \left\{ f \in H : \sum_{n=1}^{\infty} \frac{|\langle f, v_n \rangle|^2}{h(\sigma_n)^2} \le \rho^2 \right\}, \qquad (5.2)$$

where $h(\sigma_n) > 0$ for all $n \in \mathbb{N}$ and $h(\sigma_n) \to 0$ as $n \to \infty$. It can be seen that

$$X_h := \left\{ \varphi \in H : \sum_{n=1}^{\infty} \frac{|\langle \varphi, v_n \rangle|^2}{h(\sigma_n)^2} < \infty \right\}$$

is a Hilbert space with respect to the inner product

$$\langle \varphi, \psi \rangle_h := \sum_{n=1}^{\infty} \frac{\langle \varphi, v_n \rangle \langle v_n, \psi \rangle}{h(\sigma_n)^2}, \quad \varphi, \phi \in H,$$

so that the set $M_{h,\rho}$ in (5.2) is the closed ball in X_h with centre 0 and radius ρ.

5.1 *A special case*

We observe that σ_n can be written in the form

$$\sigma_n = e^{-\beta\mu_n}, \quad n \in \mathbb{N},$$

with $\beta > 0$ and

$$\mu_n := \frac{1}{\beta} \ln\left(\frac{1}{\sigma_n}\right), \quad n \in \mathbb{N}.$$

Then, taking

$$h(\sigma_n) := \left[\frac{1}{\beta} \ln\left(\frac{1}{\sigma_n}\right) \right]^{-p}, \quad n \in \mathbb{N},$$

for some $p > 0$, the set $M_{h,\rho}$ take the form

$$M_{h,\rho} := \left\{ f \in H : \sum_{n=1}^{\infty} \mu_n^{2p} |\langle f, v_n \rangle|^2 \le \rho^2 \right\}.$$

The following theorem is proved in [8] in a more general context. We give its proof here, as in the present context, the proof involves less use of advanced topics in operator theory.

Theorem 5.1. *Suppose $h : (0, a] \to (0, \infty)$ with $a \ge \|A\|$ is continuous, strictly monotonically increasing and $\lim_{\lambda \to 0} \varphi(\lambda) = 0$ such that the function $\lambda \mapsto \lambda[h^{-1}(\sqrt{\lambda})]^2$ is convex. Then, for every $f \in M_{h,\rho}$,*

$$\|f\| \le \rho \psi^{-1}\left(\frac{\|Af\|}{\rho}\right),$$

where $\psi(\lambda) := \lambda\, h^{-1}(\lambda)$ for $0 < \lambda \le a$. In particular, the quantity

$$\omega(M_{h,\rho}, \delta) := \sup\{\|f\| : f \in M_{h,\rho},\ \|Af\| \le \delta\}$$

satisfies

$$\omega(M_{h,\rho}, \delta) \le \rho \psi^{-1}\left(\frac{\delta}{\rho}\right). \qquad (5.3)$$

Proof. We observe that $f \in M_{h,\rho}$ if and only if there exists $w \in H$ such that

$$f = \sum_{n=1}^{\infty} h(\sigma_n)\langle w, v_n\rangle v_n, \quad \|w\| \le \rho.$$

Now, suppose that $f \in M_{h,\rho}$ and $w \in H$ is such that $f = \sum_{n=1}^{\infty} h(\sigma_n)\langle w, v_n\rangle v_n$ with $\|w\| \le \rho$. Note that

$$\frac{\|f\|^2}{\|w\|^2} = \sum_{n=1}^{\infty}[h(\sigma_n)]^2\omega_n, \quad \frac{\|Af\|^2}{\|w\|^2} = \sum_{n=1}^{\infty}[\sigma_n h(\sigma_n)]^2\omega_n$$

where $\omega_n := |\langle w, v_n\rangle|^2/\|w\|^2$. Since $\sum_{n=1}^{\infty}\omega_n = 1$ and $\tilde{\psi} : \lambda \mapsto \lambda[h^{-1}(\sqrt{\lambda})]^2$ is convex, Jensen's inequality implies that

$$\tilde{\psi}\left(\frac{\|f\|^2}{\|w\|^2}\right) = \tilde{\psi}\left(\sum_{n=1}^{\infty}[h(\sigma_n)]^2\omega_n\right) \le \sum_{n=1}^{\infty}\tilde{\psi}([h(\sigma_n)]^2)\omega_n.$$

Since $\tilde{\psi}(\xi^2) = [\psi(\xi)]^2$, it follows that

$$\begin{aligned}\left[\psi\left(\frac{\|f\|}{\|w\|}\right)\right]^2 &\le \sum_{n=1}^{\infty}\{\psi[h(\sigma_n)]\}^2\omega_n \\ &= \sum_{n=1}^{\infty}\sigma_n^2[h(\sigma_n)]^2\omega_n = \frac{\|Af\|^2}{\|w\|^2}.\end{aligned}$$

Thus,

$$\frac{\|f\|}{\|w\|}h^{-1}\left(\frac{\|f\|}{\|w\|}\right) = \psi\left(\frac{\|f\|}{\|w\|}\right) \le \frac{\|Af\|}{\|w\|}$$

so that

$$h^{-1}\left(\frac{\|f\|}{\|w\|}\right) \le \frac{\|Af\|}{\|f\|}.$$

Since h^{-1} is monotonically increasing, we have

$$\begin{aligned}\psi\left(\frac{\|f\|}{\rho}\right) &= \frac{\|f\|}{\rho}h^{-1}\left(\frac{\|f\|}{\rho}\right) \le \frac{\|f\|}{\rho}h^{-1}\left(\frac{\|f\|}{\|w\|}\right) \\ &\le \frac{\|Af\|}{\rho}.\end{aligned}$$

Thus, $\|f\| \le \rho\psi^{-1}(\|Af\|/\rho)$. Proof of the particular case is obvious. □

Corollary 5.2. *Suppose $f, \tilde{f} \in M_{h,\rho}$ are such that $\|Af - A\tilde{f}\| \le \delta$. Then*

$$\|f - \tilde{f}\| \le 2\rho\psi^{-1}\left(\frac{\delta}{2\rho}\right).$$

Proof. By hypothesis, we have

$$\frac{f-\tilde{f}}{2} \in M_{h,\rho}, \quad \|A\left(\frac{f-\tilde{f}}{2}\right)\| \le \frac{\delta}{2}.$$

Hence, in view of the estimate (5.3), we have

$$\|f - \tilde{f}\| \le 2\omega\left(M_{h,\rho}, \frac{\delta}{2}\right) \le 2\rho\psi^{-1}\left(\frac{\delta}{2\rho}\right)$$

completing the proof. □

Remark 5.3. It is known that the estimate given in (5.3) for the quantity $\omega(M_{h,\rho}, \delta)$ is order optimal for the source set (See, Tautenhahn [10]).

5.2 *Recourse to BHCP*

In the case of BHCP, we have

$$\sigma_n = e^{-\lambda_n^2(\tau - t_0)}, \quad \lambda_n := \frac{cn\pi}{\ell}$$

so that we may take

$$\beta := \frac{(\tau - t_0)\pi^2c^2}{\ell^2}, \quad \mu_n := n^2.$$

With these choices, and taking

$$h(\sigma_n) := \left[\frac{1}{\beta}\ln\left(\frac{1}{\sigma_n}\right)\right]^{-p}, \quad n \in \mathbb{N},$$

for $p > 0$, we have

$$X_h := \left\{f \in H : \sum_{n=1}^{\infty} n^{4p}|\langle f, v_n\rangle|^2 < \infty\right\}$$

and

$$M_{h,\rho} := \left\{f \in H : \sum_{n=1}^{\infty} n^{4p}|\langle f, v_n\rangle|^2 \le \rho^2\right\}.$$

Note that X_h is a *periodic Sobolev space*.

Thus, in this case we have

$$h(\lambda) = \left(\frac{1}{\beta}\ln\frac{1}{\lambda}\right)^{-p}, \quad 0 < \lambda \le \|A\|.$$

We observe that

$$h(\lambda) = s \iff \lambda = e^{-\beta/s^{1/p}}.$$

Hence,

$$\psi(s) = sh^{-1}(s) = se^{-\beta/s^{1/p}}.$$

Now,

$$\psi(s) = \lambda \iff \frac{1}{\lambda} = \frac{1}{s} e^{\beta/s^{1/p}} \iff \ln\frac{1}{\lambda} = \ln\frac{1}{s} + \frac{\beta}{s^{1/p}}.$$

Hence,

$$s = \left[\frac{\beta}{\ln\frac{1}{\lambda} - \ln\frac{1}{s}}\right]^p = \left(\frac{\beta}{\ln\frac{1}{\lambda}}\right)^p \left[1 - \frac{\ln\frac{1}{s}}{\ln\frac{1}{\lambda}}\right]^{-p}.$$

Note that

$$\frac{\ln\frac{1}{s}}{\ln\frac{1}{\lambda}} = \frac{\ln\frac{1}{s}}{\ln\frac{1}{s} + \beta/s^{1/p}} \to 0 \quad \text{as} \quad s \to 0.$$

Hence

$$\psi^{-1}(\lambda) = \beta^p \left[\ln\frac{1}{\lambda}\right]^{-p} [1 + o(1)].$$

Thus,

$$\omega(\delta, M_{h,\rho}) \le \rho\psi^{-1}(\delta/\rho) = \rho\beta^p \left[\ln\left(\frac{\rho}{\delta}\right)\right]^{-p} [1 + o(1)]. \tag{5.4}$$

6. A Regularization

From the last sections, it is clear that if we are having only a noisy data $\tilde{g}$ in place of g with $\|g - \tilde{g}\| \le \delta$, then it is not advisable to take $\tilde{f} = \sum_{n=1}^{\infty} (\langle \tilde{g}, v_n\rangle/\sigma_n)\, v_n$ as an approximation for the solution $f = \sum_{n=1}^{\infty} (\langle g, v_n\rangle/\sigma_n)\, v_n$ for the operator equation (3.3), unless both g and $\tilde{g}$ satisfy certain additional smoothness assumptions. Thus, if one does not have a priori knowledge on the perturbed data, then one should have some ways of computing approximate solutions. A regularization procedure is used for such situations. There are many regularization methods available for ill-posed equations (See, e.g. Engl, Hanke and Neubauer [2] and the references therein). But, in the case of backward heat conduction problem, we have full knowledge of the singular values and singular vectors of the associated forward operator. Therefore, truncated singular value decomposition (TSVD) would be an easy to apply regularization method for this case. So, we consider TSVD in the following.

Suppose $\tilde{g}$ is the noisy data available in place of $g \in R(A)$ with $\|g - \tilde{g}\| \le \delta$. Then $f = \sum_{n=1}^{\infty} (\langle g, v_n\rangle/\sigma_n) v_n$ satisfies the equation $Af = g$. For $k \in \mathbb{N}$, let

$$f_k := \sum_{n=1}^{k} \frac{\langle g, v_n\rangle}{\sigma_n} v_n, \quad \tilde{f}_k := \sum_{n=1}^{k} \frac{\langle \tilde{g}, v_n\rangle}{\sigma_n} v_n.$$

Theorem 6.1. *We assume that $\sigma_{n+1} \le \sigma_n$ for all $n \in \mathbb{N}$. Then*

$$\|f - \tilde{f}_k\| \le \sqrt{\varepsilon_k^2 + \frac{\delta^2}{\sigma_k^2}}, \tag{6.1}$$

where

$$\varepsilon_k^2 := \sum_{n=k+1}^{\infty} \frac{|\langle g, v_n\rangle|^2}{\sigma_n^2} \to 0 \quad \text{as} \quad k \to \infty.$$

Further, if

$$\ell_1 := \min\left\{k : \varepsilon_k \le \frac{\delta}{\sigma_k}\right\}, \quad \ell_2 := \max\left\{k : \frac{\delta}{\sigma_k} \le \varepsilon_k\right\},$$

and $\ell := \min\{\ell_1, \ell_2\}$, then

$$\|f - \tilde{f}_\ell\| \le \sqrt{\varepsilon_\ell^2 + \frac{\delta^2}{\sigma_\ell^2}} \le \sqrt{\varepsilon_k^2 + \frac{\delta^2}{\sigma_k^2}} \quad \forall k \in \mathbb{N}. \tag{6.2}$$

Proof. Note that

$$\|f_k - \tilde{f}_k\|^2 = \sum_{n=1}^{k} \frac{|\langle g - \tilde{g}, v_n\rangle|^2}{\sigma_n^2} \le \frac{1}{\sigma_k^2} \sum_{n=1}^{k} |\langle g - \tilde{g}, v_n\rangle|^2 \le \frac{\|g - \tilde{g}\|^2}{\sigma_k^2}.$$

Hence, if $\tilde{g}$ is close to g, then $\tilde{f}_k$ is close to f_k. Thus, $\tilde{f}_k$ can be thought of as a *regularized solution* of (3.3). We may also observe that

$$f - \tilde{f}_k = \sum_{n=1}^{\infty} \frac{\langle g, v_n\rangle}{\sigma_n} v_n - \sum_{n=1}^{k} \frac{\langle \tilde{g}, v_n\rangle}{\sigma_n} v_n = \sum_{n=k+1}^{\infty} \frac{\langle g, v_n\rangle}{\sigma_n} v_n + \sum_{n=1}^{k} \frac{\langle g - \tilde{g}, v_n\rangle}{\sigma_n} v_n.$$

Hence,

$$\|f - \tilde{f}_k\|^2 = \sum_{n=k+1}^{\infty} \frac{|\langle g, v_n\rangle|^2}{\sigma_n^2} + \sum_{n=1}^{k} \frac{|\langle g - \tilde{g}, v_n\rangle|^2}{\sigma_n^2} \le \varepsilon_k^2 + \frac{\delta^2}{\sigma_k^2},$$

where

$$\varepsilon_k^2 := \sum_{n=k+1}^{\infty} \frac{|\langle g, v_n\rangle|^2}{\sigma_n^2} \to 0 \quad \text{as} \quad k \to \infty.$$

Thus,

$$\|f - \tilde{f}_k\| \le \sqrt{\varepsilon_k^2 + \frac{\delta^2}{\sigma_k^2}}.$$

To obtain (6.2), first we note that (δ/σ_k) is an increasing sequence and (ε_k) is a decreasing sequence. Thus, $\varepsilon_k \leq \delta/\sigma_k$ for all large enough k. Assuming $\delta/\sigma_1 < \varepsilon_1$, which is true for small enough δ, it follows that there exists $\ell \in \mathbb{N}$ such that

$$\varepsilon_\ell^2 + \frac{\delta^2}{\sigma_\ell^2} \leq \varepsilon_k^2 + \frac{\delta^2}{\sigma_k^2} \quad \forall k \neq \ell.$$

In fact, it can be easily seen that

$$\ell := \min\{\ell_1, \ell_2\},$$

where

$$\ell_1 := \min\left\{k : \varepsilon_k \leq \frac{\delta}{\sigma_k}\right\}, \quad \ell_2 := \max\left\{k : \frac{\delta}{\sigma_k} \leq \varepsilon_k\right\}.$$

Thus, we obtain the estimate

$$\|f - \tilde{f}_\ell\| \leq \sqrt{\varepsilon_\ell^2 + \frac{\delta^2}{\sigma_\ell^2}} \leq \sqrt{\varepsilon_k^2 + \frac{\delta^2}{\sigma_k^2}} \quad \forall k \in \mathbb{N}.$$

□

Remark 6.2. The estimate (6.2) is best possible by the above approach. This can be seen by taking $\tilde{g} := f + \delta v_k$, for in this case, (6.1) takes the form

$$\|f - \tilde{f}_k\|^2 = \varepsilon_k^2 + \frac{\delta^2}{\sigma_k^2}.$$

Thus, the TSVD-regularized solution $\tilde{f}_\ell$ can be taken as a stable approximate solution of the given operator equation.

Note that, ε_ℓ in the estimate (6.2) is, in general, unknown. However, if we know additional smoothness of f, then it is possible to determine the value of ℓ from the knowledge of speed of convergence of the sequence (σ_n) of singular values. For instance, suppose that the solution f belongs to the source set $M_{h,\rho}$ in (5.2), that is,

$$M_{h,\rho} := \left\{ f \in H : \sum_{n=1}^{\infty} \frac{|\langle f, v_n\rangle|^2}{h(\sigma_n)^2} \leq \rho^2 \right\}$$

where $h : (0, a] \to (0, \infty)$ with $a \geq \|A\|$ is continuous, strictly monotonically increasing and $\lim_{\lambda\to 0} h(\lambda) = 0$. Then there exists $w \in H$ be such that $f = \sum_{n=1}^{\infty} h(\sigma_n)\langle w, v_n\rangle v_n$ with $\|w\| \leq \rho$. Therefore,

$$\langle g, v_n\rangle = \langle Af, v_n\rangle = \langle f, Av_n\rangle = \sigma_n\langle h(\sigma_n)w, v_n\rangle.$$

Hence, using the fact that $\sigma_{n+1} \leq \sigma_n$ for all $n \in \mathbb{N}$, we have

$$\varepsilon_k^2 = \sum_{n=k+1}^{\infty} [h(\sigma_n)]^2 |\langle w, v_n\rangle|^2 \leq \rho^2 [h(\sigma_{k+1})]^2.$$

and (6.1) gives

$$\|f - \tilde{f}_k\| \leq \sqrt{\rho^2 [h(\sigma_{k+1})]^2 + \frac{\delta^2}{\sigma_k^2}} \tag{6.3}$$

for all $k \in \mathbb{N}$.

Proposition 6.3. *For $k \in \mathbb{N}$ and $\delta > 0$, let*

$$E(\delta, k) := \sqrt{\rho^2 [h(\sigma_{k+1})]^2 + \delta^2/\sigma_k^2}.$$

For $\delta > 0$, if $k \in \mathbb{N}$ satisfies

$$\sigma_{k+1} \leq h^{-1}[\psi^{-1}(\delta/\rho)] \leq \sigma_k,$$

then

$$E(\delta, k) \leq \sqrt{2}\rho\psi^{-1}(\delta/\rho).$$

If $\delta := \rho\sigma_k h(\sigma_k)$, then we also have

$$E(\delta, k) \geq \rho\psi^{-1}(\delta/\rho).$$

Proof. Since $\psi(\lambda) := \lambda h^{-1}(\lambda)$ and $h(\cdot)$ is monotonically increasing, the condition

$$\sigma_{k+1} \leq h^{-1}[\psi^{-1}(\delta/\rho)] \leq \sigma_k$$

implies that

$$\rho h(\sigma_{k+1}) \leq \rho\psi^{-1}(\delta/\rho)$$

and

$$\frac{\delta}{\sigma_k} \leq \frac{\delta}{h^{-1}[\psi^{-1}(\delta/\rho)]} = \rho\psi^{-1}(\delta/\rho).$$

Thus,

$$E(\delta, k) \leq \sqrt{2}\,\rho\psi^{-1}(\delta/\rho).$$

To see the last part of the proposition, we observe that

$$\delta = \rho\sigma_k h(\sigma_k) \iff \delta = \rho\psi(h(\sigma_k)) \iff h(\sigma_k) = \psi^{-1}(\delta/\rho).$$

Hence, if $\delta := \rho\sigma_k h(\sigma_k)$, then

$$\rho\psi^{-1}(\delta/\rho) = \rho h(\sigma_k) = \frac{\delta}{\sigma_k} \leq E(\delta, k).$$

This completes the proof. □

Theorem 6.4. *Suppose $f \in M_{h,\rho}$. Let $k \in \mathbb{N}$ be such that*

$$\sigma_{k+1} \le h^{-1}[\psi^{-1}(\delta/\rho)] \le \sigma_k.$$

Then

$$\|f - \tilde{f}_k\| \le \sqrt{2}\rho\psi^{-1}(\delta/\rho). \tag{6.4}$$

Further, the above estimate is order optimal for the error bound in (6.3).

Proof. In view of the relation (6.3) and Proposition 6.3, we obtain the estimate in (6.4). To see this estimate is order optimal, consider $\tilde{g} := g + \delta v_k$. Then we have

$$\|f - \tilde{f}_k\| = \sqrt{\varepsilon_k^2 + \frac{\delta^2}{\sigma_k^2}} = E(\delta, k).$$

so that for the data $\tilde{g} := g + \delta v_k$ with $\delta = \rho\sigma_k h(\sigma_k)$, Proposition 6.3 shows that

$$\rho\psi^{-1}(\delta/\rho) \le \|f - \tilde{f}_k\| \le \sqrt{2}\,\rho\psi^{-1}(\delta/\rho).$$

This completes the proof of the theorem. □

7. Concluding Remarks

Stability estimate for BHCP has been derived under a general source condition which is applicable to recover from the temperature at time $t = \tau > t_0$, not only the temperature at time $t_0 > 0$, but also the temperature at time $t_0 = 0$. For obtaining stable approximate solutions, truncated singular value decomposition (TSVD) is suggested as a regularization method. It is shown that the derived error estimate for TSVD is order optimal under the general source condition.

References

[1] K. A. Ames, G. W. Clark, J. F. Epperson and S. F. Oppenheimer, A comparison of regularizations for an ill-posed problem, *Math. Comp.*, **67** No. 224 (1998) 1451–1471.

[2] H. W. Engl, M. Hanke and A. Nuebauer, Regularization of inverse problems, Kluwer, Dordrect (1996).

[3] J. Hadamard, Lectures on Cauchy's problem in linear partial differential equations, *Yale Univ. Press*, New Haven (1923).

[4] T. Hohage, Regularization of exponentially ill-posed problems, *Numer. Funct. Anal. and Optimiz.*, **21** (2000) 439–464.

[5] A. Kirsch, An introduction to the mathematical theory of inverse problems, Springer-Verlag (1996).

[6] M. T. Nair, Functional analysis: A first course, Prentice-Hall of India, New Delhi (2002).

[7] M. T. Nair, Linear operator equations: Approximation and regularization, World Scientific, Singapore (2009).

[8] M. T. Nair, E. Schock and U. Tautenhahn, Morozov's discrepancy principle under general source conditions, *Zeit. Anal. und ihre Anwen.*, **22** (2003) 199–214.

[9] S. Pereverzev and E. Schock, Morozov's discrepancy principle for Tikhonov regularization of severely ill-posed problems, *Numer. Funct. Anal. and Optimiz.*, **21** (2000) 901–920.

[10] U. Tautenhahn, Optimality for ill-posed problems under general source conditions, *Numer. Funct. Anal. and Optimiz.*, **19** (1998) 377–398.

A 125th-Birthday Gift for Srinivasa Ramanujan

(Aziz S. Inan, Ph.D., Electrical Engineering, University of Portland, Portland, Oregon)

(22 December 2012; updated 29 December 2012)

Indian mathematician and genius Srinivasa Ramanujan was born on 22 December 1887 (expressed as 22-12-1887 or simply, 22121887) in Erode, Madras, India and died on 26 April 1920 (26041920) at 32 in Kumbakonam, Madras, India [1]. Saturday, 22 December 2012 marks his 125th birthday. On this special occasion, I constructed the following numerical brainteasers involving some numbers connected to his life as a birthday gift for him:

1. Ramanujan's single full birth date expressed as 22121887 is very special. How? 22121887 is the 1396426th prime number [2] where $1396426 = 2 \times 31 \times 101 \times 223$.

These four prime factors add up to 357 and coincidentally, Ramanujan's birthday (22 December) corresponds to the 357th day of 2012 (and every other leap year)! Wow!

2. Ramanujan's birthday 22121887 can be constructed from the first ten digits of Euler's number e without using any arithmetic. (Note that the first ten digits of Euler's number are $e = 2.718281828$.) Can you figure out how? (Hint: Since you need only two 8 digits, place each pair of the four 8 digits on the top of each other.)
3. Ramanujan's birthday 22 December is also connected to the first three digits of pi. How? Add the squares of numbers 22 and 12 (which represent 22 December) and take half of the result. What comes out? (314!).
4. There is a numerical connection between Ramanujan's full name and his death day. How? If numbers 1 to 26 are assigned to the letters of the English alphabet as $A = 1$, $B = 2$, $C = 3$, etc., the numbers assigned to the letters of Srinivasa and Ramanujan each add up to 112 and 93. Interestingly enough, one fourth of the product of numbers 112 and 93 (representing Srinivasa Ramanujan) yield 2604 representing 26 April!
5. Interestingly enough, the first 21 and 22 digits of Euler's number e each add up to 93 corresponding to the sum of the numbers assigned to the letters of "Ramanujan" where 22 and reverse of 21 put side by side as 2212 represent Ramanujan's birthday, 22 December! In addition, note that twice the sum of the first twelve digits of e yields 112, corresponding to "Srinivasa."
6. Numbers 112 and 93 (representing Srinivasa Ramanujan) also mutually share an interesting numerical property: The sum of their squares is 21193 where the reverse of the leftmost three digits is 112 and the rightmost two digits equals 93!
7. Ramanujan's name is also "cryptically" connected to the Hardy-Ramanujan number 1729 [3, 4]. How? The prime factors of 1729 are 7, 13 and 19 and interestingly enough, these three prime factors add up to 39, the reverse of which is 93, corresponding to Ramanujan.
8. In addition, Ramanujan's death year can simply be produced using the prime factors of 1729. How? Primes 7 and 13 add up to 20, and 19 and 20 put side by side yield 1920, the year Ramanujan died.
9. Also, if the rightmost two digits of 1729 are switched, twice 1792 equals to the product of the digits of 22121887. In addition, 1792 times the special number 12345 yields one of Ramanujan's future birthdays 22122240 that is divisible by his death year 1920!
10. Ramanujan's 125th birthday is special not only because 125 years represent one eighth of a millennium but also for other reasons. Why? First, 22 December 2012 is Ramanujan's 125th birthday and coincidentally, the 93rd anniversary of Ramanujan's death day is to occur exactly 125 days after this day! Amazing!
11. Second, the 125th prime number 691 and its reverse 196 add up to 887 which coincide with the rightmost three digits of Ramanujan's birth year, 1887.
12. Next, using basic arithmetic, Ramanujan's birthday number 125 can be produced from the digits of the Hardy-Ramanujan number 1729 as $7 \times 2 \times 9 - 1 = 125$!
13. Also, half of the sum of the reverses of numbers 112 and 93 (representing Srinivasa Ramanujan) yields 125!
14. In addition, the sum of the digits of Ramanujan's 125th birthday expressed as 22122012 is 12 (the month number of his birthday), the sum of the squares of the digits is 22 (the day number of his birthday), and the product of its nonzero digits is 32 (signifying his death age).
15. Ramanujan's last birthday in this (21st) century (his 213th) expressed as 22122100 is unique because it is divisible by the first seven prime numbers except 3, since $22122100 = 2 \times 2 \times 5 \times 5 \times 7 \times 11 \times 13 \times 13 \times 17$. Additionally, if 22122100 is split as 22, 12, 21, and 00, the sum of these numbers yield 55, which also equals to the sum of the prime factors of 22122100.
16. Lastly, Ramanujan's palindrome birthday 22 December is to occur in the 22nd century on 22122122!

Thanks for your love for numbers and unprecedented contributions to mathematics Ramanujan, and have a happy 125th birthday!

[1] `http://en.wikipedia.org/wiki/Srinivasa_Ramanujan`

[2] `http://www.bigprimes.net/` (`http://www.bigprimes.net/cruncher/22121887/`)

[3] `http://en.wikipedia.org/wiki/1729_%28number%29`

[4] A. Inan, "Mystery number," + *Plus Magazine* puzzles, December 6, 2012 `http://plus.maths.org/content/mystery-number`

21st Mathematics Training and Talent Search Programme

(Funded by National Board for Higher Mathematics)

Website: `http://www.mtts.org.in/`

MTTS 2013

Aims: The aim of the programme is to expose bright young students to the excitement of doing mathematics, to promote independent mathematical thinking and to prepare them for higher aspects of mathematics.

Academic Programmes: The programme will be at three levels: Level O, Level I and Level II. In Level O there will be courses in Linear Algebra, Analysis and Number Theory/Combinatorics. In Levels I and II there will be courses on Algebra, Analysis and Topology. There will be seminars by students at all Levels.

The faculty will be active mathematicians with a commitment to teaching and from various leading institutions. The aim of the instructions is not to give routine lectures and presentation of theorem-proofs but to stimulate the participants to think and discover mathematical results.

Eligibility:

- **Level O:** Second year undergraduate students with Mathematics as one of their subjects.
- **Level I:** Final year undergraduate students with Mathematics as one of their subjects.
- **Level II:** First year postgraduate students with Mathematics as one of their subjects. Participants of the previous year Level I Programme.

Venues & Duration: MTTS 2013 will be held at three different places:

- Sardar Vallabhbhai National Institute of Technology (SVNIT), Surat. (Main centre) Duration: May 20 – June 15, 2013.
- Regional Institute of Education (RIE), Mysore. Duration: May 20 –- June 15, 2013.
- Panjab University, Chandigarh. Duration: May 27 – June 22, 2013.

How to Apply: Details and application forms can be had from the Head, Department of Mathematics of your Institution. The application forms can also be downloaded from the MTTS website:

`http://www.mtts.org.in/`

Students can also apply online by logging in to this site from **December 7, 2012**.

The completed application form should reach the programme director latest by **January 23, 2013**. Please go through the MTTS-FAQ at the website before sending any queries.

Programme Director:

Prof. S. Kumaresan
Director, MTTS Programme
Department of Mathematics and Statistics
University of Hyderabad
P.O. Central University
Hyderabad 500 046

Tel: (040) 6679 4059 (O)
E-mail: `kumaresa@gmail.com`
`mttsprogramme@gmail.com`

Selection: The selection will be purely on merit, based on consistently good academic record and the recommendation letter from a mathematics professor closely acquainted with the candidate.

Only selected candidates will be informed of their selection by the 2nd week of February 2013. The list of selected candidates will be posted on the MTTS website in the 2nd week of February 2013.

Candidates selected for the programme will be paid sleeper class return train fare by the shortest route and will be provided free boarding and lodging for the duration of the course.

Mathematics Year – 2012

Celebrating Mathmatics

Proceedings of Events Organized at Sri Pratap College

Dr. Nazir Ahmad Gilkar

The year 2012 has been declared the National Year of Mathematics by the Government of India, as a tribute to Mathematics wizard, Srinivasa Ramanujan. A variety of activities were organized during the year on the campus as detailed thus: (i) Extension Lecture (ii) Mathematics Quiz (iii) Memorial Lecture (iv) Regional Seminar.

The Department of Mathematics organized an Extension Lecture on the topic, "Towards Understanding Culture of Mathematics" delivered by Prof. Mohammad Amin Sofi, Department of Mathematics, University of Kashmir, on 26th April, 2012. At the end of the presentation an interactive session proved highly productive as a number of students raised a variety of pertinent issues.

A mathematics quiz, titled "Quality Mathematical Sciences" was held on 31 May, 2012. The special feature of the quiz contest was its transaction with due support of I.T. based power point presentation.

A memorial lecture was organized to pay tributes to Prof. Noor-ud-Din (former Principal Sri Pratap College and former V.C. University of Kashmir) – a celebrated mathematician. An eminent SPCian and special guest on the occasion Prof. Nisar Ali, a well-known economist of the state in his lecture illuminated the audience with the professional knowledge of mathematics in relation to economics, statistics and physics.

The role played by Prof. Noor-ud-bin in the upliftment of educational standards in the Valley was highlighted. A well known SPCian (chief guest) Prof. Satish Raina (former Principal of the college) was praised for his constructive role in the development of the college. Prof. Satish Raina in his presidential address narrated several anecdotes and shared his experiences with the audience. On this occasion, Dr. Ashrafa Jeelani made an announcement to create a corpus of Rs. 5.00 lakhs for Prof. Noor-ud-Din scholarship to be awarded to orphans and best students in mathematics every year. This humble gesture was acknowledged by all and sundry. The funds have been deposited in the J & K Bank Ltd.

The theme of the seminar "Role of Mathematics in Developing Creative Thinking", was deliberated upon in two technical sessions, wherein more than 18 papers were presented. The seminar was sponsored by Department of Science and Technology. Dr. Naseer Ahmad Shah represented Department of Science and Technology and made a significant contribution in the academic input of the seminar.

The year long proceedings culminated in several recommendations, some of which are:

- To explore ways and means for executing new course combinations with a blending of Mathematics viz;
  - (i) Physics – Economics – Mathematics
  - (ii) Forensic Chemistry – Forensic Mathematics – Forensic Accounting
  - (iii) Actuarial Sciences – Quantitative Techniques – Mathematics
- To establish an advanced Institute of studies and Research in Mathematics;

- To avail of facilities of research projects from different funding agencies;
- To enrich the process of teaching – learning – evaluation through creative curriculum design, and innovative delivery methodology;
- To forge a sustained connectivity in mathematics curriculum at school, college and university level;
- To promote an academic culture for guest lectures, faculty exchange, and student mobility in a cluster of institutions.

Dr. Nazir Ahmad Gilkar is Principal Sri Pratap College and can be visited at

gilkarna@rediffmail.com

The faculty of Mathematics of the college including Prof. Asia Siddiqui (Head), Prof. Khursheed Ahmad Thakur, Prof. Tariq A. Shikari, Dr. Kanwal Jeet Singh deserve all appreciation and encouragement for their sincere efforts towards making all the events very successful and results orientated.

Details of Workshop/Conferences in India

For details regarding Advanced Training in Mathematics Schools
Visit: http://www.atmschools.org/

Name: National Workshop on Graph Colorings
Location: Indian Statistical Institute, Chennai Centre
Date: January 25–27, 2013
Visit: http://www.isichennai.res.in/∼nwgc2013/

Name: 3rd IIMA International Conference on Advanced Data Analysis, Business Analytics and Intelligence
Location: Indian Institute of Management, Ahmedabad, India.
Date: April 13–14, 2013
Visit: http://www.iimahd.ernet.in/icadabai2013/

Name: International Conference on Mathematics (ICM-2013)
Location: Kochi, Kerala, India (Venue of the Commencement of Conference will be announced soon)
Date: August 09–10, 2013
Visit: http://www.imrf.in/icm2013.html

Name: International Conference on Mathematics, Statistics and Computer Engineering (ICMSCE)
Location: Kochi, Kerala, India (Venue of the Commencement of Conference will be announced soon)
Date: August 09–10, 2013
Visit: http://www.imrf.in/icmsce2013.html

Name: National Conference on Mathematical Sciences and Applications 2013 (NCMSA 2013)
Location: Karunya University, Coimbatore
Date: March 14–15, 2013
Visit: http://www.karunya.edu/sh/maths/NCMSA-2013/index.html

Name: MACKIE 2013
Location: IIT Madras, Chennai
Date: February 04–06, 2013
Visit: http://www.mackie2013.iitm.ac.in/

Name: Indo-Slovenia Conference on Graph Theory and Applications (Indo-Slov-2013)
Location: Thiruvananthapuram, India
Date: February 22–24, 2013
Visit: http://indoslov2013.wordpress.com/

Name: 15th Annual Conference of the Society of Statistics, Computer and Application
Location: Apaji Institute of Mathematics and Applied Computer Technology, Banasthali Vidyapith, Rajasthan
Date: February 24–26, 2013
Visit: http://www.banasthali.org/banasthali/wcms/en/home/lower-menu/FinalBrochure.pdf

Name: International conclave on innovation in engineering and management
Location: Birla Institute of Technology, Patna
Date: February 22–23, 2013
Visit: http://www.iciem2013.org/

Name: 3rd Indo-German Workshop on Adaptive Finite Element Methods (WAFEM 2013)
Location: Institute of Mathematica and Applications, Bhubaneswar
Date: February 22 – March 2, 2013
Download application from http://www.math.iitb.ac.in/~npde-tca/ and send the filled up application to npde.2012@gmail.com

The Department of Mathematics, **University of Jammu** in collaboration with Jammu Mathematical Society is organizing a National Seminar on "Recent Trends in Analysis and Topology" during **February 26–28, 2013**. This seminar is clubbed with the 23rd Annual Conference of the Jammu Mathematical Society. Tentatively, there will be about nine Plenary lectures (three on each day), equal number of half an hour invited talks, and paper presentations. Eminent Mathematicians working in Analysis and related areas are expected to participate. A refereed seminar proceedings will be published.

Contact Details:

Prof. D. S. Jamwal (President)
dalipsj@yahoo.com

Prof. Bhopinder Singh (Vice President)
bhopinder7@rediffmail.com

Dr. K. S. Charak (Secretary)
kscharak7@rediffmail.com

Dr. Shakeel Ahmed Raina (Joint Secretary)
shakeelra2002@yahoo.co.in

Dr. Sunil Gupta (Cashier)
sunilgupta276819@yahoo.com

Visit: http://jammumathsoc.org/programmes.php

Details of Workshop/Conferences in Abroad

ICTP organizes numerous international conferences, workshops, seminars and colloquiums every year.
For details **Visit:** http://www.ictp.it/scientific-calendar.aspx

Name: Modular Representation Theory of Finite and p-adic Groups
Location: Institute for Mathematical Sciences, National University of Singapore, Singapore.
Date: April 1–26, 2013
Visit: http://www2.ims.nus.edu.sg/Programs/013mod/index.php

Name: Recreational Mathematics Colloquium III
Location: University of Azores, Ponta Delgada, Portugal.
Date: April 3–6, 2013
Visit: http://ludicum.org/ev/rm/13

Name: 38th Arkansas Spring Lecture Series in the Mathematical Sciences: "Extension and Interpolation of Functions"
Location: University of Arkansas, Fayetteville, Arkansas.
Date: April 4–6, 2013
Visit: `http://math.uark.edu/3742.php`

Name: 3rd International Conference on E-Learning and Knowledge management Technology (ICEKMT 2013)
Location: Hotel Ramada, Bangkok, Thailand.
Date: April 5–6, 2013
Visit: `http://www.icekm.com`

Name: IMA Mathematics in Finance
Location: Edinburgh Conference Centre, Heriot-Watt University, Edinburgh, United Kingdom
Date: April 8–9, 2013
Visit: `http://www.ima.org.uk/`

Name: Fourteenth International Conference on Numerical Combustion (NC13)
Location: Holiday Inn Riverwalk, San Antonio, Texas.
Date: April 8–10, 2013
Visit: `http://www.siam.org/meetings/nc13/`

Name: AIM Workshop: Geometric perspectives in mathematical quantum field theory
Location: American Institute of Mathematics, Palo Alto, California.
Date: April 8–12, 2013
Visit: `http://www.aimath.org/ARCC/workshops/geometricqft.html`

Name: Interactions between Noncommutative Algebra, Representation Theory, and Algebraic Geometry
Location: Mathematical Sciences Research Institute, Berkeley, California.
Date: April 8–12, 2013
Visit: `http://www.msri.org/web/msri/scientific/workshops/all-workshops/show/-/event/Wm9063`

Name: International School and Research Workshop on Complex Systems
Location: Centre de Recerca Matemática, Bellaterra, Barcelona.
Date: April 8–13, 2013

Name: Graduate Research Conference in Algebra and Representation Theory
Location: Kansas State University, Manhattan, Kansas.
Date: April 12–14, 2013
Visit: `http://www.math.ksu.edu/events/grad_conf_2013/`

Name: Geomathematics 2013
Location: Hotel Haus am Weinberg, Sankt Martin, Palatinate, Germany.
Date: April 15–17, 2013
Visit: `http://www.geomathematics2013.de`

Name: ICERM Workshop: Combinatorics, Multiple Dirichlet Series and Analytic Number Theory
Location: ICERM, Providence, Rhode Island.

Date: April 15–19, 2013
Visit: http://icerm.brown.edu/sp-s13-w3

Name: New Frontiers in Numerical Analysis and Scientific Computing
Location: Kent State University, Kent, Ohio.
Date: April 19–20, 2013 (NEW DATE)
Visit: http://www.math.kent.edu/∼li/LR60/

Name: Underrepresented Students in Topology and Algebra Research Symposium
Location: Purdue University, West Lafayette, Indiana.
Date: April 19–21, 2013
Visit: http://www.ustars.org

Name: 2013 Great Lakes SIAM conference on "Computational Mathematics: Modeling, Algorithms and Applications"
Location: Central Michigan University, Mount Pleasant, Michigan.
Date: April 20, 2013
Visit: http://people.cst.cmich.edu/zheng1x/GLSIAM2013.html

Name: Great Lakes Geometry Conference 2013
Location: Northwestern University, Evanston, Illinois.
Date: April 20–21, 2013
Visit: http://www.math.northwestern.edu/greatlakes2013

Name: The Cape Verde International Days on Mathematics 2013 (CVIM'2013)
Location: University of Cape Verde, Praia, Cape Verde.
Date: April 22–25, 2013
Visit: http://sites.google.com/site/cvim2013

Name: Algebra, Combinatorics and Representation Theory: a conference in honor of the 60th birthday of Andrei Zelevinsky
Location: Northeastern University, Boston, Massachusetts.
Date: April 24–28, 2013
Visit: http://www.math.neu.edu/∼bwebster/ACRT/

Name: Philosophy of Information: The Value of Information
Location: American University, Washington, DC.
Date: April 26, 2013
Visit: http://www.american.edu/cas/economics/info-metrics/workshop/workshop-2013-spring.cfm

Name: Mathematics of Planet Earth 2013. Establishing the scientific foundation for Quantitative Public Health Decision-Making: Linking surveillance, disease modeling, and simulation at the Fields Institute
Location: FIELDS, Toronto, Canada.
Date: April 29 – May 1, 2013
Visit: http://www.fields.utoronto.ca/programs/scientific/12-13/public_health/

Name: Non-Smooth Geometry
Location: Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, California.

Date: April 29 – May 3, 2013
Visit: http://www.ipam.ucla.edu/programs/iagws3/

Name: J-holomorphic Curves in Symplectic Geometry, Topology and Dynamics
Location: Centre de Recherches Mathématiques, Montréal, Canada.
Date: April 29 – May 10, 2013
Visit: http://www.crm.umontreal.ca/2013/Curves13/index_e.php

Name: Third Buea International Conference on the Mathematical Sciences
Location: University of Buea, Cameroon.
Date: April 30 – May 3, 2013
Visit: http://www.bueaconference.com

Name: Atkin Memorial Lecture and Workshop: Cohen-Lenstra Heuristics
Location: University of Illinois at Chicago, Chicago, Illinois.
Date: May 3–5, 2013
Visit: http://www.math.uic.edu/~rtakloo/atkin2013.html

Name: Mathematics of Planet Earth 2013 – Pan-Canadian Thematic Program – Major and Neglected Diseases in Africa
Location: Ottawa, Canada.
Date: May 5–9, 2013
Visit: http://www.crm.umontreal.ca/act/theme/theme_2013_1_en/africa13_e.php

Name: AIM Workshop: Algorithms for lattices and algebraic automorphic forms
Location: American Institute of Mathematics, Palo Alto, California.
Date: May 6–10, 2013
Visit: http://www.aimath.org/ARCC/workshops/algoautoforms.html

Name: The Commutative Algebra of Singularities in Birational Geometry: Multiplier Ideals, Jets, Valuations, and Positive Characteristic Methods
Location: Mathematical Sciences Research Institute, Berkeley, California.
Date: May 6–10, 2013
Visit: http://www.msri.org/web/msri/scientific/workshops/all-workshops/show/-/event/Wm9000

Name: Analysis, Complex Geometry, and Mathematical Physics: A Conference in Honor of Duong H. Phong
Location: Columbia University, New York, New York.
Date: May 7–11, 2013
Visit: http://math.columbia.edu/phong2013

Name: The International Conference on Technological Advances in Electrical, Electronics and Computer Engineering (TAEECE2013)
Location: Mevlana University, Konya, Turkey.
Date: May 9–11, 2013
Visit: http://sdiwc.net/conferences/2013/taeece2013/

Name: Mathematics of Planet Earth 2013 – Pan-Canadian Thematic Program – Impact of climate change on biological invasions and population distributions
Location: BIRS, Banff, Canada.
Date: May 12–17, 2013
Visit: `http://www.birs.ca/events/2013/5-day-workshops/13w5095`

Name: AIM Workshop: Nonhomogeneous boundary-value problems for nonlinear waves
Location: American Institute of Mathematics, Palo Alto, California.
Date: May 13–17, 2013
Visit: `http://www.aimath.org/ARCC/workshops/nonlinwaves.html`

Name: SIAM Conference on Applications of Dynamical Systems (DS13)
Location: Snowbird Ski and Summer Resort, Snowbird, Utah.
Date: May 19–23, 2013
Visit: `http://www.siam.org/meetings/ds13/`

Name: FEMTEC 2013 – 4th International Congress on Computational Engineering and Sciences
Location: Stratosphere Hotel, Las Vegas, Nevada.
Date: May 19–24, 2013
Visit: `http://femtec2013.femhub.com/`

Name: 15th International Conference on Functional Equations and Inequalities
Location: Ustron (near the borders with the Czech Republic and Slovakia), Poland.
Date: May 19–25, 2013
Visit: `http://mat.up.krakow.pl/icfei/15ICFEI/`

Name: Mathematics of Planet Earth 2013 – Pan-Canadian Thematic Program – Summer School on Mathematics of Infectious Disease
Location: York University, Toronto, Canada.
Date: May 19–27, 2013
Visit: `http://www.fields.utoronto.ca/programs/scientific/12-13/infectious/`

Name: Quasiconformal Geometry and Elliptic PDEs
Location: Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, California.
Date: May 20–24, 2013
Visit: `http://www.ipam.ucla.edu/programs/iagws4/`

Name: Questions, Algorithms, and Computations in Abstract Group Theory
Location: University of Braunschweig, Germany.
Date: May 21–24, 2013
Visit: `http://www.icm.tu-bs.de/~beick/conf/qac.html`

Name: Algebra and Topology: A conference celebrating Lionel Schwartz's 60th birthday
Location: University of Nantes, France.

Date: May 22–24, 2013

Visit: http://www.math.sciences.univ-nantes.fr/∼LS60/

Name: 10th Hstam International Congress on Mechanics

Location: Technical University of Crete, Chania, Crete, Greece.

Date: May 25–27, 2013

Visit: http://www.10hstam.tuc.gr

Name: The 19th International Conference on Difference Equations and Applications

Location: Sultan Qaboos University, Muscat, Oman.

Date: May 26–30, 2013

Visit: http://www.squ.edu.om/Portals/87/Conference/Conference2013/index.htm

Name: Topological, Symplectic and Contact Spring in Toulouse

Location: Université Paul Sabatier, Toulouse, France.

Date: May 26 – June 27, 2013

Visit: http://www.math.univ-toulouse.fr/∼barraud/Juin2013/

Name: Control, index, traces and determinants: The journey of a probabilist. A conference related to the work of Jean-Michel Bismut

Location: Amphithéâtre Lehman Bâtiment 200 Université Paris-Sud F-91405, Orsay, France.

Date: May 27–31, 2013

Visit: http://www.math.u-psud.fr/∼repsurf/ERC/Bismutfest/Bismutfest.html

Name: Introductory week to the Topological, Symplectic and Contact Spring in Toulouse

Location: Université Paul Sabatier, Toulouse, France.

Date: May 27–31, 2013

Visit: http://www.math.univ-toulouse.fr/∼barraud/Juin2013

Name: Summer School on Topics in Space-Time Modeling and Inference

Location: Aalborg University, Department of Mathematical Sciences, Aalborg, Denmark.

Date: May 27–31, 2013

Visit: http://csgb.dk/activities/2013/space-timemodeling/

Name: Masterclass: $(u, v, w$ knots$)x$(topology, combinatorics, low and high algebra) by Dror Bar-Natan (University of Toronto)

Location: QGM, Aarhus Univrsity, Aarhus, Denmark.

Date: May 27 – June 7, 2013

Visit: http://qgm.au.dk/currently/events/show/artikel/masterclass-uvw-knotsxtopology-combinatorics-low-and-high-algebra/

Name: "The first International Western Balkan Conference of Mathematical Sciences"

Location: Elbasan University, Elbasan, Albania.

Date: May 30, 2013

Name: Dynamical systems and statistical physics (91th Encounter between Mathematicians and Theoretical Physicists)

Location: Institut de Recherche Mathématique Avancée, Strasbourg, France.

Date: May 30 – June 1, 2013
Visit: `http://www-irma.u-strasbg.fr/article1321.html`

Name: GESTA 2013 (Topological, Symplectic and Contact Spring in Toulouse)
Location: Université Paul Sabatier, Toulouse, France.
Date: June 3–7, 2013
Visit: `http://www.math.univ-toulouse.fr/~barraud/Juin2013`

Name: 8th Spring School on Analysis
Location: PasekynadJizerou, Czech Republic.
Date: June 2–8, 2013
Visit: `http://www.karlin.mff.cuni.cz/katedry/kma/ss/jun13`

Name: MEGA 2013: Effective Methods in Algebraic Geometry
Location: Goethe University, Frankfurt am Main, Germany.
Date: June 3–7, 2013
Visit: `http://www.math.uni-frankfurt.de/mega2013`

Name: Summer school on Finsler geometry with applications to low-dimensional geometry and topology
Location: Department of Mathematics, University of the Aegean, Karlovassi, Island of Samos, Greece.
Date: June 3–9, 2013
Visit: `http://myria.math.aegean.gr/conferences/finsler13/index.html`

Name: Moduli Spaces and their Invariants in Mathematical Physics
Location: Centre de RecherchesMathématiques, Montréal, Canada.
Date: June 3–14, 2013
Visit: `http://www.crm.umontreal.ca/2013/Moduli13/index_e.php`

Name: Nonlinear expectations, stochastic calculus under Knightian uncertainty, and related topics
Location: Institute for Mathematical Sciences, National University of Singapore, Singapore.
Date: June 3 – July 12, 2013

Name: Focus Program on Noncommutative Geometry and Quantum Groups
Location: Fields Institute for Research in Mathematical Sciences, Toronto, Ontario, Canada.
Date: June 3–28, 2013
Visit: `http://www.fields.utoronto.ca/programs/scientific/12-13/quantumgroups`

Name: Conference on Nonlinear Mathematical Physics: Twenty Years of JNMP
Location: The Sophus Lie Conference Center, Nordfjordeid, Norway.
Date: June 4–14, 2013
Visit: `http://staff.www.ltu.se/~johfab/jnmp/index.html`

Name: 4th Novi Sad Algebraic Conference-NSAC 2013
Location: Department of Mathematics and Informatics, Faculty of Science, University of Novi Sad, Novi Sad, Serbia.
Date: June 5–9, 2013
Visit: `http://sites.dmi.rs/events/2013/nsac2013`

Name: XVth International Conference on Geometry, Integrability and Quantization
Location: Sts. Constantine and Elena resort, near Varna, Bulgaria.
Date: June 7–12, 2013
Visit: `http://www.bio21.bas.bg/conference`

Name: 39th International Conference "Applications of Mathematics in Engineering and Economics" – AMEE'13
Location: Technical University Leisure House, Sozopol, Bulgaria
Date: June 8–13, 2013
Visit: `http://www.aimath.org/ARCC/workshops/autoformcovergp.html`

Name: AIM Workshop: Automorphic forms and harmonic analysis on covering groups
Location: American Institute of Mathematics, Palo Alto, California.
Date: June 10–14, 2013
Visit: `http://www.aimath.org/ARCC/workshops/autoformcovergp.html`

Name: ApplMath13, 8th Conference on Applied Mathematics and Scientific Computing
Location: Sibenik, Croatia.
Date: June 10–14, 2013
Visit: `http://applmath13.math.hr`

Name: Computational Methods and Function Theory 2013
Location: Shantou University, Shantou, Guangdong, China.
Date: June 10–14, 2013
Visit: `http://math.stu.edu.cn/cmft/index.asp`

Name: Pde's, Dispersion, Scattering theory and Control theory
Location: University of Monastir, Monastir, Tunisia.
Date: June 10–14, 2013
Contact: Kais Ammari, `kais.ammari@fsm.rnu.tn`, Gilles Lebeau, `Gilles.Lebeau@unice.fr`

Name: Recent Advances in Hodge Theory: Period Domains, Algebraic Cycles, and Arithmetic
Location: UBC Campus, Vancouver, B.C., Canada.
Date: June 10–20, 2013
Visit: `http://icmca2013.cbu.edu.tr/`

Name: 4th International Conference on Mathematical and Computational Applications
Location: Celal Bayar University, Applied Mathematics and Computation Center, Manisa, Turkey.
Date: June 11–13, 2013
Visit: `http://icmca2013.cbu.edu.tr/`

Name: 6th Chaotic Modeling and Simulation International Conference (CHAOS2013)
Location: Yeditepe University, Istanbul, Turkey.
Date: June 11–14, 2013
Visit: `http://www.cmsim.org`

Name: Tenth edition of the Advanced Course in Operator Theory and Complex Analysis
Location: Sevilla, Spain.
Date: June 12–14, 2013
Visit: `http://congreso.us.es/ceacyto/2013`

Name: 51st International Symposium on Functional Equations
Location: Rzeszów, Poland.
Date: June 16–23, 2013
Contact: Józef Tabor, `tabor@univ.rzeszow.pl`

Name: AIM Workshop: Exponential random network models
Location: American Institute of Mathematics, Palo Alto, California.
Date: June 17–21, 2013
Visit: `http://www.aimath.org/ARCC/workshops/exprandnetwork.html`

Name: Summer school on Donaldson hypersurfaces (in Topological, Symplectic and Contact Spring in Toulouse)
Location: La Llagonne, France.
Date: June 17–21, 2013
Visit: `http://www.math.univ-toulouse.fr/~barraud/Juin2013`

Name: Numerical Computations: Theory and Algorithms (International conference and Summer School NUMTA2013)
Location: Eurolido Hotel, Falerna (CZ), Tyrrhenian Sea, Italy.
Date: June 17–23, 2013
Visit: `http://wwwinfo.deis.unical.it/~yaro/numta2013`

Name: Algebraic Graph Theory
Location: University of Wyoming, Laramie, Wyoming.
Date: June 17–28, 2013
Visit: `http://www.uwyo.edu/jwilliford/rmmc2013/rmmc_2013.html`

Name: Algebraic Topology
Location: Mathematical Sciences Research Institute, Berkeley, California.
Date: June 17–28, 2013
Visit: `http://www.msri.org/web/msri/scientific/workshops/summer-graduate-workshops/show/-/event/Wm9603`

Name: SUMMER@ICERM: 2013 Undergraduate Summer Research Program Geometry and Dynamics
Location: ICERM, Providence, Rhode Island.
Date: June 17 – August 9, 2013
Visit: `http://icerm.brown.edu/summerug_2013`

Name: The 9th East Asia SIAM (EASIAM) Conference & The 2nd Conference on Industrial and Applied Mathematics (CIAM)
Location: Bandung Institute of Technology, Bandung, Indonesia.
Date: June 18–20, 2013
Visit: `http://www.math.itb.ac.id/~easiam2013/`

Name: Nonlinear Elliptic and Parabolic Partial Differential Equations
Location: Dipartimento di Matematica, Politecnico di Milano, Italy.
Date: June 19–21, 2013
Visit: `http://www.mate.polimi.it/nep2de/`

Name: 4th International conference "Nonlinear Dynamics-2013"
Location: Sevastopol, Ukraine.
Date: June 19–22, 2013
Visit: `https://sites.google.com/site/ndkhpi2013/home`

Name: "Experimental and Theoretical Methods in Algebra, Geometry and Topology". Held in the honor of AlexandruDimca and Stefan Papadima on the occasion of their 60th birthday
Location: Mangalia (near Constanta), Romania.
Date: June 21–24, 2013
Visit: `http://math.univ-ovidius.ro/Conference/ETMAGT60/GeneralInfo.htm`

Name: Physics and Mathematics of Nonlinear Phenomena 2013
Location: Hotel Le Sirene, Gallipoli, South of Italy.
Date: June 22–29, 2013
Visit: `http://pmnp2013.dmf.unisalento.it/`

Name: Numerical Analysis and Scientific Comlputation with Applications (NASCA13)
Location: University of Littoral Cote d'Opale, Calais, France.
Date: June 24–26, 2013
Visit: `http://www-lmpa.univ-littoral.fr/NASCA13/`

Name: EACA'S Second International School on Computer Algebra and Applications
Location: Faculty of Sciences, University of Valladolid, Spain.
Date: June 24–28, 2013
Visit: `http://monica.unirioja.es/web_2EACA/index.html`

Name: Low-dimensional Topology and Geometry in Toulouse
Location: Université Paul Sabatier, Institut de Mathématiques de Toulouse, Toulouse, France.
Date: June 24–28, 2013
Visit: `http://www.math.univ-toulouse.fr/top-geom-conf-2013/en/ldtg-mb/`

Name: 5th Conference for Promoting the Application of Mathematics in Technical and Natural Sciences (AMiTaNS'13)
Location: Resort of Albena, Bulgaria.
Date: June 24–29, 2013
Visit: `http://2013.eac4amitans.eu/`

Name: IAS/PCMI Summer 2013: Geometric Analysis
Location: Mathematical Sciences Research Institute, Berkeley, California.
Date: June 30 – July 20, 2013
Visit: `http://www.msri.org/web/msri/scientific/workshops/summer-graduate-workshops/show/-/event/Wm9754`

Name: Seminaire de MathematiquesSuperieures 2013: Physics and Mathematics of Link Homology
Location: Montreal, Canada.
Date: June 24 – July 5, 2013
Visit: http://www.msri.org/web/msri/scientific/workshops/summer-graduate-workshops/show/-/event/Wm9461

Name: International Symposium on Symbolic and Algebraic Computation (ISSAC)
Location: Northeastern University, Boston, Massachussetts.
Date: June 26–29, 2013
Visit: http://www.issac-conference.org/2013/

Name: 5th National Dyscalculia and Maths Learning Difficulties Conference, UK
Location: Cumberland Hotel, London, England.
Date: June 27, 2013
Visit: http://www.dyscalculia-maths-difficulties.org.uk/

Name: British Combinatorial Conference 2013
Location: Royal Holloway, University of London, Egham, United Kingdom.
Date: June 30 – July 5, 2013
Visit: http://bcc2013.ma.rhul.ac.uk/

Name: 11th International Conference on Vibration Problems (ICOVP-2013)
Location: Instituto Superior Técnico, Lisbon, Portugal
Date: September 9–1, 2013
Visit: http://www.icovp.com/

Name: International Conference on Complex Analysis & Dynamical Systems VI
Location: Nahariya, Israel
Date: May 19–24, 2013
Visit: http://conferences.braude.ac.il/math/Math2013/default.aspx

Name: 4th International Symposium on Inverse problems, Design and Optimization (IPDO-2013)
Location: Albi, France
Date: June 26–28, 2013
Visit: http://ipdo2013.congres-scientifique.com/

Computational Methods and Function Theory 2013

June 10–14; Shantou University, China

Description: The general theme of the meeting concerns various aspects of interaction of complex variables and scientific computation, including related topics from function theory, approximation theory and numerical analysis. Another important aspect of the CMFT meetings, previously held in Valparaiso 1989, Penang 1994, Nicosia 1997, Aveiro 2001, Joensuu 2005 and Ankara 2009, is to promote the creation and maintenance of contacts with scientists from diverse cultures.

Information:

E-mail: `ptli@stu.edu.cn`;
`http://math.stu.edu.cn/cmft/index.asp`

Department of Mathematics

Visvesvaraya National Institute of Technology, Nagpur 440010, Maharastra, India

Advertisement Date: 28-12-2012

Applications are invited for the position of full-time Fellowship (JRF) for a period of Three years to work under the National Board for Higher Mathematics (NBHM), Department of Atomic Energy sponsored research project entitled "Numerical methods for solving Singularly Perturbed Delay Differential Equations with large Delays" under Dr. P. Pramod Chakravarthy, Associate Professor in Mathematics.

Eligibility: Candidates are expected to have post graduate degree in Mathematics/Applied Mathematics, with a good academic background and sound knowledge in Numerical Analysis as well as knowledge in programming languages such as FORTRAN/C++.

Junior Research Fellow (JRF)/Senior Research Fellow (SRF)

Designation & Qualification	Emoluments per month for first 2 years	Emoluments per month after 2 years/SRF
Junior Research Fellow (JRF) leading to Ph.D.	Rs. 16,000/-	Rs. 18,000/-

All other allowances are as per guidelines of NBHM (DAE).

Mode of Selection: By Interview
(The selected candidate will be permitted to register for Ph.D. at VNIT Nagpur. He/she will have a chance to work/interact with National Board for Higher Mathematics (NBHM).

The applications (Bio-data) may be submitted to the following address by Post/E-mail.

Dr. P. Pramod Chakravarthy
Associate Professor (Principal Investigator)
Department of Mathematics
V.N.I.T. Nagpur 440010, Maharastra
E-mail: `ppchakravarthy@mth.vnit.ac.in`

The selected candidate shall be required to work in the Department of Mathematics, Visvesvaraya National Institute Technology, Nagpur as full time fellow.

The last date of receipt of application is 31st January 2013.

(Dr. P. Pramod Chakravarthy)
Principal Investigator

The National Academy of Sciences, India

5, Lajpatrai Road, Allaahbad

National Level Scientific Writing Contest

The National Academy of Sciences, India invites entries (scientific write-up on "**Nanotechnology**") from the students of B.Sc./B.Tech./M.B.B.S./B.U.M.S./B.A.S.M. of Indian Universities by January 25, 2013 for evaluation and further participation (for selected entries/write-up only) in On the Spot Contest to be held on February 26, 2013 at National Academy of Sciences, India, 5 Lajpatrai Road Allahabad. The idea behind, is to generate scientific temperament among the students and also make them aware about the current scientific and environmental issues. The Rules for Entries/Scientific Write-up on "**Nanotechnology**" has been given below:

Rules for National Level Scientific Writing Contest

1. Topic for the Scientific Writing Contest is "**Nanotechnology**".

2. The Write-up should be either in English or in Hindi; entries be precise and confined to 3000 words only.
3. It should be neatly typed with double space on A4 size paper.
4. Proper attention must be paid on Syntax and Grammar.
5. The participants are requested to write an "Abstract" in the beginning of the write-up in approximately 150 words.
6. It would be appreciated if the participants cite the references (if any) in the end of the text, which would not be counted in 3000 words.
7. The participants must write their University/College and Home Addresses along with their telephone number(s) and E-mail I.D. in the end of the write-up.
8. The write-up prepared as per above guidelines be sent to the Office of the National Academy of Sciences, India by January 25, 2013 on the address given below* through proper channel (duly forwarded by either the Head of the concerned Department or Head of the Institution).
9. 5 best Scientific Write-ups will be selected from entries received in the Academy. The selected students will be invited to participate in "On the Spot Contest" on February 26, 2013 at Allahabad. The participants will have to write on a related topics in 3 hours.
10. Out of these five contestants, finally three would be given cash awards on National Science Day (28th February 2013); and the rest would be given the participation certificates.
11. The Academy will bear the travel expenses (second class train fare) of these five students; and modest arrangement of their stay at Allahabad during these days (26–28, February, 2013) will also be made by the Academy.

---

*The National Academy of Sciences, India. 5, Lajpatrai Road, Allahabad 211 002. Phones: 0532-2640224 (O); 09415306124(M);
E-mail: `allahabad.nasi@gmail.com`

The Mathematics Newsletter may be download from the RMS website at
`www.ramanujanmathsociety.org`